Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Relative motion
Exercise A, Question 1

Question:

The velocity vectors of two particles P and Q are $\mathbf{v}_P$ and $\mathbf{v}_Q$ respectively. Find the velocity of P relative to Q and the relative speed of Q to P in each of the following cases:

a $\mathbf{v}_P = (5\mathbf{i} + 6\mathbf{j})\text{m s}^{-1}$, $\quad \mathbf{v}_Q = (4\mathbf{i} - 3\mathbf{j})\text{m s}^{-1}$

b $\mathbf{v}_P = 6\mathbf{j}\text{m s}^{-1}$, $\quad \mathbf{v}_Q = (-2\mathbf{i} + \mathbf{j})\text{m s}^{-1}$

c $\mathbf{v}_P = (5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})\text{m s}^{-1}$, $\quad \mathbf{v}_Q = (\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})\text{m s}^{-1}$.

Solution:

a $\;{}_P\mathbf{v}_Q = \mathbf{v}_P - \mathbf{v}_Q = (5\mathbf{i} + 6\mathbf{j}) - (4\mathbf{i} - 3\mathbf{j}) = (\mathbf{i} + 9\mathbf{j})\text{ m s}^{-1}$

$|{}_Q\mathbf{v}_P| = |{}_P\mathbf{v}_Q| = |\mathbf{i} + 9\mathbf{j}| = \sqrt{82}\text{ m s}^{-1}$

b $\;{}_P\mathbf{v}_Q = \mathbf{v}_P - \mathbf{v}_Q = 6\mathbf{j} - (-2\mathbf{i} + \mathbf{j}) = (2\mathbf{i} + 5\mathbf{j})\text{ m s}^{-1}$

$|{}_Q\mathbf{v}_P| = |{}_P\mathbf{v}_Q| = |(2\mathbf{i} + 5\mathbf{j})| = \sqrt{2^2 + 5^2} = \sqrt{29}\text{ m s}^{-1}$

c $\;{}_P\mathbf{v}_Q = \mathbf{v}_P - \mathbf{v}_Q = (5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) - (\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

$= (4\mathbf{i} + 12\mathbf{j} - 5\mathbf{k})\text{ m s}^{-1}$

$|{}_Q\mathbf{v}_P| = |{}_P\mathbf{v}_Q| = \sqrt{4^2 + 12^2 + (-5)^2} = \sqrt{16 + 144 + 25}$

$= \sqrt{185}\text{ m s}^{-1}$

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Relative motion
Exercise A, Question 2

Question:

A man is driving due north at 40 km h^{-1} along a straight road when he notices that the wind appears to be coming from N60°W with a speed of 40 km h^{-1}. Find the actual velocity of the wind.

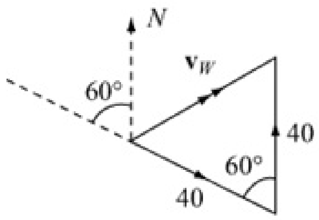
Solution:

$\mathbf{v}_M = 40 \text{ km h}^{-1}$ due N

${}_W\mathbf{v}_M = 40 \text{ km h}^{-1}$ from N60°W

${}_W\mathbf{v}_M = \mathbf{v}_W - \mathbf{v}_M \Rightarrow \mathbf{v}_W = {}_W\mathbf{v}_M + \mathbf{v}_M$

Draw the vector Δ:



Vector Δ is equilateral so $|\mathbf{v}_W| = 40 \text{ km h}^{-1}$ in direction N60°E

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Relative motion
Exercise A, Question 3

Question:

The velocity of A relative to B is $(2\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$ and the velocity of B relative to C is $(-\mathbf{i}+4\mathbf{j})\text{m s}^{-1}$. Find the velocity of A relative to C.

Solution:

$$\left.\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ \text{and } {}_B\mathbf{v}_C &= \mathbf{v}_B - \mathbf{v}_C \end{aligned}\right\} \text{ adding}$$

$${}_A\mathbf{v}_B + {}_B\mathbf{v}_C = \mathbf{v}_A - \mathbf{v}_C = {}_A\mathbf{v}_C$$

Hence, ${}_A\mathbf{v}_C = (2\mathbf{i}+3\mathbf{j}) + (-\mathbf{i}+4\mathbf{j}) = (\mathbf{i}+7\mathbf{j})\text{ m s}^{-1}$

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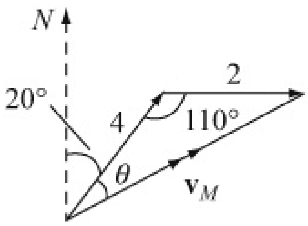
Relative motion
Exercise A, Question 4

Question:

A man who can row at 4 km h^{-1} in still water rows with his boat steering in the direction N20°E. There is a current of 2 km h^{-1} flowing due E. With what speed and in what direction does the boat actually move?

Solution:

${}_M\mathbf{v}_W$ is 4 km h^{-1} in N20°E
$\mathbf{v}_W$ is 2 km h^{-1} due E
${}_M\mathbf{v}_W = \mathbf{v}_M - \mathbf{v}_W \Rightarrow \mathbf{v}_M = {}_M\mathbf{v}_W + \mathbf{v}_W$



Draw the vector Δ:
by cosine rule,

$$|\mathbf{v}_M|^2 = 4^2 + 2^2 - 2 \times 4 \times 2 \cos 110°$$

$$|\mathbf{v}_M| = \sqrt{20 - 16\cos 110°} = 5.05 \text{ km h}^{-1}$$

by sine rule

$$\frac{\sin\theta}{2} = \frac{\sin 110°}{5.047}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{2\sin 110°}{5.047}\right) = 21.9°$$

The boat moves at 5.05 km h^{-1} in N41.9° E

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Relative motion
Exercise A, Question 5

Question:

A woman is walking along a road with a speed of 4 km h^{-1}. The rain is falling vertically at 7 km h^{-1}. At what angle to the vertical should she hold her umbrella?

Solution:

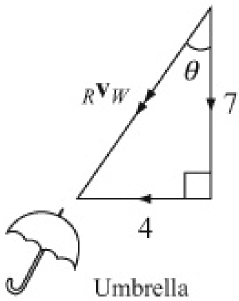
$\mathbf{v}_W$ is 4 km h^{-1} horizontally $(\rightarrow)$

$\mathbf{v}_R$ is 7 km h^{-1} vertically $(\downarrow)$

${}_R\mathbf{v}_W = \mathbf{v}_R - \mathbf{v}_W$ Draw the vector Δ:

$\tan\theta = \frac{4}{7} \Rightarrow \theta = 29.7°$

Angle is $29.7°$



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Relative motion

Exercise A, Question 6

Question:

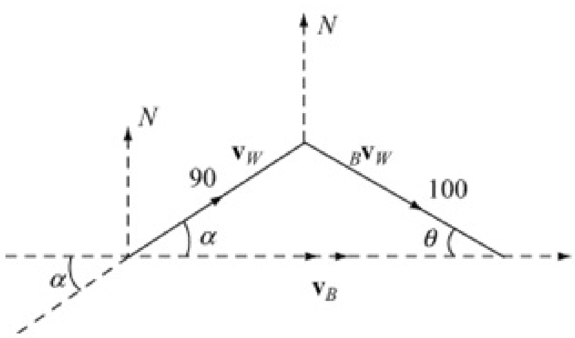
A bird can fly in still air at 100 km h^{-1}. The wind blows at 90 km h^{-1} from W$\alpha°$S, where $\tan \alpha = \frac{3}{4}$. The bird wishes to return to its nest which is due E of its present position. In which direction, relative to the air, should it fly?

Solution:

	Mag	Dir
${}_B\mathbf{v}_W$	100	?
$\mathbf{v}_W$	90	From α W of S $(\tan \alpha = \frac{3}{4})$
$\mathbf{v}_B$	?	due E

${}_B\mathbf{v}_W = \mathbf{v}_B - \mathbf{v}_W \Rightarrow \mathbf{v}_B = {}_B\mathbf{v}_W + \mathbf{v}_W$

Draw the vector Δ: (Draw $\mathbf{v}_W$ FIRST, since we have both its magnitude and direction)



$\frac{\sin \theta}{90} = \frac{\sin \alpha}{100}$

$\sin \theta = \frac{9}{10} \times \frac{3}{5} = 0.54$

$\Rightarrow \theta = 32.68°$

Hence, the bird should fly on a bearing of $122.68°$ or $32.68°$ S of E.

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Relative motion
Exercise A, Question 7

Question:

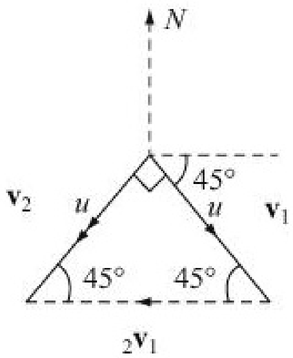
Two cars are moving at the same speed. The first is moving SE while the other appears to be approaching it from the east. Find the direction in which the second car is moving.

Solution:

	Mag	Dir
$\mathbf{v}_1$	u	SE
${}_2\mathbf{v}_1$	?	From E
$\mathbf{v}_2$	u	?

${}_2\mathbf{v}_1 = \mathbf{v}_2 - \mathbf{v}_1$

$\Rightarrow \mathbf{v}_2 = \mathbf{v}_1 + {}_2\mathbf{v}_1$



Draw the vector Δ:
Triangle is isosceles
Direction of $\mathbf{v}_2$ is SW

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Relative motion
Exercise A, Question 8

Question:

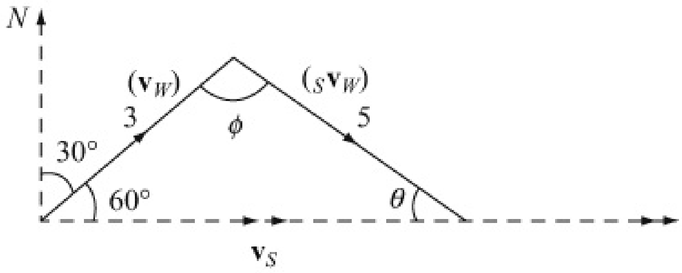
A ship has to travel 20 km due E. If the speed of the ship in still water is 5 km h^{-1} and if there is a current of 3 km h^{-1} in the direction N30°E, find how long it will take.

Solution:

	Mag	Dir
$\mathbf{v}_S$	?	E
${}_S\mathbf{v}_W$	5	?
$\mathbf{v}_W$	3	N30°E

$${}_S\mathbf{v}_W = \mathbf{v}_S - \mathbf{v}_W$$
$$\Rightarrow \mathbf{v}_S = \mathbf{v}_W + {}_S\mathbf{v}_W$$

Draw vector Δ:



$$\frac{\sin\theta}{3} = \frac{\sin 60°}{5} \Rightarrow \sin\theta = \frac{3\sqrt{3}}{10} \Rightarrow \theta = 31.3°$$
$$\Rightarrow \phi = 180° - (60° + 31.3°) = 88.7°$$
$$\therefore |\mathbf{v}_S|^2 = 3^2 + 5^2 - 2\times3\times5\cos 88.7° = 34 - 30\cos 88.7°$$
$$|\mathbf{v}_S| = 5.772\text{ km h}^{-1}$$
$$\therefore \text{Time} = \frac{20}{5.772}\text{ h} = 3.464$$
$$= 3\text{ h }28\text{ minutes (nearest minute)}$$

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Relative motion
Exercise A, Question 9

Question:

An aeroplane can fly at 600 km h^{-1} in still air. It has to fly to an airport which is SW of its current position. There is a wind of 90 km h^{-1} blowing from N20°W .

a What course should the aeroplane set?
b What is the ground speed of the aeroplane?

Solution:

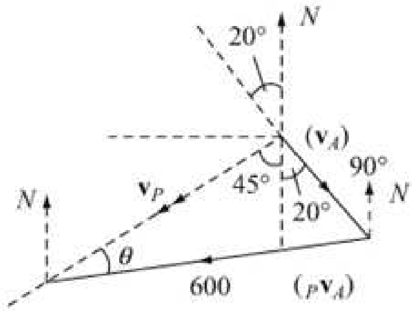
	Mag	Dir
$_P\mathbf{v}_A$	600	?
$\mathbf{v}_A$	90	From N20°W
$\mathbf{v}_P$	?	SW

$_P\mathbf{v}_A$ is the velocity of the plane relative to the air.

$_P\mathbf{v}_A = \mathbf{v}_P - \mathbf{v}_A$

$\Rightarrow \mathbf{v}_P = \mathbf{v}_A + {_P\mathbf{v}_A}$

a Draw the vector Δ:



$$\frac{\sin\theta}{90} = \frac{\sin 65°}{600}$$

$$\Rightarrow \sin\theta = \frac{9\sin 65°}{60}$$

$$\Rightarrow \theta = 7.813°$$

Course is ≤S 52.8°W

b 3rd angle of vector Δ

$= 180° - (65° + 7.813°)$

$= 107.187°$

$$\frac{|\mathbf{v}_P|}{\sin 107.187°} = \frac{600}{\sin 65°}$$

$$\Rightarrow |\mathbf{v}_P| = \frac{600\sin 107.187°}{\sin 65°} = 632.46.$$

i.e. ground speed of aeroplane is 632 km h^{-1} (nearest km h^{-1})

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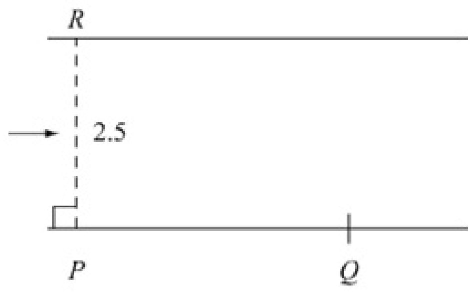
Relative motion
Exercise A, Question 10

Question:

A river flows at 2.5 m s^{-1}. A fish swims from a point P to a point Q which is directly upstream from P, and then back to P with speed 6.5 m s^{-1} relative to the water. A second fish, in the same time and with the same relative speed as the first fish, swims to the point R on the bank directly opposite to P and back to P. Find the ratio $PQ : PR$.

Solution:



$|\mathbf{v}_R| = 2.5$

$|_F\mathbf{v}_R| = 6.5$

$_F\mathbf{v}_R = \mathbf{v}_F - \mathbf{v}_R$

$\Rightarrow \mathbf{v}_F = {}_F\mathbf{v}_R + \mathbf{v}_R$

Fish 1

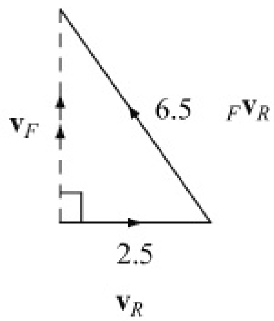
$$_P t_Q = \left(\frac{PQ}{6.5+2.5}\right) = \frac{PQ}{9}$$

$$_Q t_P = \left(\frac{PQ}{6.5-2.5}\right) = \frac{PQ}{4}$$

$$\therefore \text{Total time} = \frac{PQ}{9} + \frac{PQ}{4} = \frac{13PQ}{36}$$

Fish 2 (P to R)

	Mag	Dir
$\mathbf{v}_R$	2.5	→
$_F\mathbf{v}_R$	6.5	?
$\mathbf{v}_F$	?	↑



$|\mathbf{v}_F| = \sqrt{6.5^2 - 2.5^2} = 6$

$|\mathbf{v}_F| = 6$ for R to P also. $\quad \therefore \text{total time} = \frac{2PR}{6} = \frac{PR}{3}$

so, $\frac{13PQ}{36} = \frac{PR}{3} \Rightarrow PQ : PR = 12 : 13$

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Relative motion
Exercise A, Question 11

Question:

A man is cruising in a boat which is capable of a speed of 10 km h^{-1} in still water. He is heading towards a marker buoy which is NE of his position and 6 km away. The current is running at a speed of 3 km h^{-1} due E.

a What course should he set?

b How long will take to reach the buoy?

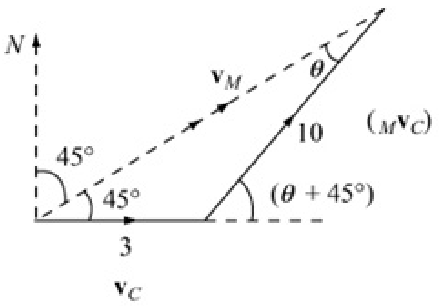
Solution:

a

	Mag	Dir
${}_M\mathbf{v}_C$	10	?
$\mathbf{v}_C$	3	due E
$\mathbf{v}_M$	?	NE

$${}_M\mathbf{v}_C = \mathbf{v}_M - \mathbf{v}_C$$
$$\Rightarrow \mathbf{v}_M = \mathbf{v}_C + {}_M\mathbf{v}_C$$

Draw the vector Δ:



$$\frac{\sin\theta}{3} = \frac{\sin 45°}{10}$$
$$\sin\theta = \frac{3\sqrt{2}}{20}$$
$$\Rightarrow \theta = 12.247°$$

Course is N $(90 - \theta - 45°)$E

i.e. N(32.753°)E

i.e. N32.8°E

b

$$\frac{|\mathbf{v}_M|}{\sin(\theta + 45°)} = \frac{10}{\sin 45°}$$

$$\Rightarrow |\mathbf{v}_M| = \frac{10\sin 57.247°}{\sin 45°} = 11.8936\ldots$$

$$\therefore \text{time} = \frac{6}{11.8936} = 30 \text{ minutes} \quad \text{(nearest minute)}$$

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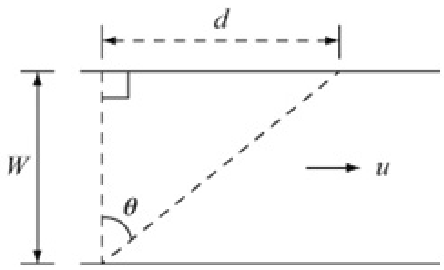
Relative motion
Exercise A, Question 12

Question:

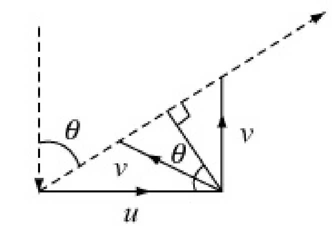
A river flows at a speed u. A boat is rowed with speed v relative to the river. The width of the river is w and the boat is to reach the opposite bank at a distance d downstream. Show that, if $\frac{uw}{\sqrt{w^2+d^2}} < v < u$, there are two directions in which the boat may be steered.

Solution:



	Mag	Dir
$\mathbf{v}_W$	u	$\rightarrow$
${}_B\mathbf{v}_W$	v	?
$\mathbf{v}_B$	?	θ

$${}_B\mathbf{v}_W = \mathbf{v}_B - \mathbf{v}_W$$
$$\therefore \mathbf{v}_B = \mathbf{v}_W + {}_B\mathbf{v}_W$$



Draw vector Δ:

Two possible positions for a vector of length v if $u > v > u\cos\theta$

From top diagram, $\cos\theta = \frac{w}{\sqrt{w^2+d^2}}$

as required.

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Relative motion
Exercise A, Question 13

Question:

A car is moving due W and the wind appears, to the driver, to be coming from a direction N60°W. When he drives due E at the same speed the wind appears to be coming from a direction N30°E. If he now travels due S at the same speed, find the apparent direction of the wind.

Solution:

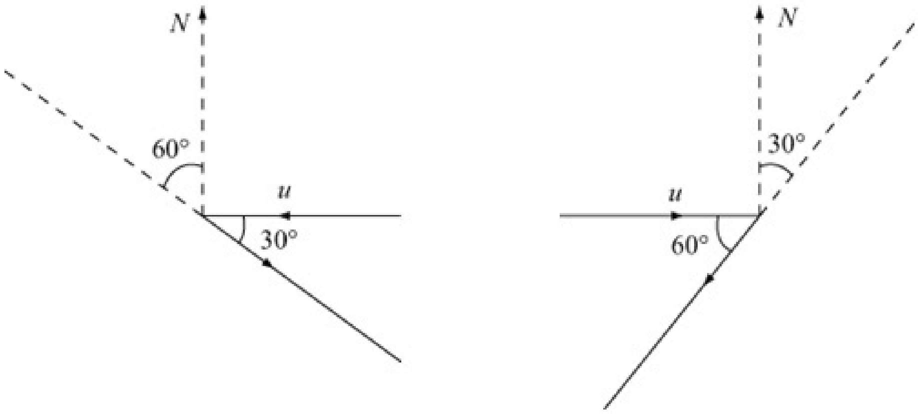
Scenario 1

	Mag	Dir
$\mathbf{v}_C$	u	due W
${}_W\mathbf{v}_C$	?	From N60°W
$\mathbf{v}_W$	?	?

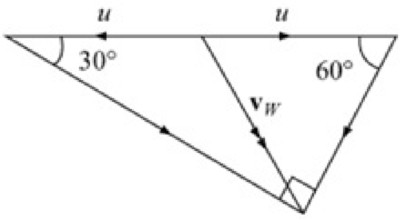
Scenario 2

	Mag	Dir
$\mathbf{v}_C$	u	due E
${}_W\mathbf{v}_C$	?	From N30°E
$\mathbf{v}_W$	?	?

$${}_W\mathbf{v}_C = \mathbf{v}_W - \mathbf{v}_C \Rightarrow \mathbf{v}_W = \mathbf{v}_C + {}_W\mathbf{v}_C$$



Now, put the two triangles together, bearing in mind that the resultant, in both cases, is $\mathbf{v}_W$ i.e. will be a common side:



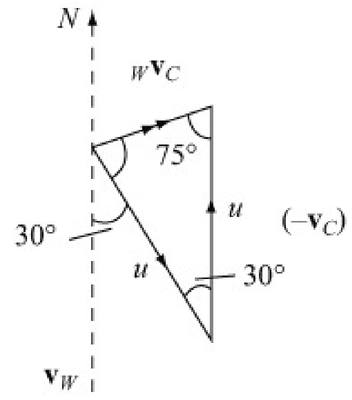
Using angle in a semi-circle is $90°$ property $|\mathbf{v}_W| = u$ (radius of circle). Then $RH\Delta$ is equilateral. Hence, direction of wind is on a bearing of $150°$(S30°E)

Scenario 3

	Mag	Dir
$\mathbf{v}_C$	u	due S
${}_W\mathbf{v}_C$	?	?
$\mathbf{v}_W$	u	S30°E

Draw vector Δ

${}_W\mathbf{v}_C = \mathbf{v}_W - \mathbf{v}_C$



Vector Δ is isosceles.

∴ base angles are both 75°

∴ direction of ${}_W\mathbf{v}_C$ is N75°E

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Relative motion
Exercise A, Question 14

Question:

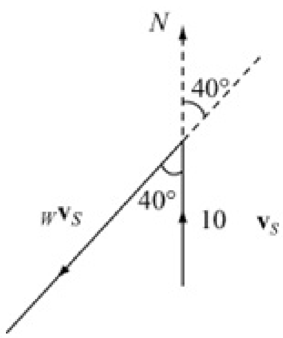
When a ship travels at 10 km h^{-1} due N the wind appears to be coming from a direction N40°E. When the speed is increased to 25 km h^{-1} the wind appears to be coming from a direction N25°E. Find the true speed and direction of the wind.

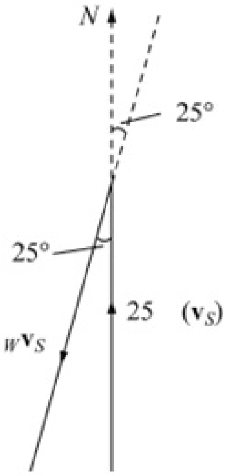
Solution:

	Mag	Dir
$\mathbf{v}_S$	10	due N
${}_W\mathbf{v}_S$	?	From N40°E
$\mathbf{v}_W$	?	?

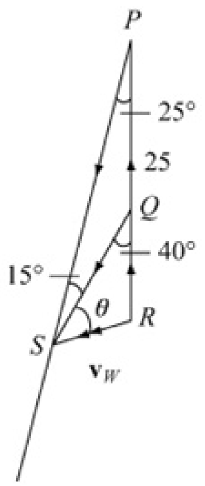
	Mag	Dir
$\mathbf{v}_S$	25	due N
${}_W\mathbf{v}_S$	?	N25°E
$\mathbf{v}_W$	?	?

$${}_W\mathbf{v}_S = \mathbf{v}_W - \mathbf{v}_S \Rightarrow \mathbf{v}_W = \mathbf{v}_S + {}_W\mathbf{v}_S$$





We now put the two triangles together:



In ΔPQS, $PQ = 15$

$$\frac{QS}{\sin 25^\circ} = \frac{15}{\sin 15^\circ}$$

$$\Rightarrow QS = 24.493$$

In ΔQRS,

$$|\mathbf{v}_W|^2 = 24.493^2 + 10^2 - 2 \times 24.493 \times 10 \cos 40^\circ$$

$$|\mathbf{v}_W| = 18.02 \text{ km h}^{-1}$$

In ΔQRS,

$$\frac{\sin\theta}{10} = \frac{\sin 40^\circ}{18.02}$$

$$\Rightarrow \sin\theta = \frac{10\sin 40^\circ}{18.02}$$

$$\Rightarrow \theta = 20.9^\circ$$

$\therefore$ Speed of wind is 18.0 km h^{-1} from N60.9°E

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Relative motion
Exercise A, Question 15

Question:

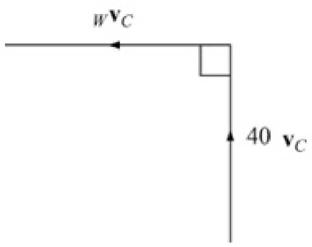
A woman cycles due N at 40 km h^{-1} and the wind seems to be blowing from the East. When she cycles due S at 50 km h^{-1}, the wind seems to be blowing from the South East. Find the true velocity of the wind.

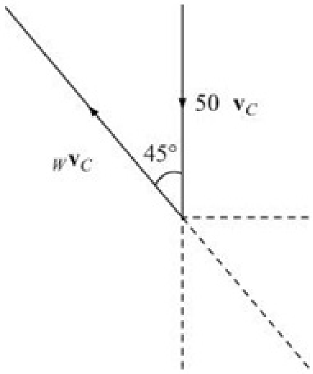
Solution:

	Mag	Dir
$\mathbf{v}_C$	40	due N
$_W\mathbf{v}_C$	?	From E
$\mathbf{v}_W$	?	?

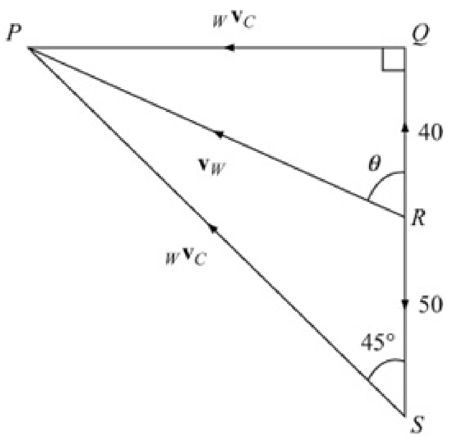
	Mag	Dir
$\mathbf{v}_C$	50	due S
$_W\mathbf{v}_C$	?	From SE
$\mathbf{v}_W$	?	?

$$_W\mathbf{v}_C = \mathbf{v}_W - \mathbf{v}_C \Rightarrow \mathbf{v}_W = \mathbf{v}_C + {_W\mathbf{v}_C}$$





We now put the two vector triangles together using the common side $(\mathbf{v}_W)$



$$Q\hat{P}S = 45^\circ \text{ (From } \Delta PQS)$$
$$\Rightarrow PQ = 90$$
$$\Rightarrow |\mathbf{v}_W| = \sqrt{40^2 + 90^2}$$
$$= 10\sqrt{97} \simeq 98.5 \text{ km h}^{-1}$$
$$\tan\theta = \frac{90}{40}$$
$$\Rightarrow \theta = 66.0^\circ$$

$\therefore$ Velocity of wind is 98.5 km h^{-1} from S66°E

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Relative motion
Exercise A, Question 16

Question:

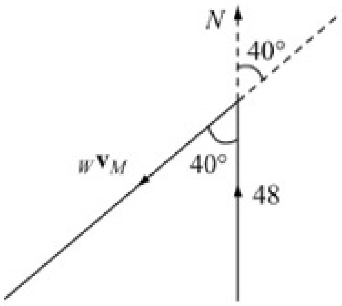
When a motorcyclist travels along a straight road at 48 km h^{-1} due N, the wind seems to be blowing from a direction N40°E. When he returns along the same road at the same speed, the wind seems to be blowing from a direction S30°E. Find the true speed and direction of the wind.

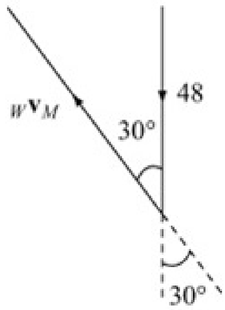
Solution:

	Mag	Dir
$\mathbf{v}_M$	48	due N
${}_W\mathbf{v}_M$	?	From N40°E
$\mathbf{v}_W$	?	?

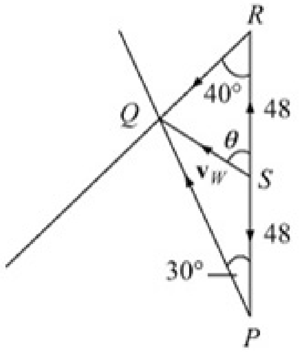
	Mag	Dir
$\mathbf{v}_M$	48	due S
${}_W\mathbf{v}_M$	?	From S30°E
$\mathbf{v}_W$	?	?

$${}_W\mathbf{v}_M = \mathbf{v}_W - \mathbf{v}_M \Rightarrow \mathbf{v}_W = \mathbf{v}_M + {}_W\mathbf{v}_M$$





Putting the two triangles together, using the common side $(\mathbf{v}_W)$



Let $Q\hat{S}R = \theta$

So $Q\hat{S}P = 180° - \theta$

In ΔPQR, $\dfrac{PQ}{\sin 40°} = \dfrac{QR}{\sin 30°} = \dfrac{96}{\sin 110°}$

$$\Rightarrow PQ = \frac{96 \sin 40°}{\sin 110°} = 65.67 \text{ and } QR = \frac{96 \sin 30°}{\sin 110°} = 51.08$$

In ΔPQS, $PQ^2 = 48^2 + QS^2 - 2 \times 48 \times QS \cos(180° - \theta)$ ①

In ΔQSR, $QR^2 = 48^2 + QS^2 - 2 \times 48 \times QS \cos\theta$ ②

① + ②: $PQ^2 + QR^2 = 2 \times (48^2 + QS^2)$ ← since $\cos(180° - \theta) = -\cos\theta$

$$\Rightarrow QS = |\mathbf{v}_W| = \sqrt{\frac{65.67^2 + 51.08^2}{2} - 48^2}$$

$$= 34.0 \text{ km h}^{-1}$$

ΔQRS, $\dfrac{\sin\theta}{51.08} = \dfrac{\sin 40°}{34.01} \Rightarrow \sin\theta = \dfrac{51.08 \sin 40°}{34.01}$

$$\Rightarrow \theta = 74.9°$$

∴ Velocity of wind is 34.0 km h^{-1} from S74.9°E

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Relative motion
Exercise B, Question 1

Question:

At 10.30 a.m. an aeroplane has position vector $(-100\mathbf{i}+220\mathbf{j})$ km and is moving with constant velocity $(300\mathbf{i}+400\mathbf{j})\text{km h}^{-1}$. At 10.45 a.m. a cargo plane has position vector $(-60\mathbf{i}+355\mathbf{j})$ km and is moving with constant velocity $(400\mathbf{i}+300\mathbf{j})\text{km h}^{-1}$.

a Show that the planes will crash if they maintain these velocities.
b Find the time at which the crash will occur.
c Find the position vector of the point at which the crash takes place.

Solution:

a position vector of aeroplane at 10.45 $=\begin{pmatrix}-100\\220\end{pmatrix}+\frac{1}{4}\begin{pmatrix}300\\400\end{pmatrix}=\begin{pmatrix}-25\\320\end{pmatrix}$

At time t h after 10.45 am:

$$\mathbf{r}_A=\begin{pmatrix}-25\\320\end{pmatrix}+t\begin{pmatrix}300\\400\end{pmatrix}$$

$$\mathbf{r}_C=\begin{pmatrix}-60\\355\end{pmatrix}+t\begin{pmatrix}400\\300\end{pmatrix}$$

$${}_A\mathbf{r}_C=\mathbf{r}_A-\mathbf{r}_C=\begin{pmatrix}35\\-35\end{pmatrix}+t\begin{pmatrix}-100\\100\end{pmatrix}=\begin{pmatrix}35-100t\\-35+100t\end{pmatrix}$$

Hence, ${}_A\mathbf{r}_C=0$ when

$t=\frac{35}{100}$ h

$=21$ minutes

b They collide at 11.06 a.m.

c $\mathbf{r}_A=\begin{pmatrix}-25\\320\end{pmatrix}+\frac{35}{100}\begin{pmatrix}300\\400\end{pmatrix}=\begin{pmatrix}80\\460\end{pmatrix}$

They collide at the point with position vector $(80\mathbf{i}+460\mathbf{j})$km

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Relative motion
Exercise B, Question 2

Question:

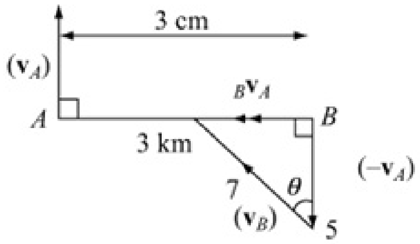
Hiker A is 3 km due W of hiker B. Hiker A walks due N at 5 km h^{-1}. Hiker B starts at the same time and walks at 7 km h^{-1}.

a In what direction should B walk in order to meet A?

b How long will it take to do so?

Solution:

Fix A (i.e. consider motion relative to A).



In velocity Δ,

$$\cos\theta = \frac{5}{7}$$

$$\Rightarrow \theta = 44.4°$$

a B should walk N44.4° W

b $|{}_B\mathbf{v}_A| = \sqrt{7^2 - 5^2} = \sqrt{24}$

$$\therefore \text{Time} = \frac{3}{\sqrt{24}} = \frac{3}{2\sqrt{6}} = \frac{\sqrt{6}}{4} \text{ h}$$

$$= 36.7 \text{ minutes}$$

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Relative motion
Exercise B, Question 3

Question:

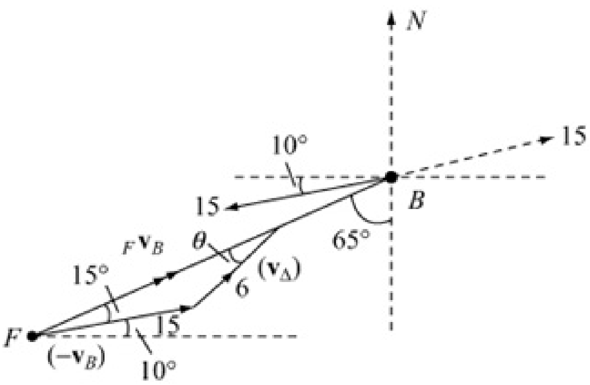
A batsman strikes a cricket ball at 15 m s^{-1} on a bearing of $260°$. A fielder is standing 45 m from the batsman on a bearing of $245°$. He runs at 6 m s^{-1} to intercept the ball.

a Find the direction in which the fielder should run in order to intercept the ball as quickly as possible.

b Find the time, to 1 decimal place, that it takes him to do so.

Solution:

a



Fix the ball (i.e. consider motion relative to the ball)
Using sine rule on vector Δ

$$\frac{\sin\theta}{15} = \frac{\sin 15°}{6}$$

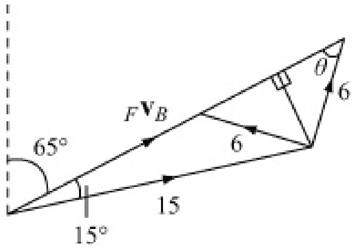
$$\sin\theta = \frac{5\sin 15°}{2}$$

$\theta = 40.32°$ (assuming θ is acute)

θ could be $180° - 40.32°$ (see below)

$\therefore$ Direction of $\mathbf{v}_F$ is $\text{N}(65° - 40.32°)\text{E}$

i.e. N24.7°E



There are 2 possible directions for $\mathbf{v}_F$, as shown is the diagram; the RH one will give the shortest interception time.

b Third angle in the vector Δ is $180° - (15° + \theta) = 124.68°$

$$\frac{|_F\mathbf{v}_B|}{\sin 124.68°} = \frac{6}{\sin 15°}$$

$$\Rightarrow |_F\mathbf{v}_B| = \frac{6\sin 124.68°}{\sin 15°}$$

$$= 19.0637\ldots$$

$$\text{Time} = \frac{45}{19.0637} = 2.4 \text{ s (1 d.p.)}$$

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Relative motion
Exercise B, Question 4

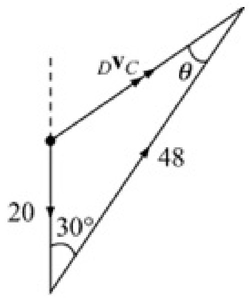
Question:

A destroyer, moving at 48 km h^{-1} in a direction N30°E, observes, at 12 noon, a cargo ship which is steaming due N at 20 km h^{-1}. The destroyer intercepts the cargo ship at 12.45 pm. Find

a the distance of the cargo ship from the destroyer at 12 noon,
b the bearing of the cargo ship from the destroyer at 12 noon.

Solution:

a Fix the cargo ship (i.e. consider motion relative to the cargo ship)
i.e. apply a vector of magnitude 20 due S to both.



by cosine rule,

$$|_D\mathbf{v}_C|^2 = 20^2 + 48^2 - 2 \times 20 \times 48 \cos 30°$$
$$|_D\mathbf{v}_C| = 32.268 \text{ km h}^{-1}$$
$$\text{Distance} = 0.75 \times 32.268$$
$$= 24.2 \text{ km}$$

b $\frac{\sin\theta}{20} = \frac{\sin 30°}{32.268} \Rightarrow \sin\theta = \frac{10}{32.268}$

$\Rightarrow \theta = 18.053°$

$\therefore$ Bearing is $(30° + \theta) = 48.1°$

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Relative motion
Exercise B, Question 5

Question:

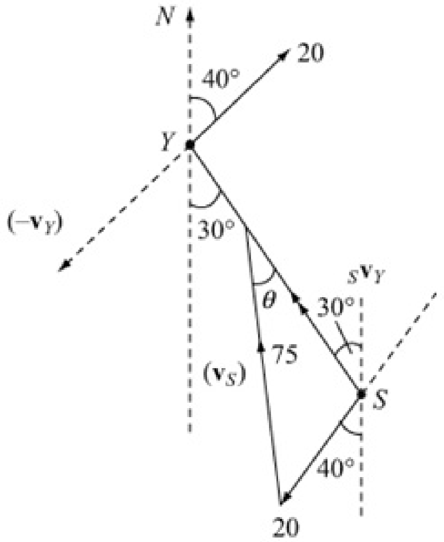
A speedboat moving at 75 km h^{-1} wishes to intercept a yacht which is moving at 20 km h^{-1} in a direction $040°$. Initially the speedboat is 10 km from the yacht on a bearing of $150°$.

a Find the course that the speedboat should set in order to intercept the yacht.

b Find how long the journey will take.

Solution:

a



Fix the yacht (i.e. consider the motion relative to the yacht)

$$\frac{\sin 110°}{75} = \frac{\sin\theta}{20}$$

$$\frac{4\sin 110°}{15} = \sin\theta \Rightarrow \theta = 14.512°$$

Third angle of vector Δ is

$$180° - 110° - 14.512° = 55.488°$$

Course is N15.5°W

b

$$\frac{|_S\mathbf{v}_Y|}{\sin 55.488°} = \frac{75}{\sin 110°} \Rightarrow |_S\mathbf{v}_Y| = 65.7667\ldots$$

$$\therefore \text{Time} = \frac{10}{65.7667} \text{ h} = 9.1 \text{ minutes (1 d.p.)}$$

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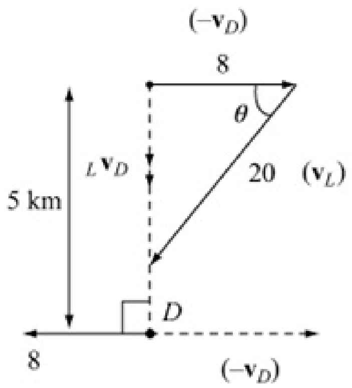
Relative motion
Exercise B, Question 6

Question:

A lifeboat sets out from a harbour at 10.10 a.m. to go to the assistance of a dinghy which is, at that time, 5 km due S of the harbour and drifting at 8 km h^{-1} due W. The lifeboat can travel at 20 km h^{-1}. Find the course that it should set in order to reach the yacht as quickly as possible and find the time when it arrives.

Solution:

Fix the dinghy (i.e. consider the motion relative to the dinghy)



$$\cos\theta = \frac{8}{20} = 0.4$$

$$\Rightarrow \theta = 66.42°$$

$$90° - \theta = 23.58°$$

Course is S23.6° W

$$|{}_L\mathbf{v}_D| = \sqrt{20^2 - 8^2} = \sqrt{336}$$

$$\therefore \text{Time} = \frac{5}{\sqrt{336}} = 16.4 \text{ minutes.}$$

$\therefore$ Arrives at 10.26 a.m.

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Relative motion
Exercise B, Question 7

Question:

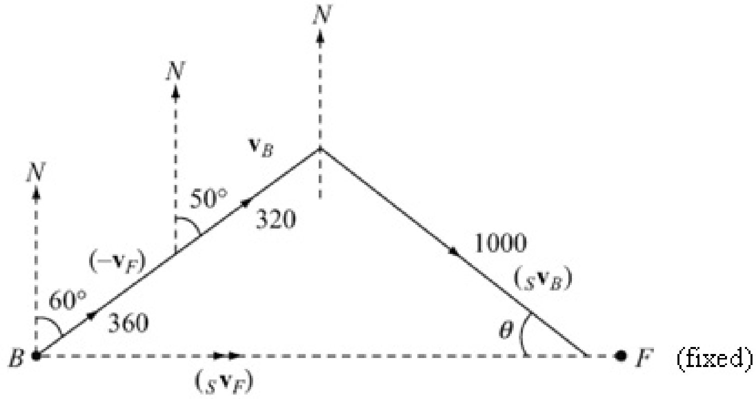
A gunner in a bomber, which is flying N50°E at 320 m s^{-1} wishes to fire at a fighter plane which is flying S60°W at 360 m s^{-1}. If the gun fires its shell at 1000 m s^{-1}, in what direction should the gun be aimed when the fighter is due E of the bomber?

Solution:

Fix the fighter by applying a vector 360 m s^{-1} N60°E

Then ${}_B\mathbf{v}_F + {}_S\mathbf{v}_B = {}_S\mathbf{v}_F$

i.e. $\mathbf{v}_B - \mathbf{v}_F + {}_S\mathbf{v}_B = {}_S\mathbf{v}_F$



$360\cos 60° + 320\cos 50° - 1000\sin\theta = 0$

$$\Rightarrow \sin\theta = \frac{180 + 320\cos 50°}{1000}$$

$$\Rightarrow \theta = 22.7° \Rightarrow 90° - \theta = 67.3°$$

Direction of gun is S67.3° E

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Relative motion
Exercise C, Question 1

Question:

The position vectors and velocity vectors of two ships P and Q at 9 a.m. are as follows

$\mathbf{r}_P = (2\mathbf{i}+\mathbf{j})\text{km} \qquad \mathbf{v}_P = (3\mathbf{i}+\mathbf{j})\text{km h}^{-1}$

$\mathbf{r}_Q = (-\mathbf{i}-4\mathbf{j})\text{km} \qquad \mathbf{v}_Q = (11\mathbf{i}+3\mathbf{j})\text{km h}^{-1}$

Assuming that these velocities remain constant, find

a the least distance between P and Q in the subsequent motion,

b the time at which this least separation occurs.

Solution:

a

$$\mathbf{r}_P = \begin{pmatrix}2\\1\end{pmatrix} + t\begin{pmatrix}3\\1\end{pmatrix}, t \text{ hrs after 9a.m.}$$

$$\mathbf{r}_Q = \begin{pmatrix}-1\\-4\end{pmatrix} + t\begin{pmatrix}11\\3\end{pmatrix}, t \text{ hrs after 9a.m.}$$

$$\Rightarrow {}_P\mathbf{r}_Q = \begin{pmatrix}3\\5\end{pmatrix} + t\begin{pmatrix}-8\\-2\end{pmatrix} = \begin{pmatrix}3-8t\\5-2t\end{pmatrix}$$

$$\Rightarrow |{}_P\mathbf{r}_Q|^2 = (3-8t)^2 + (5-2t)^2 = X \text{ say}$$

$$\frac{dX}{dt} = -16(3-8t) - 4(5-2t) = 0 \quad \text{for a minimum}$$

$$\Rightarrow 12 - 32t + 5 - 2t = 0$$

$$\Rightarrow \quad 17 = 34t$$

$$\Rightarrow \quad \frac{1}{2} = t$$

a and **b** $\therefore X_{\min} = (-1)^2 + 4^2 = 17$

$\therefore$ closest distance is $\sqrt{17}$ km at 9.30 a.m.

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Relative motion
Exercise C, Question 2

Question:

The position vectors and velocity vectors of two ships P and Q at certain times are as follows

$\mathbf{r}_P = (\mathbf{i} + 4\mathbf{j})\text{km} \quad \mathbf{v}_P = (4\mathbf{i} + 8\mathbf{j})\text{km h}^{-1}$ at 9 a.m.

$\mathbf{r}_Q = (20\mathbf{j})\text{km} \quad \mathbf{v}_Q = (9\mathbf{i} - 2\mathbf{j})\text{km h}^{-1}$ at 8 a.m.

Assuming that these velocities remain constant, find

a the least distance between P and Q in the subsequent motion,

b the time at which this least separation occurs.

Solution:

At t hours after 9 a.m.,

$$\mathbf{r}_P = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 1+4t \\ 4+8t \end{pmatrix}$$

$$\mathbf{r}_Q = \begin{pmatrix} 0 \\ 20 \end{pmatrix} + (t+1)\begin{pmatrix} 9 \\ -2 \end{pmatrix} = \begin{pmatrix} 9+9t \\ 18-2t \end{pmatrix}$$

$${}_P\mathbf{r}_Q = \begin{pmatrix} -8-5t \\ -14+10t \end{pmatrix} \Rightarrow {}_P\mathbf{v}_Q = \begin{pmatrix} -5 \\ 10 \end{pmatrix} \quad \text{(Differentiating with respect to } t\text{)}$$

Closest when ${}_P\mathbf{r}_Q \cdot {}_P\mathbf{v}_Q = 0$

i.e. $\begin{pmatrix} -8-5t \\ -14+10t \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 10 \end{pmatrix} = 0$

$$40 + 25t - 140 + 100t = 0$$
$$125t = 100$$
$$t = 0.8\text{ h} \quad (48\text{ minutes})$$

so, ${}_P\mathbf{r}_Q = \begin{pmatrix} -8-4 \\ -14+8 \end{pmatrix} = \begin{pmatrix} -12 \\ -6 \end{pmatrix} \Rightarrow |{}_P\mathbf{r}_Q| = 6\sqrt{1^2 + 2^2} = 6\sqrt{5}$

a $\therefore$ Least distance between P and Q is $6\sqrt{5}$ km

b This occurs at 9.48 a.m.

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Relative motion
Exercise C, Question 3

Question:

The position vectors and velocity vectors of two ships P and Q at certain times are as follows

$\mathbf{r}_P = (8\mathbf{i} - \mathbf{j})\text{km} \quad \mathbf{v}_P = (3\mathbf{i} + 7\mathbf{j})\text{km h}^{-1}$ at 3 p.m.

$\mathbf{r}_Q = (3\mathbf{i} + \mathbf{j})\text{km} \quad \mathbf{v}_Q = (2\mathbf{i} + 3\mathbf{j})\text{km h}^{-1}$ at 2 p.m.

Assuming that these velocities remain constant, find

a the least distance between P and Q in the subsequent motion,

b the time at which this least separation occurs.

Solution:

At t hours after 3 p.m.:

$$\mathbf{r}_P = \begin{pmatrix} 8 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 8+3t \\ -1+7t \end{pmatrix}$$

$$\mathbf{r}_Q = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + (t+1)\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5+2t \\ 4+3t \end{pmatrix}$$

$${}_P\mathbf{r}_Q = \begin{pmatrix} +3+t \\ -5+4t \end{pmatrix} \Rightarrow {}_P\mathbf{v}_Q = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} +3+t \\ -5+4t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 0 \text{ for closest approach}$$

$$+3+t-20+16t = 0$$

$$17t = 17$$

$$t = 1$$

Then $${}_P\mathbf{r}_Q = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \Rightarrow |{}_P\mathbf{r}_Q| = \sqrt{17}\text{ km}$$

Least distance is $\sqrt{17}$ km at 4 p.m.

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Relative motion
Exercise C, Question 4

Question:

The position vectors and velocity vectors of two ships P and Q at 3 p.m. are as follows

$\mathbf{r}_P = (3\mathbf{i} - 5\mathbf{j})\text{km} \quad \mathbf{v}_P = (15\mathbf{i} + 14\mathbf{j})\text{km h}^{-1}$

$\mathbf{r}_Q = (13\mathbf{i} + 5\mathbf{j})\text{km} \quad \mathbf{v}_Q = (3\mathbf{i} - 10\mathbf{j})\text{ km h}^{-1}$

Assuming that these velocities remain constant,

a find the least distance between P and Q in the subsequent motion.

Ship Q has guns with a range of up to 5 km.

b Find the length of time for which ship P is within the range of ship Q's guns.

Solution:

At t hours after 3 p.m.:

$$\mathbf{r}_P = \begin{pmatrix} 3 \\ -5 \end{pmatrix} + t\begin{pmatrix} 15 \\ 14 \end{pmatrix} = \begin{pmatrix} 3+15t \\ -5+14t \end{pmatrix}$$

$$\mathbf{r}_Q = \begin{pmatrix} 13 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ -10 \end{pmatrix} = \begin{pmatrix} 13+3t \\ 5-10t \end{pmatrix}$$

$${}_P\mathbf{r}_Q = \begin{pmatrix} -10+12t \\ -10+24t \end{pmatrix} \Rightarrow {}_P\mathbf{v}_Q = \begin{pmatrix} 12 \\ 24 \end{pmatrix}$$

$$\begin{pmatrix} -10+12t \\ -10+24t \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 24 \end{pmatrix} = 0, \quad \text{for closest approach}$$

$$-120 + 144t - 240 + 576t = 0$$

$$720t = 360$$

$$t = \frac{1}{2}$$

a Then, ${}_P\mathbf{r}_Q = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \Rightarrow |{}_P\mathbf{r}_Q|_{\min} = \sqrt{20} = 2\sqrt{5}$ km

b

$$\text{Need } |{}_P\mathbf{r}_Q| \le 5$$

$$\Rightarrow |{}_P\mathbf{r}_Q|^2 \le 25$$

$$\Rightarrow (12t-10)^2 + (24t-10)^2 \le 25$$

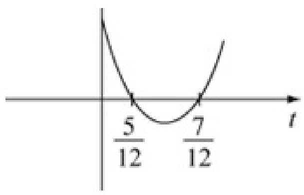
$$\Rightarrow 144t^2 - 240t + 100 + 576t^2 - 480t + 100 - 25 \le 0$$

$$\Rightarrow 720t^2 - 720t + 175 \le 0$$

$$\Rightarrow 144t^2 - 144t + 35 \le 0$$

$$\Rightarrow (12t-7)(12t-5) \le 0$$

$$\Rightarrow \frac{5}{12} \le t \le \frac{7}{12}.$$



$$\therefore \text{Length of time} = \frac{7}{12} - \frac{5}{12} = \frac{1}{6} \text{ hour}$$

i.e. 10 minutes

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Relative motion
Exercise C, Question 5

Question:

The position vectors and velocity vectors of two ships P and Q at certain times are as follows

$\mathbf{r}_P = (-2\mathbf{i}+3\mathbf{j})\text{km}$ $\quad \mathbf{v}_P = (12\mathbf{i}-4\mathbf{j})\text{km h}^{-1}$ at 2.45 p.m.

$\mathbf{r}_Q = (8\mathbf{i}+7\mathbf{j})\text{km}$ $\quad \mathbf{v}_Q = (2\mathbf{i}-14\mathbf{j})\text{km h}^{-1}$ at 3 p.m.

Assuming that these velocities remain constant,

a find the least distance between P and Q in the subsequent motion.

Ship Q has guns with a range of up to 2 km.

b Find the length of time for which ship P is within the range of ship Q's guns.

Solution:

At t hours after 2.45 p.m.,

$$\mathbf{r}_P = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 12 \\ -4 \end{pmatrix} = \begin{pmatrix} -2+12t \\ 3-4t \end{pmatrix}$$

$$\mathbf{r}_Q = \begin{pmatrix} 8 \\ 7 \end{pmatrix} + \left(t - \frac{1}{4}\right)\begin{pmatrix} 2 \\ -14 \end{pmatrix} = \begin{pmatrix} 7\frac{1}{2}+2t \\ 10\frac{1}{2}-14t \end{pmatrix}$$

$${}_P\mathbf{r}_Q = \begin{pmatrix} -9\frac{1}{2}+10t \\ -7\frac{1}{2}+10t \end{pmatrix} \Rightarrow {}_P\mathbf{v}_Q = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} -9\frac{1}{2}+10t \\ -7\frac{1}{2}+10t \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 10 \end{pmatrix} = 0 \quad \text{for closest approach}$$

$$\Rightarrow -95+100t-75+100t = 0$$

$$200t = 170$$

$$t = \frac{17}{20}$$

Then $${}_P\mathbf{r}_Q = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow |{}_P\mathbf{r}_Q|_{\min} = \sqrt{2}\text{ km}$$

b Need $|{}_P\mathbf{r}_Q| \le 2$

$$\Rightarrow |{}_P\mathbf{r}_Q|^2 \le 4$$

$$\Rightarrow \left(10t - 9\frac{1}{2}\right)^2 + \left(10t - 7\frac{1}{2}\right)^2 \le 4$$

$$\Rightarrow 100t^2 - 190t + 90.25 + 100t^2 - 150t + 56.25 - 4 \le 0$$

$$\Rightarrow 200t^2 - 340t + 142.5 \le 0$$

Roots given by $$t = \frac{340 \pm \sqrt{(340)^2 - 4 \times 200 \times 142.5}}{400}$$

$$= \frac{340 \pm 40}{400} = \frac{15}{20} \quad \text{or} \quad \frac{19}{20}$$

$$\therefore \frac{15}{20} \le t \le \frac{19}{20}$$

$$\therefore \text{Length of time} = \frac{4}{20} = \frac{1}{5}\text{ h} = 12\text{ minutes}$$

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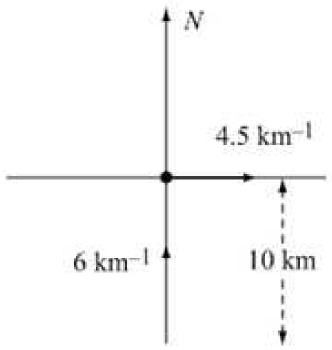
Relative motion
Exercise D, Question 1

Question:

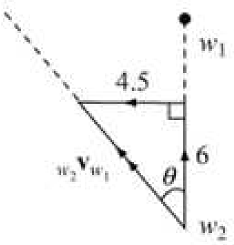
Two straight roads cross at right angles. A woman leaves the cross-roads and walks due E at 4.5 km h^{-1}. At the same time another woman leaves a point 10 km due S of the cross-roads and walks due N at 6 km h^{-1}.

a After how long will they be closest together?
b How far apart will they then be?

Solution:

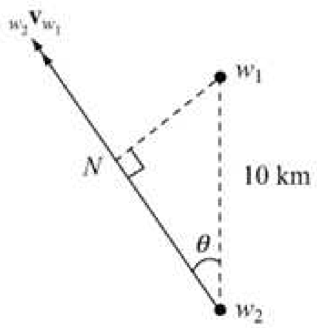


Fix first woman
(i.e. apply a velocity 4.5 km h^{-1} due W to both)



$\tan\theta = \frac{4.5}{6} = \frac{3}{4}$

a



N is closest approach position.

$w_2N = 10\cos\theta = 8 \text{ km}$

$\therefore \text{Time} = \frac{8}{\sqrt{4.5^2 + 6^2}} = \frac{8}{7.5} = \frac{16}{15} \text{ h}$

$\therefore$ Closest after 1 hr 4 minutes

b $w_1N = 10\sin\theta = 10 \times \frac{3}{5} = 6 \text{ km}$

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Relative motion
Exercise D, Question 2

Question:

Two trains are travelling on railway lines which cross at right angles. The first train is travelling at 45 km h^{-1} and the second is travelling at 108 km h^{-1}.

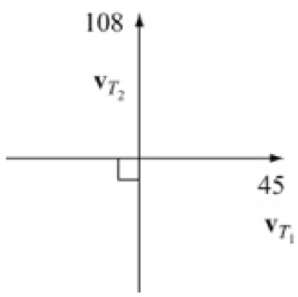
a Find their relative speed.

The slower train passes the point where the lines cross one minute before the faster train.

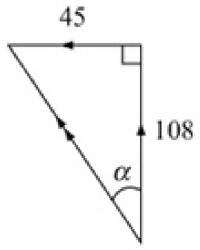
b Find the shortest distance between the trains.

Solution:

a



$_{T_2}\mathbf{v}_{T_1} = \mathbf{v}_{T_2} - \mathbf{v}_{T_1}$



$\therefore \text{Relative speed} = |_{T_2}\mathbf{v}_{T_1}|$

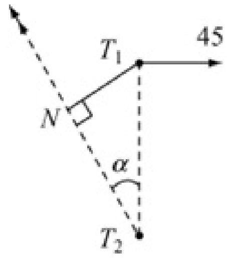
$= \sqrt{45^2 + 108^2}$

$= 117 \text{ km h}^{-1}$

b At $t = 0$:

$T_1T_2 = \frac{108}{60}$

$= 1.8 \text{ km}$



T_1N is shortest distance

$= 1.8 \sin\alpha = 1.8 \times \frac{45}{117} = \frac{9}{13}$ km

$\simeq 0.692$ km (3 s.f.)

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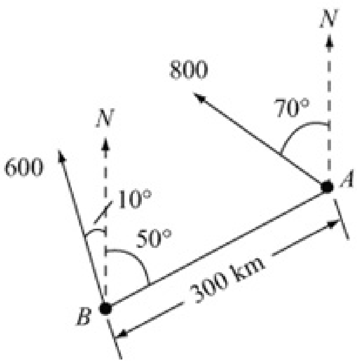
Relative motion
Exercise D, Question 3

Question:

At 10 a.m. an aircraft A is 300 km N50°E of another aircraft B. Aircraft A is flying at 800 km h^{-1} in the direction N70°W and aircraft B is flying at 600 km h^{-1} in the direction N10°W.

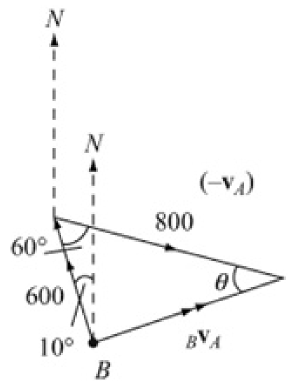
a Find the least distance between the aircraft in the subsequent motion.
b Find the time when they are closest to each other.

Solution:



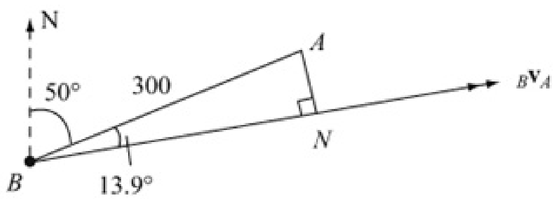
Fix A (i.e. consider motion relative to A)
Apply a vector 800 S70°E to both:
by cos rule,

$$|_B\mathbf{v}_A|^2 = 600^2 + 800^2 - 2\times 600\times 800\cos 60°$$
$$= 520\,000$$
$$|_B\mathbf{v}_A| = 100\sqrt{52}$$



$$\frac{\sin\theta}{600} = \frac{\sin 60°}{100\sqrt{52}}$$
$$\sin\theta = \frac{3\sqrt{3}}{\sqrt{52}}$$
$$\theta = 46.1°$$

Third angle is
$180° - 60° - 46.1° = 73.9°$
Direction of $_B\mathbf{v}_A$ is N63.9°E



N is the point of closest approach.
$AN = 300\sin 13.9° = 72.1$ km (3 s.f.)
$BN = 300\cos 13.9° = 291.21\ldots$

$$\text{Time} = \frac{291.21}{100\sqrt{52}}\text{ h} = 0.4038..$$
$$= 24.2 \text{ minutes}$$

Least distance between then is 72.1 km at 10.24 (nearest minute)

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Relative motion
Exercise D, Question 4

Question:

A ship P steams at 20 km h^{-1} on a bearing of $015°$. Another ship Q steams at 12 km h^{-1} on a bearing of $330°$.

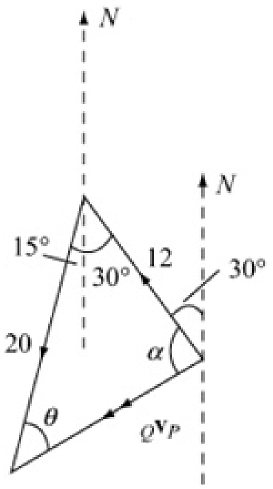
a Find the velocity of Q relative to P.

At 12 noon Q is 5 km due E of P. If they maintain their velocities,

b find the shortest distance between the ships.

Solution:

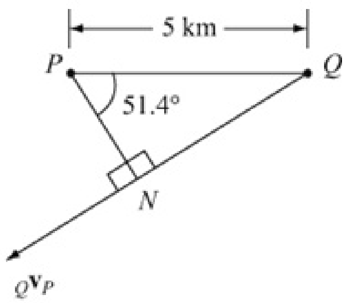
a ${}_Q\mathbf{v}_P = \mathbf{v}_Q - \mathbf{v}_P$



$$|{}_Q\mathbf{v}_P|^2 = 20^2 + 12^2 - 2\times 20\times 12\cos 45°$$
$$= 544 - 240\sqrt{2}$$
$$|{}_Q\mathbf{v}_P| = 14.3\text{ km h}^{-1}$$
$$\frac{\sin\theta}{12} = \frac{\sin 45°}{14.303\ldots} \Rightarrow \sin\theta = \frac{12\sin 45°}{14.303\ldots}$$
$$\Rightarrow \theta = 36.4°$$
$$\alpha = 180° - 45° - 36.4° = 98.6°$$

$\therefore$ Direction of ${}_Q\mathbf{v}_P$ is on a bearing $(180° + 51.4°)$ i.e. $231.4°$.

b At noon:



N is the point of closest approach.

Shortest distance between P and $Q = PN = 5\cos 51.4°$
$= 3.12\text{ km}$

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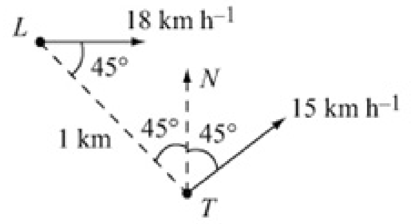
Relative motion
Exercise D, Question 5

Question:

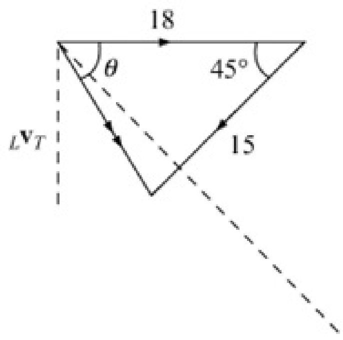
At a particular instant a liner is 1 km NW of a tanker. The liner is moving at 18 km h^{-1} due E and the tanker is moving at 15 km h^{-1} NE.

a Find the shortest distance between the ships.
b Find the interval of time that passes until they are at the point of closest approach.

Solution:



Fix the tanker
(i.e. apply a vector
of 15 km h^{-1} SW to both)



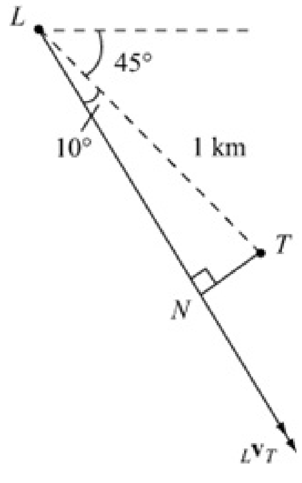
by cos rule,

$|{}_L\mathbf{v}_T|^2 = 18^2 + 15^2 - 2 \times 18 \times 15 \cos 45°$

$= 549 - 270\sqrt{2}$

$|{}_L\mathbf{v}_T| = 12.9\ (291)$

$\frac{\sin\theta}{15} = \frac{\sin 45°}{12.9291} \Rightarrow \sin\theta = \frac{15 \sin 45°}{12.9291} \Rightarrow \theta = 55°$



N is the point of closest approach.

$TN = 1 \sin 10°$ km

$= 0.174$ km

$\text{Time} = \frac{LN}{|{}_L\mathbf{v}_T|}$

$= \frac{1 \cos 10°}{12.929}$ h

$= 4.6$ minutes

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Relative motion
Exercise E, Question 1

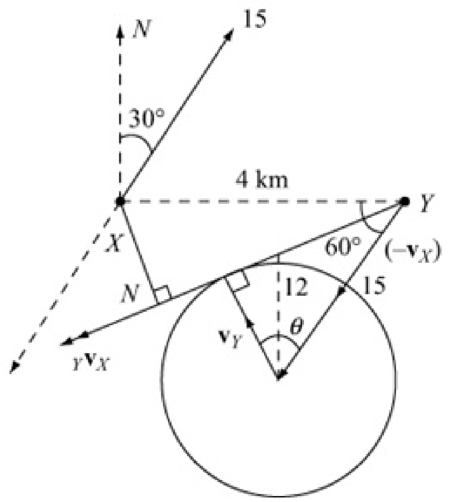
Question:

X and Y are two yachts and X is sailing at a constant speed of 15 km h^{-1} in a direction N30°E . At 2 p.m. Y is 4 km due E of X. Given that Y travels at a constant speed of 12 km h^{-1},

a show that it is not possible for Y to intercept X,
b find the course that Y should set in order to get as close as possible to X,
c find the shortest distance between the yachts,
d find the time when they are closest.

Solution:

a



Fix X (i.e. consider motion relative to X)
Since $15 \sin 60° > 12$, impossible for Y to catch X.

b $\cos\theta = \frac{12}{15} = \frac{4}{5}$

$\Rightarrow \theta = 36.87°$

$\therefore$ course is $\theta - 30° = 6.87°$ W of N

Course for Y is N6.87°W

c N is the point of closest approach.

$X\hat{Y}N = 60° - (90° - \theta) = \theta - 30° = 6.87°$

$\therefore XN = 4 \sin 6.87° = 0.48$ km

d Time $= \frac{NY}{|_Y\mathbf{v}_X|} = \frac{4\cos 6.87°}{\sqrt{15^2 - 12^2}} = \frac{4\cos 6.87°}{9}$ h

$= 26.5$ minutes

Time is $2.26\frac{1}{2}$ p.m.

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Relative motion
Exercise E, Question 2

Question:

Two aircraft P and Q are flying at the same altitude. At 12 noon aircraft Q is 5 km due S of aircraft P, and is flying at a constant 300 m s^{-1} in the direction N60°E. If aircraft P flies at a constant speed of 200 m s^{-1}, find

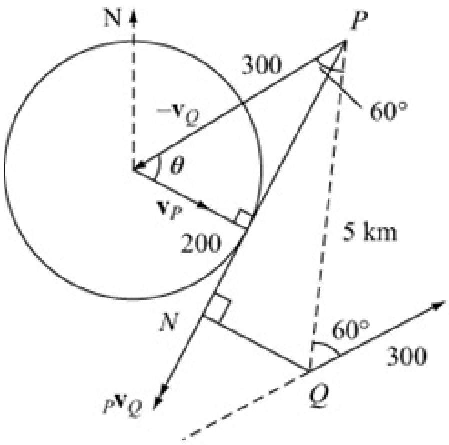
a the direction in which it must fly in order to pass as close to aircraft Q as possible,
b the distance between the planes when they are closest,
c the time when they are closest.

5 km due
f aircraft
ossible,

Solution:

At noon,
Fix Q (i.e. consider motion relative to Q)
by applying a vector of magnitude 300 m s^{-1}
in S60°W direction.
N is the point of closest approach,

$$\cos\theta = \frac{200}{300} \Rightarrow \theta = 48.19°$$



a Bearing of
$$\mathbf{v}_P = 60° + \theta$$
$$= 108° \text{(nearest degree)}$$

b Angle between ${}_P\mathbf{v}_Q$ and $PQ = 60° - (90° - \theta) = \theta - 30°$
$$= 18.19°$$
$$\therefore \text{Closest approach, } QN = 5\sin 18.19°$$
$$= 1.56 \text{ km}$$

c
$$\text{Time} = \frac{PN}{|{}_P\mathbf{v}_Q|}$$
$$= \frac{5\cos 18.19° \times 1000}{\sqrt{300^2 - 200^2}} = \frac{5\cos 18.19° \times 1000}{100\sqrt{5}} \text{ s}$$
$$= 21.2S \text{ (after 12 noon)}$$

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Relative motion
Exercise E, Question 3

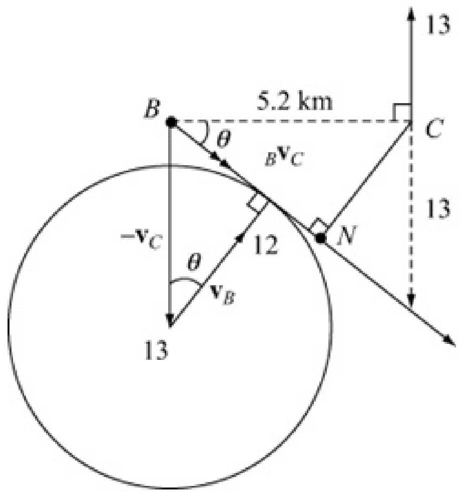
Question:

At 3 p.m. boat C is due E of boat B and $BC = 5.2$ km. Boat C is travelling due N at a constant speed of 13 km h^{-1}. Given that boat B travels at 12 km h^{-1}, find

a the course that B should set in order to get as close as possible to C,
b the shortest distance between the boats,
c the time when this occurs,
d the distance from the closest position of the boats to the initial position of B.

Solution:

a



Fix C (i.e. consider motion relative to C)

$_B\mathbf{v}_C = \sqrt{13^2 - 12^2} = 5 \text{ km h}^{-1}$

$\cos\theta = \frac{12}{13} \Rightarrow \theta = 22.62°$

Direction of B is N 22.6° E.

b Angle between $_B\mathbf{v}_C$ and $BC = 90° - (90° - \theta) = \theta$

$\therefore$ Least distance, $CN = 5.2 \sin\theta = 2$ km

c $\text{Time} = \frac{BN}{|_B\mathbf{v}_C|} = \frac{5.2\cos 22.62°}{5}$

$= 0.96$ h

$= 57.6$ minutes

$\therefore$ Time is 3.58 p.m. (nearest minute)

d Distance moved by

$B = 12 \times 0.96$

$= 11.52$ km

$= 11.5$ km (3 s.f.)

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Relative motion
Exercise E, Question 4

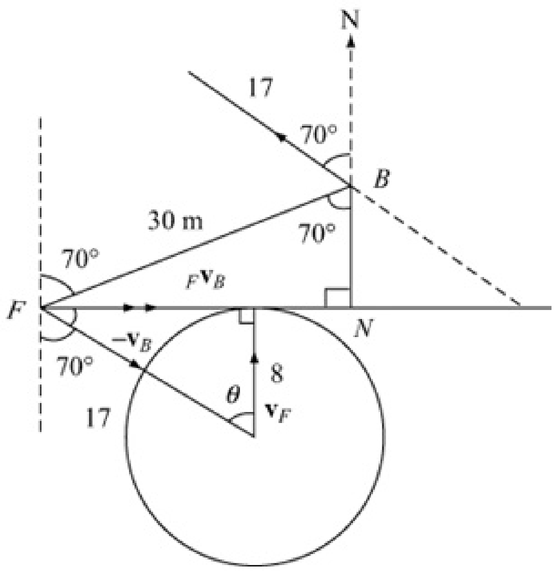
Question:

A fielder is placed at a distance of 30 m from a batsman and on a bearing of $250°$. The batsman hits the ball at 17 m s^{-1} in the direction N70°W. Given that the fielder runs at 8 m s^{-1} from the moment the ball is struck, and ignoring any change in the speed of the ball, find

a how close the fielder gets to the ball,

b the time, from the instant when the ball was struck, that it takes the fielder to get to the closest position.

Solution:



Fix the ball, by applying a vector of magnitude 17 m s^{-1} in direction S70°E.
N is the point of closest approach.

$\cos\theta = \frac{8}{17} \Rightarrow \theta = 61.93°$

$\therefore$ Bearing of F's course is

$360° - (70 - 61.93°) = 351.93°$

$= 352°$ (3 s.f.)

a Angle $B\hat{F}N = 40° - (90° - \theta)$

$= \theta - 50° = 11.93°$

$\therefore$ Closest distance, $BN = 30 \sin 11.93° = 6.2$ m

b $\text{Time} = \frac{FN}{\text{relative speed}} = \frac{30\cos 11.93°}{\sqrt{17^2 - 8^2}} = \frac{30\cos 11.93}{15}$

$= 2\cos 11.93°$

$= 1.96$ s

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Relative motion
Exercise E, Question 5

Question:

At 10 a.m. a frigate F is 16 km due E of a cruiser C. The cruiser is moving at a constant speed of 40 km h^{-1} on a bearing of 030° and the frigate is moving at a constant speed of 20 km h^{-1}. Find

a the course that F should set in order to get as close as possible to C,
b the closest distance between them,
c the time when this occurs.

The guns on the frigate have a range of up to 10 km.

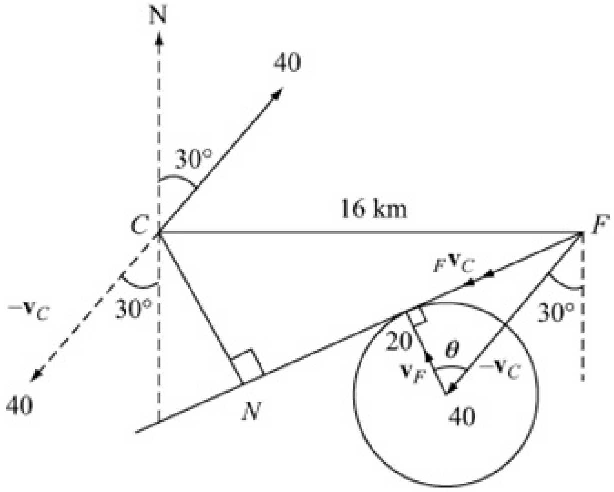
d Find the length of time for which C is within the range of ship F's guns.

The guns on the cruiser have a range of up to 9 km.

e Find the length of time for which F is within the range of ship C's guns.

Solution:

a



Fix C i.e. consider motion relative to C

$$\cos\theta = \frac{20}{40} = \frac{1}{2}$$
$$\Rightarrow \theta = 60°$$

b $\therefore$ Frigate sails an a bearing of 330°
N is the point of closest approach.

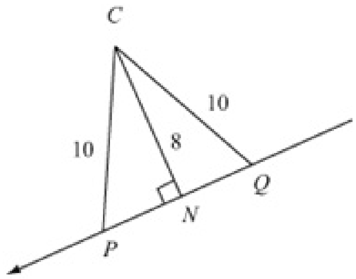
$$C\hat{F}N = 90° - (90° - \theta) - 30° = \theta - 30° = 30°$$

$\therefore CN$, closest approach $= 16\sin 30° = 8$ km

c
$$\text{Time} = \frac{FN}{|_F\mathbf{v}_C|} = \frac{16\cos 30°}{\sqrt{40^2 - 20^2}} = \frac{8\sqrt{3}}{10\sqrt{12}} = \frac{4}{5}\times\frac{1}{2}$$
$$= \frac{2}{5}\text{ h}$$
$$= 24 \text{ minutes}$$

Closest at 10.24 a.m.

d



$$PQ = 2PN = 2\sqrt{10^2 - 8^2}$$
$$= 12 \text{ km}$$
$$\text{Time} = \frac{12}{10\sqrt{12}}\text{ h} = 0.3464\text{ h}$$
$$= 20.8 \text{ minutes}$$

e Similarly,
$$\text{time} = \frac{2\sqrt{10^2 - 9^2}}{10\sqrt{12}}$$
$$= \frac{1}{5}\frac{\sqrt{19}}{\sqrt{12}} = 0.2516\ldots\text{ h}$$
$$= 15.1 \text{ minutes}$$

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Relative motion
Exercise F, Question 1

Question:

Particles P, Q and R move in a plane with constant velocities. At time $t = 0$ the position vectors of P, Q and R, relative to a fixed origin O, are $(\mathbf{i} + 3\mathbf{j})$km, $(9\mathbf{i} + 9\mathbf{j})$km and $(6\mathbf{i} + 13\mathbf{j})$ km respectively. The velocity of R relative to P is $(7\mathbf{i} - 10\mathbf{j})$km h^{-1} and the velocity of R relative to Q is $(9\mathbf{i} - 12\mathbf{j})$km h^{-1}.

a Find the velocity of Q relative to P.
b Show that P and Q do not collide.
c Find the shortest distance between P and Q.
d Find the time taken to reach the position of closest approach.
e Show that Q and R do collide.
f Find the distance between P and R when this collision occurs.

Solution:

a $_R\mathbf{v}_P = \mathbf{v}_R - \mathbf{v}_P = \begin{pmatrix} 7 \\ -10 \end{pmatrix}$ ①

$_R\mathbf{v}_Q = \mathbf{v}_R - \mathbf{v}_Q = \begin{pmatrix} 9 \\ -12 \end{pmatrix}$ ②

$_Q\mathbf{v}_P = \mathbf{v}_Q - \mathbf{v}_P = (\mathbf{v}_R - \mathbf{v}_P) - (\mathbf{v}_R - \mathbf{v}_Q)$

$= \begin{pmatrix} 7 \\ -10 \end{pmatrix} - \begin{pmatrix} 9 \\ -12 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$_Q\mathbf{v}_P = (-2\mathbf{i} + 2\mathbf{j})\text{ km h}^{-1}$

b $\mathbf{r}_P = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$, at $t = 0$

$\overrightarrow{QP} = -\mathbf{r}_Q + \mathbf{r}_P = -\begin{pmatrix} 9 \\ 9 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -8 \\ -6 \end{pmatrix}$

Since $_Q\mathbf{v}_P \neq k\overrightarrow{QP}$, P and Q will not collide.

c At time t,

$_Q\mathbf{r}_P = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + t\,_Q\mathbf{v}_P = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8-2t \\ 6+2t \end{pmatrix}$

Closest when

$$_Q\mathbf{r}_P \cdot {}_Q\mathbf{v}_P = 0$$

$$\begin{pmatrix} 8-2t \\ 6+2t \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 0$$

$$-16 + 4t + 12 + 4t = 0$$

$$8t = 4$$

$$t = \frac{1}{2}\text{ hr}$$

At $t = \frac{1}{2}$, $_Q\mathbf{r}_P = \begin{pmatrix} 7 \\ 7 \end{pmatrix} \Rightarrow |_Q\mathbf{r}_P| = 7\sqrt{2}$ km

d $\frac{1}{2}$ h

e At $t = 0, \mathbf{r}_R - \mathbf{r}_Q = \begin{pmatrix} 6 \\ 13 \end{pmatrix} - \begin{pmatrix} 9 \\ 9 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$_Q\mathbf{v}_R = -_R\mathbf{v}_Q = \begin{pmatrix} -9 \\ 12 \end{pmatrix} = 3(\mathbf{r}_R - \mathbf{r}_Q)$ $\therefore$ collision occurs

f Collision when $t = \frac{1}{3}$

$${}_R\mathbf{r}_P = \left\{\begin{pmatrix}6\\13\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix}\right\} + t\begin{pmatrix}7\\-10\end{pmatrix}$$

When

$$t = \frac{1}{3}, {}_R\mathbf{r}_P = \begin{pmatrix}5\\10\end{pmatrix} + \frac{1}{3}\begin{pmatrix}7\\-10\end{pmatrix}$$

$$= \begin{pmatrix}\frac{22}{3}\\ \frac{20}{3}\end{pmatrix} = \frac{2}{3}\begin{pmatrix}11\\10\end{pmatrix}$$

$$|{}_R\mathbf{r}_P| = \frac{2}{3}\sqrt{11^2+10^2} = \frac{2}{3}\sqrt{221} \simeq 9.91\,\text{km}$$

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Relative motion
Exercise F, Question 2

Question:

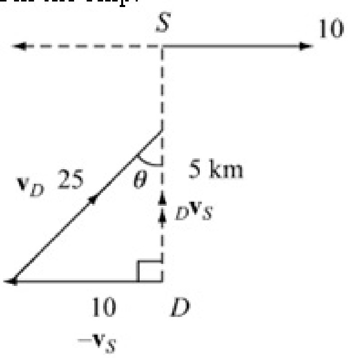
A ship is steaming due E at 10 km h^{-1}. A destroyer is 5 km due S of the ship and wishes to intercept it. If the destroyer can travel at 25 km h^{-1},

a in which direction will it travel,

b how long will it take?

Solution:

a Fix the ship.



$$\sin\theta = \frac{10}{25} = 0.4 \Rightarrow \theta = 23.6°$$

The destroyer should steer N23.6°E

b $\text{Time} = \frac{5}{\sqrt{25^2 - 10^2}} = \frac{5}{5\sqrt{5^2 - 2^2}} = \frac{1}{\sqrt{21}}\text{ h}$

$\cong 0.218\text{ h}$

$\cong 13.1\text{ minutes}$

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Relative motion
Exercise F, Question 3

Question:

Two trains S and T are moving at constant speed, S at 50 km h^{-1} NW and T at a speed $v\text{ km h}^{-1}$ due W. If the velocity of S relative T is NE in direction,

a show that it is 50 km h^{-1} in magnitude,

b find the value of v.

If the speeds of S and T are interchanged,

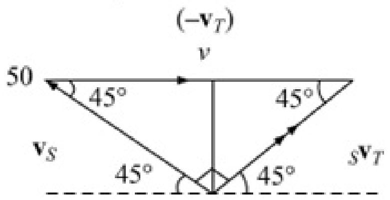
c find the velocity of S relative to T in magnitude and direction.

Solution:

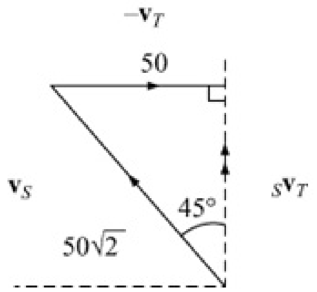
a ${}_S\mathbf{v}_T = \mathbf{v}_S - \mathbf{v}_T$: Vector Δ is isosceles,

$|{}_S\mathbf{v}_T| = 50$

b $\therefore v = 50\sqrt{2}$



c



$|{}_S\mathbf{v}_T| = 50\text{ km h}^{-1}$
due N

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Relative motion
Exercise F, Question 4

Question:

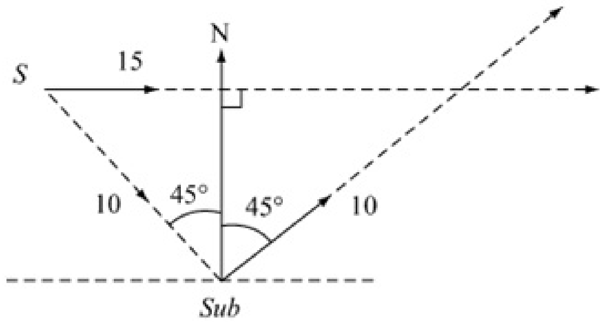
A ship is travelling due E at 15 km h^{-1} and is 10 km NW of a submarine. The submarine submerges immediately and travels at 10 km h^{-1} NE underwater.

a Show that when it crosses the ship's track, it is nearly 1 km behind.

b Find the nearest distance to which it has approached the ship.

Solution:

a



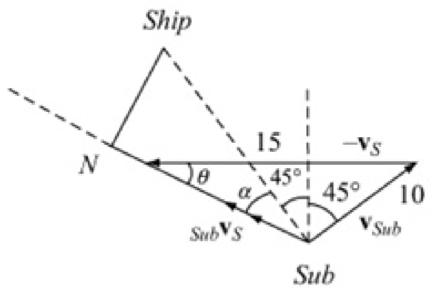
Time for sub to cross ship's track $= \frac{10}{10} = 1\text{ h}$

Distance travelled East $= 10\sin 45° = 5\sqrt{2} \simeq 7.07\text{ km}$.

In 1 h, ship travels 15 km. $\therefore$distance of ship from sub $= 15 - 10\cos 45° - 5\sqrt{2}$

$15 - 10\sqrt{2} \simeq 1\text{ km}$ i.e. sub is approximately 1 km behind.

b Fix ship; N is the point of closest approach.



cosine rule:

$$|{}_{SUB}\mathbf{v}_S|^2 = 10^2 + 15^2 - 2 \times 10 \times 15\cos 45°$$
$$= 325 - 150\sqrt{2}$$
$$|{}_{SUB}\mathbf{v}_S| = 10.624$$
$$\frac{\sin\theta}{10} = \frac{\sin 45°}{10.624}$$
$$\Rightarrow \theta = 41.73°$$

So,

$$\alpha = 180° - 135° - \theta$$
$$= 3.27°$$
$$\therefore\text{closest approach} = 10\sin\alpha = 0.571\text{ km}$$

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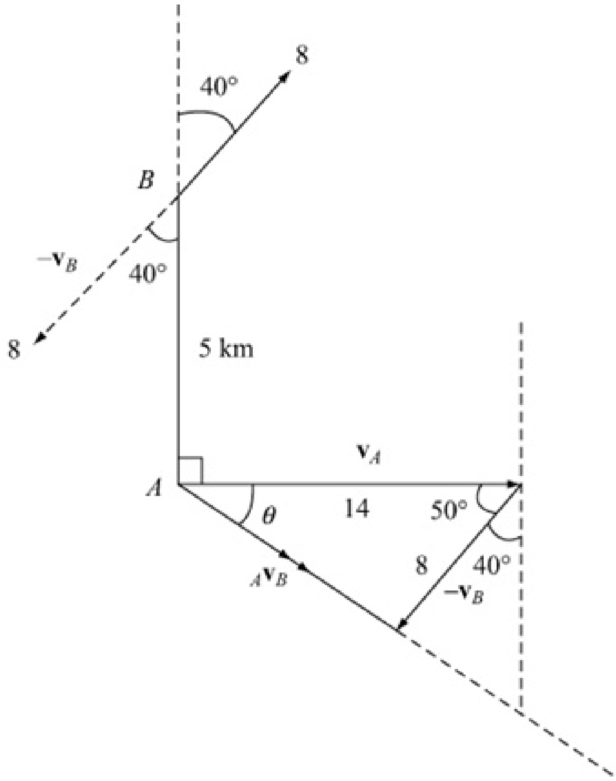
Relative motion
Exercise F, Question 5

Question:

A ship A is moving at 14 km h^{-1} due E and a ship B is moving at 8 km h^{-1} on a bearing of $040°$. At 2 p.m., A is 5 km due S of B. If the limit of visibility is 12 km, for how long after 2 p.m. is B visible to A?

Solution:



Fix B i.e. consider motion relative to B.

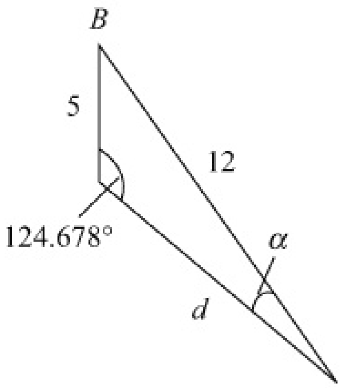
$$|_A\mathbf{v}_B|^2 = 14^2 + 8^2 - 2\times 14\times 8\cos 50^\circ$$
$$= 260 - 224\cos 50^\circ$$
$$|_A\mathbf{v}_B| = 10.771 \text{ km h}^{-1}$$

Velocity Δ

sine rule

$$\frac{\sin\theta}{8} = \frac{\sin 50^\circ}{10.771}$$
$$\Rightarrow \theta = 34.678^\circ$$

Displacement Δ



$$\frac{\sin\alpha}{5} = \frac{\sin 124.678^\circ}{12}$$
$$\Rightarrow \sin\alpha = \frac{5\sin 124.678^\circ}{12}$$
$$\Rightarrow \alpha = 20.04^\circ$$
$$\therefore \frac{d}{\sin 35.284} = \frac{12}{\sin 124.678}$$
$$\Rightarrow d = 8.4288$$
$$\therefore \text{Time} = \frac{8.4288}{|_A\mathbf{v}_B|}\text{h} = 47 \text{ minutes}$$

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Relative motion
Exercise F, Question 6

Question:

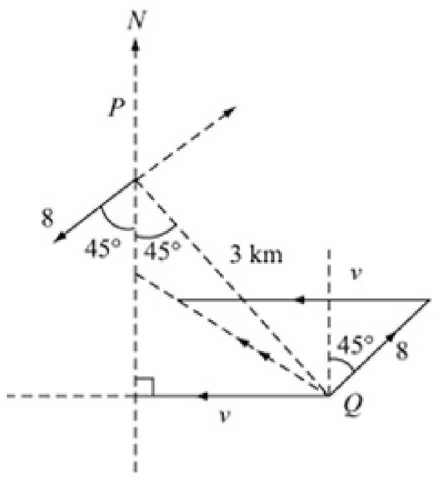
A ship P is steaming on a bearing of $225°$ at a constant speed of 8 km h^{-1}. A second ship Q is sighted, 3 km SE of P, steaming due W at a constant speed. After a certain time, Q is sighted 1 km due S of P. Find

a the time taken, from the instant when Q is first sighted, to the instant when Q is due W of P,

b the distance the ships are then apart,

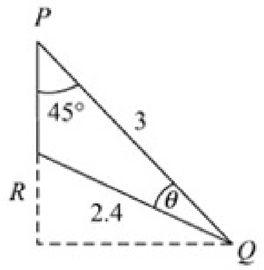
c the velocity of Q relative to P.

Solution:



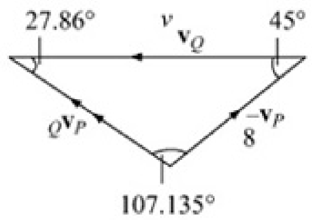
Fix P.

Displacement Δ



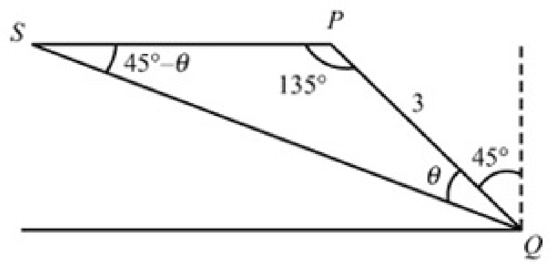
$$RQ^2 = 1^2 + 3^2 - 2\times1\times3\cos 45^\circ$$
$$= 10 - 3\sqrt{2}$$
$$RQ = 2.4$$
$$\sin\theta = \frac{\sin 45^\circ}{2.4}$$
$$\theta = 17.14^\circ$$

Velocity Δ



$$\frac{|{}_Q\mathbf{v}_P|}{\sin 45^\circ} = \frac{8}{\sin 27.86^\circ} \Rightarrow |{}_Q\mathbf{v}_P| = 12.1$$

Displacement Δ



$$\frac{QS}{\sin 135^\circ} = \frac{3}{\sin(45^\circ - \theta)}$$
$$\Rightarrow QS = \frac{3\sin 135^\circ}{\sin 27.865^\circ}$$
$$= 4.539$$
$$\therefore \text{Time} = \frac{4.539}{12.1}$$
$$= 0.375\text{ h}$$
$$\simeq 22.5\text{ minutes}$$

$$\frac{PS}{\sin\theta} = \frac{4.539}{\sin 135^\circ} \Rightarrow PS = 1.89\text{ km}$$

a 22.5 minutes

b 1.89 km

c 12.1 km h^{-1} on a bearing 298°

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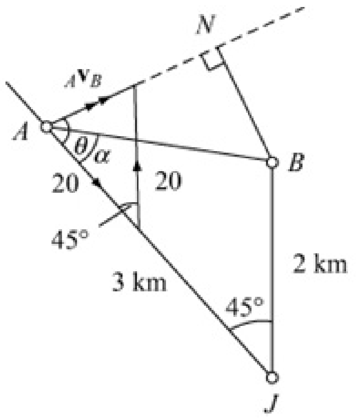
Relative motion
Exercise F, Question 7

Question:

A side road running NW joins a main road which runs due N. Two cars, A and B, each travelling at 20 km h^{-1}, are approaching the junction between the two roads. At a particular instant, A is on the side road at a distance of 3 km from the junction and B is on the main road at a distance of 2 km from the junction. Given that the speeds of the cars remain constant, find

a how close to one another they get,
b the distance of A from the junction when this occurs.

Solution:



Fix B (apply 20 km h^{-1} due N to both)
Since velocity Δ is isosceles,
$|_A\mathbf{v}_B| = 40\sin 22.5° = 15.307 \text{ km h}^{-1}$
$\theta = \frac{1}{2}(180° - 45°) = 67.5°$
N is the point of closest approach.

$$AB^2 = 3^2 + 2^2 - 2\times3\times2\cos 45° = 13 - 6\sqrt{2}$$
$$AB = 2.124786 \quad \text{Let } J\hat{A}B = \alpha$$
$$\frac{\sin\alpha}{2} = \frac{\sin 45°}{AB}$$
$$\sin\alpha = \frac{\sqrt{2}}{2.124786} \Rightarrow \alpha = 41.72676\ldots$$
$$\text{so, } B\hat{A}N = \theta - \alpha = 25.773°$$
$$\text{so, } BN = AB\sin 25.773° = 0.924 \text{ km}$$
$$\text{Time} = \frac{AN}{|_A\mathbf{v}_B|} = \frac{BN\cos 25.773°}{15.307} \text{ h}$$
$$= 0.05435\ldots$$

$$\therefore \text{Distance of } A \text{ from } J = 3 - (20\times.05435..)$$
$$= 1.91 \text{ km (3 s.f.)}$$

a 0.924 km
b 1.91 km

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Relative motion
Exercise F, Question 8

Question:

A ship is moving due W at 40 km h^{-1} and the wind appears to blow from 67.5° west of south. The ship then steams due S at the same speed and the wind then appears to blow from 22.5° east of south. Find

a the true speed of the wind,

b the true direction of the wind.

Solution:

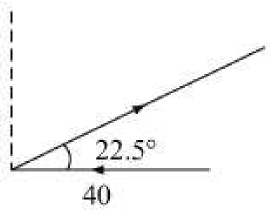
Scenario 1

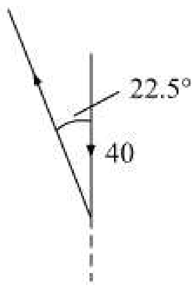
	Mag	Dir
$\mathbf{v}_S$	40	due W
$_W\mathbf{v}_S$	?	From S67.5° W
$\mathbf{v}_W$	?	?

Scenario 2

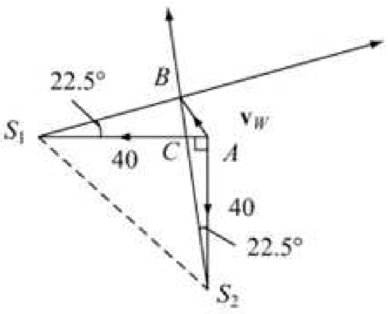
	Mag	Dir
$\mathbf{v}_S$	40	due S
$_W\mathbf{v}_S$	?	From S22.5°E
$\mathbf{v}_W$	?	?

$_W\mathbf{v}_S = \mathbf{v}_W - \mathbf{v}_S \Rightarrow \mathbf{v}_W = \mathbf{v}_S + {}_W\mathbf{v}_S$





Putting the two triangles together:



$S_2\hat{S}_1B = 45° + 22.5° = 67.5°$

$S_1\hat{S}_2B = 22.5° \Rightarrow S_1\hat{B}S_2 = 90°$

ΔABC is isosceles and

$A\hat{C}B = \frac{1}{2}(360° - 135°) = 112.5°$

$\therefore C\hat{A}B = \frac{1}{2} \times 67.5° = 33.75°$

$\therefore$ Direction of wind is N56.25° W. **(b)**

$S_1S_2 = 40\sqrt{2} \Rightarrow S_1B = 40\sqrt{2}\cos 67.5° = 21.6478$

$\frac{|\mathbf{v}_W|}{\sin 22.5°} = \frac{21.6478}{\sin 33.75°} \Rightarrow |\mathbf{v}_W| = 14.9 \text{ km h}^{-1}$ (a)

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Relative motion
Exercise F, Question 9

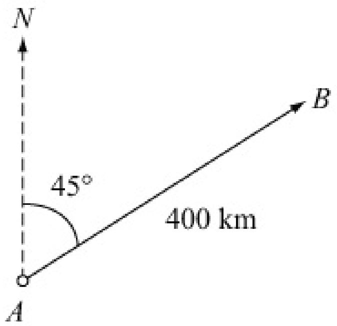
Question:

An aeroplane, which can fly at 160 km h^{-1} in still air, starts from the point A to fly to the point B which is 400 km NE of A. If there is a wind of 40 km h^{-1} blowing from the north, find

a the direction in which the aeroplane must fly,

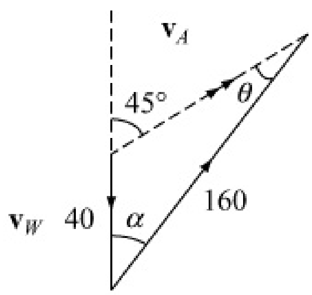
b the time taken to reach B.

Solution:



	Mag	Dir
${}_A\mathbf{v}_W$	160	?
$\mathbf{v}_A$	?	NE
$\mathbf{v}_W$	40	From N

$${}_A\mathbf{v}_W = \mathbf{v}_A - \mathbf{v}_W$$
$$\Rightarrow \mathbf{v}_A = \mathbf{v}_W + {}_A\mathbf{v}_W$$



$$\frac{\sin\theta}{40} = \frac{\sin 135°}{160}$$
$$\sin\theta = \frac{1}{4}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{8}$$
$$\Rightarrow \theta = 10.182°$$
$$\alpha = 45° - \theta = 34.818°$$

a Aeroplane must fly N34.8°E

$$\frac{|\mathbf{v}_A|}{\sin\alpha} = \frac{160}{\sin 135°} \Rightarrow |\mathbf{v}_A| = \frac{160\sin\alpha}{\sin 135°} = 129.2 \text{ km h}^{-1}$$

b $\text{Time} = \frac{400}{129.2}\text{ h} = 3.096\text{ h}$

$= 3\text{ h } 6 \text{ minutes (nearest minute)}$

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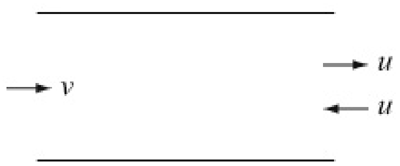
Relative motion
Exercise F, Question 10

Question:

A man can swim at a speed u relative to the water in a river which is flowing with speed v. Assuming that $u > v$, prove that it will take him $\frac{u}{\sqrt{u^2 - v^2}}$ times as long to swim a certain distance d upstream and back as it will to swim the same distance d and back in a direction perpendicular to the current, assuming that d is less than the width of the river.

Solution:

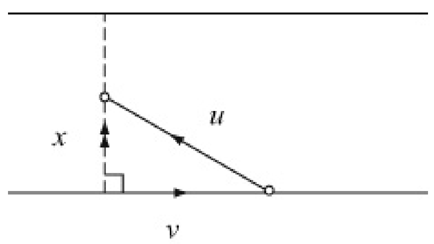


i Downstream : $t_1 = \frac{d}{u+v}$

back : $t_2 = \frac{d}{u-v}$

$$\text{Total time} = \frac{d}{u+v} + \frac{d}{u-v} = d\left(\frac{u-v+u+v}{u^2-v^2}\right)$$
$$= \frac{2du}{u^2-v^2}$$

ii



$$x = \sqrt{u^2 - v^2}$$
$$t = \frac{d}{\sqrt{u^2 - v^2}}$$
$$\therefore \text{Total Time} = \frac{2d}{\sqrt{u^2 - v^2}}$$

$$\therefore \text{Ratio of times} = \frac{2du}{u^2 - v^2} \div \frac{2d}{\sqrt{u^2 - v^2}}$$
$$= \frac{u}{u^2 - v^2} \times \sqrt{u^2 - v^2}$$
$$= \frac{u}{\sqrt{u^2 - v^2}} \text{ as required}$$

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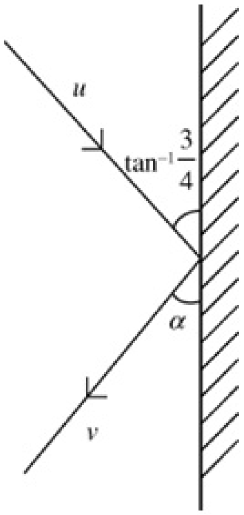
Elastic collisions in two dimensions
Exercise A, Question 1

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $\tan^{-1}\frac{3}{4}$ with the wall. The coefficient of restitution between S and the wall is $\frac{1}{3}$.

Find the speed of S immediately after the collision.

Solution:



R ↑: $v\cos\alpha = u \times \frac{4}{5}$

law of restitution ↔: $v\sin\alpha = e \times u \times \frac{3}{5} = \frac{1}{3} \times u \times \frac{3}{5} = u \times \frac{1}{5}$

squaring and adding,

$v^2 = u^2\left(\frac{16}{25} + \frac{1}{25}\right) = u^2 \times \frac{17}{25}$

$v = \frac{u\sqrt{17}}{5}$

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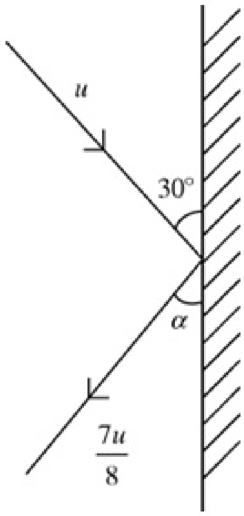
Elastic collisions in two dimensions
Exercise A, Question 2

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $30°$ with the wall. Immediately after the collision the speed of S is $\frac{7}{8}u$.

Find the coefficient of restitution between S and the wall.

Solution:



R $\uparrow$: $\frac{7u}{8}\cos\alpha = u\cos 30°$

law of restitution $\leftrightarrow$: $\frac{7u}{8}\sin\alpha = eu\sin 30°$

squaring and adding:

$$\frac{49u^2}{64} = u^2\left(\frac{3}{4} + \frac{e^2}{4}\right)$$

$$\frac{49}{16} = 3 + e^2$$

$$\frac{1}{16} = e^2, e = \frac{1}{4}$$

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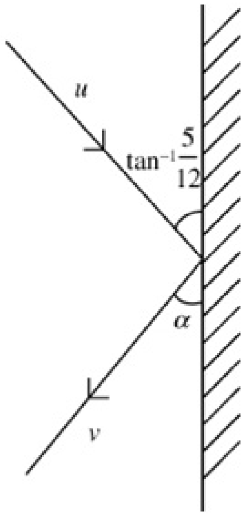
Elastic collisions in two dimensions
Exercise A, Question 3

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $\tan^{-1}\frac{5}{12}$ with the wall. The coefficient of restitution between S and the wall is $\frac{3}{5}$.

Find the speed of S immediately after the collision.

Solution:



R ↑: $v\cos\alpha = u \times \frac{12}{13}$

law of restitution ↔: $v\sin\alpha = e \times u \times \frac{5}{13} = \frac{3}{5} \times u \times \frac{5}{13} = u \times \frac{3}{13}$

squaring and adding,

$$v^2 = u^2\left(\frac{144}{169} + \frac{9}{169}\right) = u^2 \times \frac{153}{169}$$

$$v = \frac{3\sqrt{17}u}{13}$$

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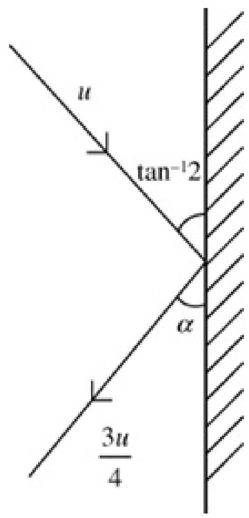
Elastic collisions in two dimensions
Exercise A, Question 4

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $\tan^{-1} 2$ with the wall. Immediately after the collision the speed of S is $\frac{3}{4}u$.

Find the coefficient of restitution between S and the wall.

Solution:



R ↑: $\frac{3u}{4}\cos\alpha = u \times \frac{1}{\sqrt{5}}$

law of restitution ↔: $\frac{3u}{4}\sin\alpha = eu \times \frac{2}{\sqrt{5}}$

squaring and adding:

$$\frac{9u^2}{16} = u^2\left(\frac{1}{5} + \frac{4e^2}{5}\right)$$

$$\frac{45}{16} = 1 + 4e^2$$

$$\frac{29}{16} = 4e^2, e = \frac{\sqrt{29}}{8}$$

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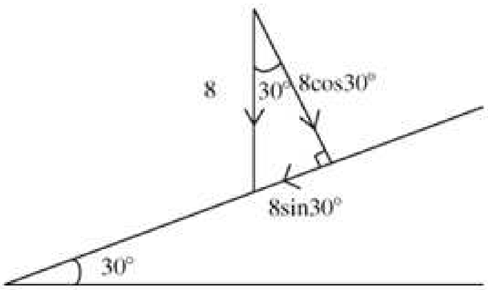
Elastic collisions in two dimensions
Exercise A, Question 5

Question:

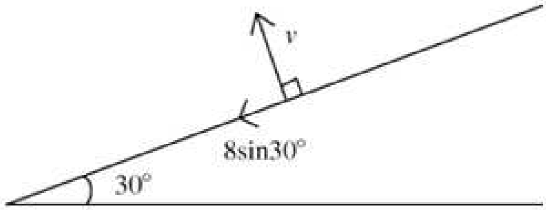
A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle $30°$ to the horizontal. Immediately before striking the plane the ball has speed 8 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{1}{4}$. Find the exact value of the speed of the ball immediately after the impact.

Solution:

Before the impact



After the impact



The component of velocity parallel to the slope $= 8\sin 30° = 8 \times \frac{1}{2} = 4$

Perpendicular to the slope:

$v = e \times 8\cos 30° = \frac{1}{4} \times 8 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

Therefore the speed immediately after impact $= \sqrt{4^2 + \sqrt{3}^2} = \sqrt{19} \text{ m s}^{-1}$

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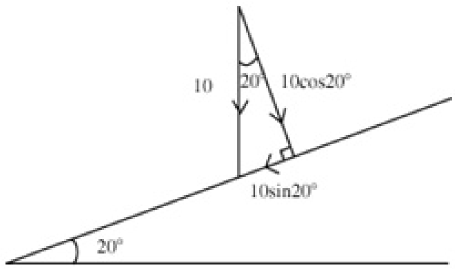
Elastic collisions in two dimensions
Exercise A, Question 6

Question:

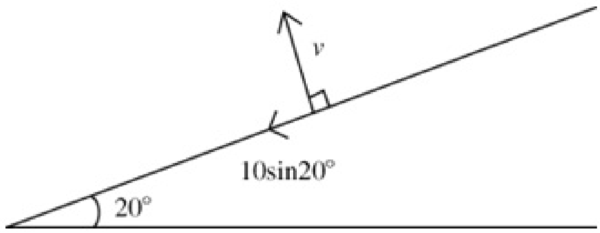
A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle 20° to the horizontal. Immediately before striking the plane the ball has speed 10 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{2}{5}$. Find the speed, to 3 significant figures, of the ball immediately after the impact.

Solution:

Before the impact



After the impact



The component of velocity parallel to the slope $= 10\sin 20^\circ$

Perpendicular to the slope:

$v = e \times 10\cos 20^\circ = \frac{2}{5} \times 10\cos 20^\circ = 4\cos 20^\circ$

Therefore the speed immediately after impact $= \sqrt{(10\sin 20^\circ)^2 + (4\cos 20^\circ)^2}$

$= \sqrt{25.826\ldots} = 5.08\text{ m s}^{-1}$

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Elastic collisions in two dimensions
Exercise A, Question 7

Question:

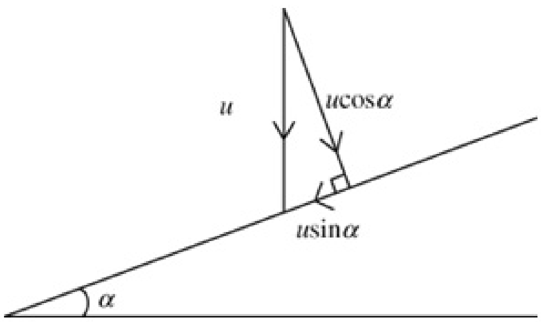
A small smooth ball of mass 750 g is falling vertically. The ball strikes a smooth plane which is inclined at an angle $45°$ to the horizontal. Immediately before striking the plane the ball has speed $5\sqrt{2}$ m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{1}{2}$. Find

a the speed, to 3 significant figures, of the ball immediately after the impact,

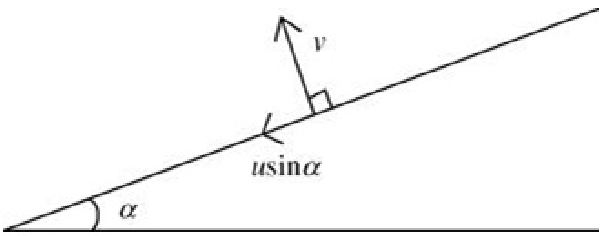
b the magnitude of the impulse received by the ball as it strikes the plane.

Solution:

Before the impact



After the impact



a The component of velocity parallel to the slope $= u\sin\alpha = 5\sqrt{2}\sin 45° = 5\sqrt{2}\times\frac{1}{\sqrt{2}} = 5$

Perpendicular to the slope:

$$v = e\times u\cos\alpha = \frac{1}{2}\times 5\sqrt{2}\cos 45°$$
$$= \frac{1}{2}\times 5\sqrt{2}\times\frac{1}{\sqrt{2}} = \frac{5}{2}$$

Therefore the speed immediately after impact $= \sqrt{5^2 + 2.5^2} = \sqrt{31.25} = 5.59$ m s^{-1}

b The impulse is perpendicular to the surface:

$$I = \frac{3}{4}\left(\frac{5}{2} - (-5)\right) = \frac{3}{4}\times\frac{15}{2} = 5.625 \text{ Ns}$$

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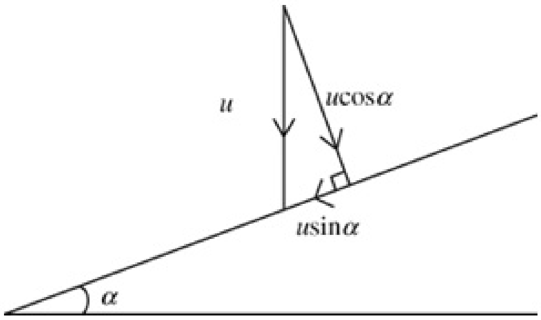
Elastic collisions in two dimensions
Exercise A, Question 8

Question:

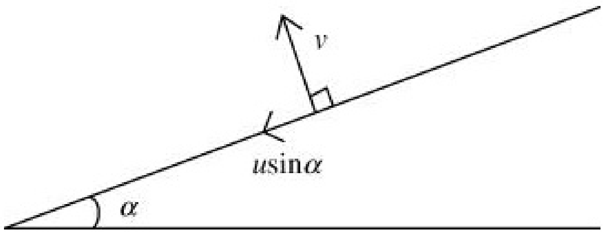
A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. Immediately before striking the plane the ball has speed 7.5 m s^{-1}. Immediately after the impact the ball has speed 5 m s^{-1}.
Find the coefficient of restitution to 2 significant figures, between the ball and the plane.

Solution:

Before the impact



After the impact



The component of velocity parallel to the slope $= u\sin\alpha = 7.5 \times \frac{3}{5} = 4.5$
Perpendicular to the slope:
$v = eu\cos\alpha = e \times 7.5 \times \frac{4}{5} = 6e$
Combining the two components:
$5^2 = 4.5^2 + 36e^2$

$\therefore e^2 = \frac{25 - 20.25}{36} = 0.1319\ldots$
$e = 0.36$

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Elastic collisions in two dimensions
Exercise A, Question 9

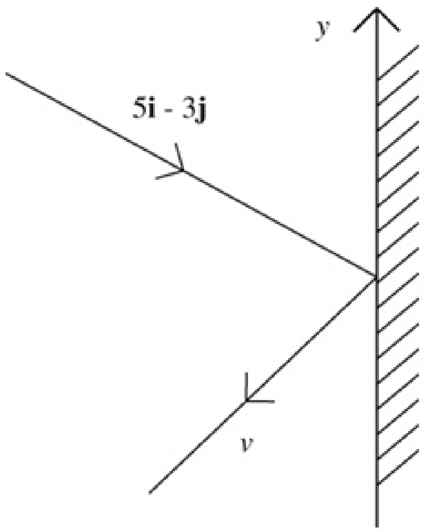
Question:

A small smooth ball of mass 800 g is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the y-axis. The velocity of the ball just before impact is $(5\mathbf{i}-3\mathbf{j})\text{m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{2}$. Find

a the velocity of the ball immediately after the impact,

b the kinetic energy lost as a result of the impact.

Solution:



a Suppose that the velocity of the ball immediately after the impact is $p\mathbf{i}+q\mathbf{j}$

$\uparrow -3 = q$

$\leftrightarrow -p = e \times 5 = \frac{5}{2}$

so $v = -2.5\mathbf{i} - 3\mathbf{j}$

b $\text{K.E. before} = \frac{1}{2} \times \frac{4}{5} \times (5^2 + 3^2) = 13.6$

$\text{K.E. after} = \frac{1}{2} \times \frac{4}{5} \times \left(2.5^2 + 3^2\right) = 6.1$

$\text{K.E. lost} = 13.6 - 6.1 = 7.5\text{ J}$

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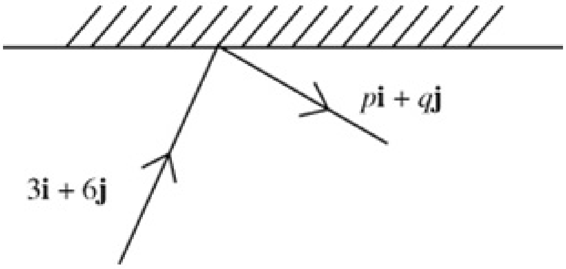
Elastic collisions in two dimensions
Exercise A, Question 10

Question:

A small smooth ball of mass 1 kg is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the x-axis. The velocity of the ball just before impact is $(3\mathbf{i}+6\mathbf{j})\text{m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. Find

a the speed of the ball immediately after the impact,
b the kinetic energy lost as a result of the impact.

Solution:



a Suppose that the velocity of the ball immediately after the impact is $p\mathbf{i}+q\mathbf{j}$

$\leftrightarrow 3 = p$ (parallel to the wall)

$\uparrow -q = \frac{1}{3}\times 6 = 2$ (perpendicular to the wall)

Speed $= \sqrt{3^2+2^2} = \sqrt{13}\text{ m s}^{-1}$.

b K.E. before impact $= \frac{1}{2}\times 1\times(3^2+6^2) = 22.5$, K.E. after impact $= \frac{1}{2}\times 1\times 13 = 6.5$

K.E. lost $= 22.5 - 6.5 = 16\text{ J}$

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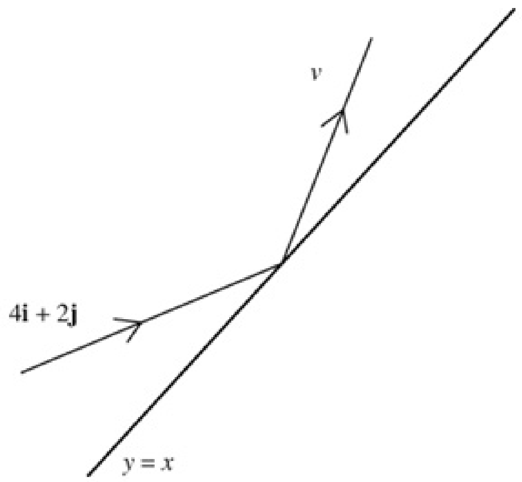
Elastic collisions in two dimensions
Exercise A, Question 11

Question:

A small smooth ball of mass 2 kg is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the line $y = x$. The velocity of the ball just before impact is $(4\mathbf{i} + 2\mathbf{j})\text{m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. Find

a the velocity of the ball immediately after the impact,
b the kinetic energy lost as a result of the impact.

Solution:



Suppose that $v = \mathbf{a} + \mathbf{b}$ where $\mathbf{a}$ is parallel to the wall and $\mathbf{b}$ is perpendicular to the wall.

$\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$ is a unit vector in the direction of the wall.

$\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$ is a unit vector perpendicular to the wall.

$$\nearrow \left[(4\mathbf{i}+2\mathbf{j}).\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})\right]\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = \mathbf{a}$$

$$= \frac{1}{\sqrt{2}} \times 6 \times \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = 3\mathbf{i}+3\mathbf{j}$$

$$\nwarrow \frac{1}{3}\left[(4\mathbf{i}+2\mathbf{j}).\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})\right]\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j}) = \mathbf{b}$$

$$= \frac{1}{3} \times \frac{1}{\sqrt{2}} \times (4-2) \times \frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$$

$$= \frac{1}{3}(-\mathbf{i}+\mathbf{j})$$

So $\mathbf{v} = 3\mathbf{i}+3\mathbf{j} - \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} = \frac{8}{3}\mathbf{i} + \frac{10}{3}\mathbf{j}$

b K.E. before $= \frac{1}{2} \times 2 \times (4^2 + 2^2) = 20$, K.E. after $= \frac{1}{2} \times 2 \times \left(\frac{64}{9} + \frac{100}{9}\right) = \frac{164}{9}$

K.E. lost $= 20 - \frac{164}{9} = \frac{16}{9}$ J

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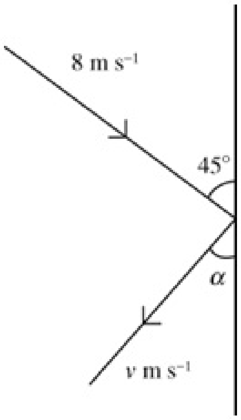
Elastic collisions in two dimensions
Exercise A, Question 12

Question:

A smooth snooker ball strikes a smooth cushion with speed 8 m s^{-1} at an angle of $45°$ to the cushion. Given that the coefficient of restitution between the ball and the cushion is $\frac{2}{5}$, find the magnitude and direction of the velocity of the ball after the impact.

Solution:



↑: no change parallel to the cushion

$8\cos 45° = v\cos\alpha = 4\sqrt{2}$

↔: Using the law of restitution

$\frac{2}{5}\times 8\sin 45° = v\sin\alpha = \frac{8\sqrt{2}}{5}$

Squaring and adding:

$v^2 = (4\sqrt{2})^2 + \left(\frac{8\sqrt{2}}{5}\right)^2 = 37.12, v \approx 6.09$

Dividing: $\tan\alpha = \frac{\frac{8\sqrt{2}}{5}}{4\sqrt{2}} = \frac{2}{5}, \alpha = \tan^{-1} 0.4$

Velocity is 6.09 m s^{-1} at $21.8°$ to the cushion.

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Elastic collisions in two dimensions
Exercise A, Question 13

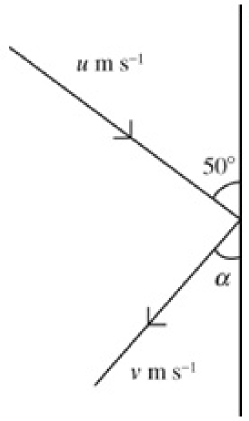
Question:

A smooth snooker ball strikes a smooth cushion with speed u m s^{-1} at an angle of $50°$ to the cushion. The coefficient of restitution between the ball and the cushion is e.

a Show that the angle between the cushion and the rebound direction is independent of u.

b Find the value of e given that the ball rebounds at right angles to its original direction.

Solution:



a ↑: no change parallel to the cushion

$u\cos 50° = v\cos\alpha$

↔: Using the law of restitution, $e \times u\sin 50° = v\sin\alpha$

Dividing: $\dfrac{v\sin\alpha}{v\cos\alpha} = \dfrac{eu\sin 50°}{u\cos 50°}$

$\Rightarrow \tan\alpha = e\tan 50°$, which is independent of the value of u

b If $\alpha = 40°$ then $\tan 40° = e\tan 50°$

$e = \dfrac{\tan 40°}{\tan 50°} \approx 0.7$

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Elastic collisions in two dimensions
Exercise A, Question 14

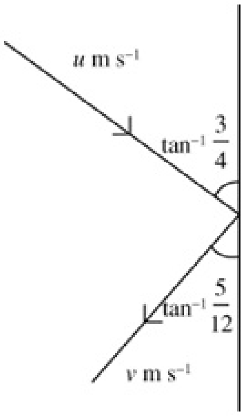
Question:

A smooth billiard ball strikes a smooth cushion at an angle of $\tan^{-1}\frac{3}{4}$ to the cushion.

The ball rebounds at an angle of $\tan^{-1}\frac{5}{12}$ to the cushion. Find

a the fraction of the kinetic energy of the ball lost in the collision,
b the coefficient of restitution between the ball and the cushion.

Solution:



a ↑: no change parallel to the cushion

$$u\times\frac{4}{5} = v\times\frac{12}{13}$$

$$v = u\times\frac{4}{5}\times\frac{13}{12} = \frac{13}{15}u$$

$$\text{K.E. lost} = \frac{1}{2}\times m\times u^2 - \frac{1}{2}\times m\times\frac{169}{225}u^2 = \frac{1}{2}\times m\times\frac{56}{225}u^2\text{J}$$

$$\text{Fraction of K.E. lost} = \frac{\frac{1}{2}\times m\times\frac{56}{225}u^2}{\frac{1}{2}\times m\times u^2} = \frac{56}{225}$$

b ↔: Using the law of restitution, $e\times u\times\frac{3}{5} = v\times\frac{5}{13}$

$$e\times u\times\frac{3}{5} = \frac{13}{15}u\times\frac{5}{13}$$

$$e = \frac{5}{15}\times\frac{5}{3} = \frac{5}{9}$$

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Elastic collisions in two dimensions
Exercise A, Question 15

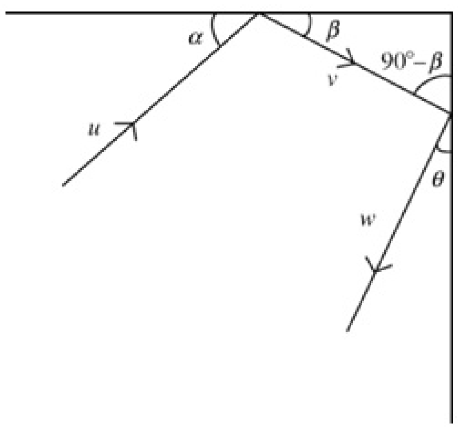
Question:

Two vertical walls meet at right angles at the corner of a room. A small smooth disc slides across the floor and bounces off each wall in turn. Just before the first impact the disc is moving with speed u m s^{-1} at an acute angle α to the wall. The coefficient of restitution between the disc and the wall is e. Find

a the direction of the motion of the disc after the second collision,

b the speed of the disc after the second collision.

Solution:



a First collision:

$\uparrow$: $e \times u \sin\alpha = v \sin\beta$

$\leftrightarrow$: $u \cos\alpha = v \cos\beta$

Second collision:

$\uparrow$: $v \cos(90-\beta) = v \sin\beta = w \cos\theta$

$\leftrightarrow$: $e \times v \sin(90-\beta) = ev \cos\beta = w \sin\theta$

$$\Rightarrow \tan\theta = \frac{e\cos\beta}{\sin\beta} = \frac{e}{\tan\beta}$$

$$= \frac{e}{e\tan\alpha} = \frac{1}{\tan\alpha}$$

$\Rightarrow \theta = 90° - \alpha$, so the path is parallel to the original path but in the opposite direction

b $eu \sin\alpha = v \sin\beta = w \cos\theta = w \cos(90° - \alpha) = w \sin\alpha$

speed after second collision $= w = eu$

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Elastic collisions in two dimensions
Exercise A, Question 16

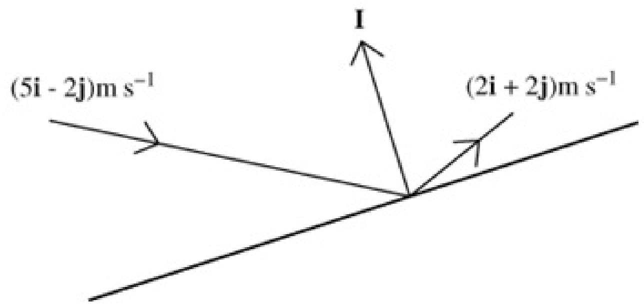
Question:

A small smooth sphere of mass m is moving with velocity $(5\mathbf{i}-2\mathbf{j})\text{m s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(2\mathbf{i}+2\mathbf{j})\text{m s}^{-1}$. Find

a the magnitude and direction of the impulse received by the sphere,

b the coefficient of restitution between the sphere and the wall.

Solution:



a $\mathbf{I} = m\mathbf{v} - m\mathbf{u} = m((2\mathbf{i}+2\mathbf{j})-(5\mathbf{i}-2\mathbf{j}))$

$= m(-3\mathbf{i}+4\mathbf{j})$

The impulse has magnitude $5m$ Ns in the direction parallel to the unit vector $\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$.

b Component of $(5\mathbf{i}-2\mathbf{j})$ in the direction of the impulse

$= [(5\mathbf{i}-2\mathbf{j}).\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})]\frac{1}{5}(-3\mathbf{i}+4\mathbf{j}) = \frac{1}{5}\times(-15-8)\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$

$= \frac{-23}{5}\times\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$

Component of $(2\mathbf{i}+2\mathbf{j})$ in the direction of the impulse

$= [(2\mathbf{i}+2\mathbf{j}).\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})]\frac{1}{5}(-3\mathbf{i}+4\mathbf{j}) = \frac{1}{5}\times(-6+8)\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$

$= \frac{2}{5}\times\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$

law of restitution

$\Rightarrow \frac{2}{5} = e\times\frac{23}{5}$

$e = \frac{2}{23}$

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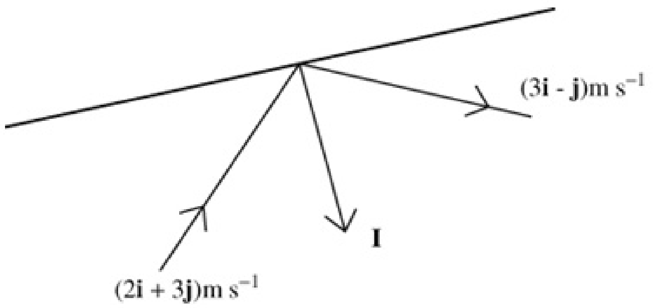
Elastic collisions in two dimensions
Exercise A, Question 17

Question:

A small smooth sphere of mass 2 kg is moving with velocity $(2\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(3\mathbf{i}-\mathbf{j})\text{m s}^{-1}$. Find

a the magnitude and direction of the impulse received by the sphere,
b the coefficient of restitution between the sphere and the wall,
c the kinetic energy lost by the sphere in the collision.

Solution:



a $\mathbf{I} = m\mathbf{v} - m\mathbf{u} = m\{(3\mathbf{i}-\mathbf{j})-(2\mathbf{i}+3\mathbf{j})\}$
$= 2(\mathbf{i}-4\mathbf{j})$
The impulse has magnitude
$2\sqrt{17}$ Ns in
the direction parallel to the unit vector
$\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})$

b Component of $(2\mathbf{i}+3\mathbf{j})$ in the direction of the impulse =
$[(2\mathbf{i}+3\mathbf{j}).\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})]\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j}) = \frac{1}{\sqrt{17}}(2-12)\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j}) = \frac{-10}{\sqrt{17}}\times\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})$
Component of $(3\mathbf{i}-\mathbf{j})$ in the direction of the impulse =
$[(3\mathbf{i}-\mathbf{j}).\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})]\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j}) = \frac{1}{\sqrt{17}}(3+4)\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j}) = \frac{7}{\sqrt{17}}\times\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})$
law of restitution $\Rightarrow \frac{7}{\sqrt{17}} = e\times\frac{10}{\sqrt{17}}, e = \frac{7}{10}$

c K.E. just before the impact $= \frac{1}{2}\times 2\times(2^2+3^2) = 13$
K.E. just after the impact $= \frac{1}{2}\times 2\times(3^2+1^2) = 10$
K.E. lost $= 13-10 = 3$ J

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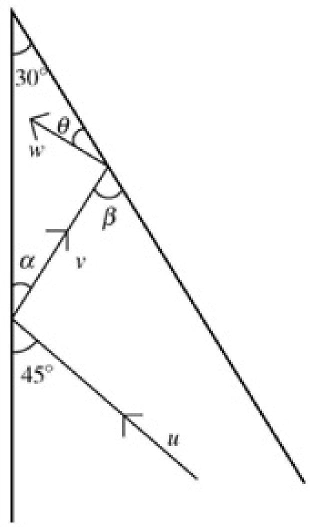
Elastic collisions in two dimensions
Exercise A, Question 18

Question:

Two smooth vertical walls stand on a smooth horizontal floor and intersect at an angle of 30°. A particle is projected along the floor with speed u m s^{-1} at 45° to one of the walls and towards the intersection of the walls. The coefficient of restitution between the particle and each wall is $\frac{1}{\sqrt{3}}$. Find the speed of the particle after one impact with each wall.

Solution:



For the first impact:

$u\cos 45° = \frac{u}{\sqrt{2}} = v\cos\alpha$

$eu\sin 45 = \frac{1}{\sqrt{3}} \times \frac{u}{\sqrt{2}} = v\sin\alpha$

By dividing, $\tan\alpha = \frac{1}{\sqrt{3}}$, $\alpha = 30°$, and $\beta = 60°$

Squaring and adding, $v^2 = \frac{u^2}{2} + \frac{u^2}{6} = \frac{4u^2}{6}$

For the second impact:

$v\cos 60° = \frac{v}{2} = w\cos\theta$

$ev\sin 60 = v \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{v}{2} = w\sin\theta$

Squaring and adding,

$w^2 = \left(\frac{v}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = \frac{v^2}{2} = \frac{u^2}{3}, \quad w = \frac{\sqrt{3}u}{3}$ m s^{-1}

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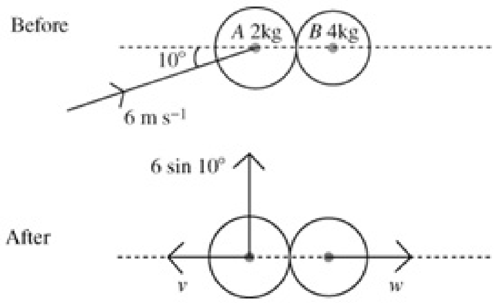
Elastic collisions in two dimensions
Exercise B, Question 1

Question:

A smooth sphere A, of mass 2 kg and moving with speed 6 m s^{-1} collides obliquely with a smooth sphere B of mass 4 kg. Just before the impact B is stationary and the velocity of A makes an angle of $10°$ with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

Solution:



Parallel to the line of centres conservation of momentum:

$2 \times 6\cos 10° = 4w - 2v, 2w - v = 6\cos 10°$

law of restitution:

$w + v = \frac{1}{2} \times 6\cos 10° = 3\cos 10°$

$\Rightarrow 3w = 9\cos 10°, w = 3\cos 10°, v = 0$

So, after the impact, A has velocity $6\sin 10° \approx 1.04 \text{ m s}^{-1}$ perpendicular to the line of centres, and B has velocity $3\cos 10° \approx 2.95 \text{ m s}^{-1}$ parallel to the line of centres

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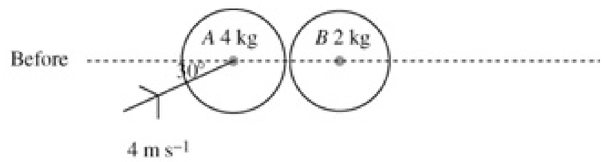
Elastic collisions in two dimensions
Exercise B, Question 2

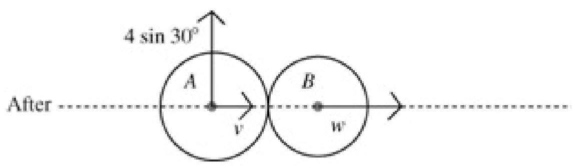
Question:

A smooth sphere A, of mass 4 kg and moving with speed 4 m s^{-1} collides obliquely with a smooth sphere B of mass 2 kg. Just before the impact B is stationary and the velocity of A makes an angle of $30°$ with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{3}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

Solution:



Perpendicular to the line of centres, component of the velocity of A is
$4\sin 30° = 2 \text{ m s}^{-1}$



Parallel to the line of centres:

$$4 \times 4\cos 30 = 4v + 2w, 4\sqrt{3} = 2v + w$$

$$w - v = \frac{1}{3} \times 4\cos 30°, 2w - 2v = \frac{4\sqrt{3}}{3}$$

$$\Rightarrow 3w = \frac{16}{3}\sqrt{3}, w = \frac{16\sqrt{3}}{9}$$

$$\text{and } v = \frac{10\sqrt{3}}{9}$$

B has speed $\frac{16\sqrt{3}}{9} \text{ m s}^{-1}$ along the line of centres.

A has speed $\sqrt{(2)^2 + \left(\frac{10\sqrt{3}}{9}\right)^2} = \sqrt{4 + \frac{100}{27}} = \sqrt{\frac{208}{27}} = \frac{4\sqrt{13}}{3\sqrt{3}} = \frac{4\sqrt{39}}{9} \text{ m s}^{-1}$ at an angle of $\tan^{-1}\left(\frac{2}{10\frac{\sqrt{3}}{9}}\right) = \tan^{-1}\left(\frac{18}{10\sqrt{3}}\right) = \tan^{-1}\left(\frac{3\sqrt{3}}{5}\right) = 46.1°$ to the line of centres

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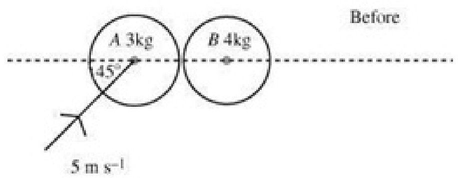
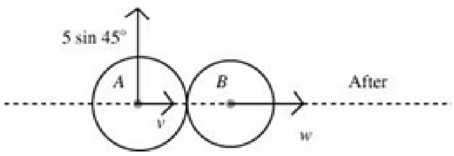
Elastic collisions in two dimensions
Exercise B, Question 3

Question:

A smooth sphere A, of mass 3 kg and moving with speed 5 m s^{-1} collides obliquely with a smooth sphere B of mass 4 kg. Just before the impact B is stationary and the velocity of A makes an angle of $45°$ with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

Solution:





Perpendicular to the line of centres, component of the velocity of A is

$5\sin 45° = \frac{5\sqrt{2}}{2}\text{ m s}^{-1}$

Parallel to the line of centres:

$3\times 5\cos 45 = 3v + 4w$

$w - v = \frac{1}{2}\times 5\cos 45°, 3w - 3v = \frac{15\sqrt{2}}{4}$

$\Rightarrow 7w = \frac{45\sqrt{2}}{4}, w = \frac{45\sqrt{2}}{28}$

and $v = \frac{10\sqrt{2}}{28} = \frac{5\sqrt{2}}{14}$

B has speed $\frac{45\sqrt{2}}{28}\text{ m s}^{-1}$ along the line of centres

A has speed $\sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{14}\right)^2} = \frac{5\sqrt{2}}{2}\sqrt{1^2 + \left(\frac{1}{7}\right)^2} = \frac{5\sqrt{2}}{2}\sqrt{\frac{50}{49}} = \frac{50}{14} = \frac{25}{7}\text{ m s}^{-1}$ at an angle of $\tan^{-1}\left(\frac{5\sin 45°}{v}\right) = \tan^{-1}\left(\frac{\frac{5\sqrt{2}}{2}}{\frac{5\sqrt{2}}{14}}\right) = \tan^{-1}7 \approx 81.9°$ to the line of centres

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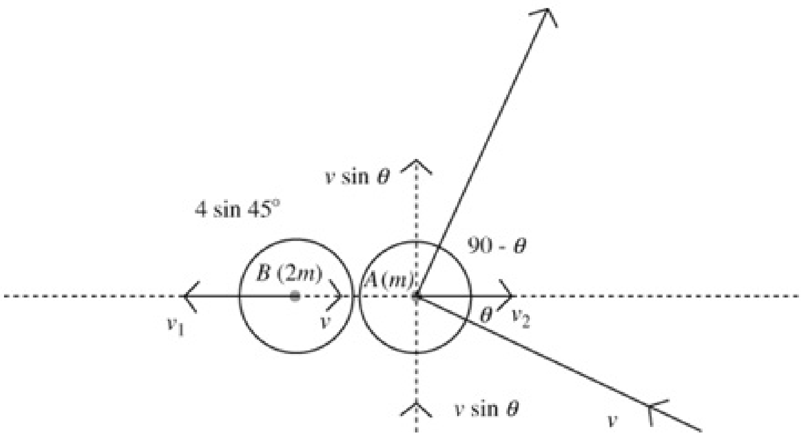
Elastic collisions in two dimensions
Exercise B, Question 4

Question:

A small smooth sphere A of mass m and a small smooth sphere B of the same radius but mass $2m$ collide. At the instant of impact, B is stationary and the velocity of A makes an angle θ with the line of centres. The direction of motion of A is turned through $90°$ by the impact. The coefficient of restitution between the spheres is e. Show that

$$\tan^2\theta = \frac{2e-1}{3}.$$

Solution:



Components perpendicular to the line of centres are unchanged. For A, the component perpendicular to the line of centres is $v\sin\theta$.
Parallel to the line of centres:
conservation of momentum $\Rightarrow mv\cos\theta = 2mv_1 - mv_2$
law of restitution

$$\Rightarrow v_1 + v_2 = ev\cos\theta$$

$$\Rightarrow v\cos\theta = 2(ev\cos\theta - v_2) - v_2 = 2ev\cos\theta - 3v_2$$

$$v_2 = \frac{v\cos\theta(2e-1)}{3}$$

$$\Rightarrow \tan(90-\theta) = \frac{1}{\tan\theta} = \frac{v\sin\theta}{v_2} = \frac{3v\sin\theta}{v\cos\theta(2e-1)},\ \therefore \frac{1}{\tan\theta} = \frac{3\tan\theta}{2e-1}\quad \therefore \tan^2\theta = \frac{2e-1}{3}$$

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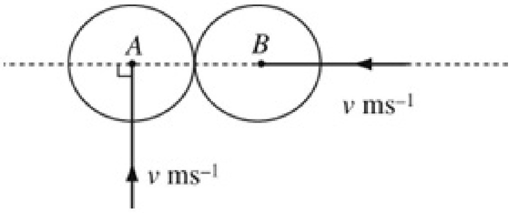
Elastic collisions in two dimensions
Exercise B, Question 5

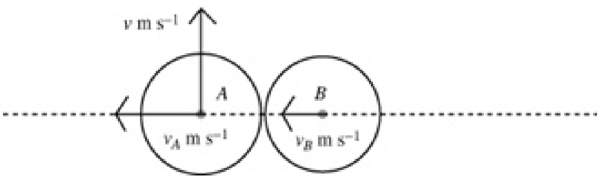
Question:

Two smooth spheres A and B are identical and are moving with equal speeds on a smooth horizontal surface. In the instant before impact, A is moving in a direction perpendicular to the line of centres of the spheres, and B is moving along the line of centres, as



shown in the diagram. The coefficient of restitution between the spheres is $\frac{2}{3}$. Find the speeds and directions of motion of the spheres after the collision.

Solution:



Components perpendicular to the line of centres are unchanged.
Conservation of momentum: $mv = mv_A + mv_B, \quad v = v_A + v_B$
law of restitution:

$$v_A - v_B = \frac{2}{3}v$$

$$\Rightarrow 2v_A = \frac{5}{3}v, v_A = \frac{5}{6}v, \quad v_B = \frac{1}{6}v$$

A has speed $\sqrt{1^2 + \left(\frac{5}{6}\right)^2}v = \sqrt{\frac{61}{36}}v = \frac{\sqrt{61}v}{6}$ m s^{-1} and is moving at

$$\tan^{-1}\left(\frac{1}{\left(\frac{5}{6}\right)}\right) = \tan^{-1}\frac{6}{5} = 50.2^\circ$$ to the line of centres.

B is moving along the line of centres with speed $\frac{1}{6}v$ m s^{-1}.

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Elastic collisions in two dimensions
Exercise B, Question 6

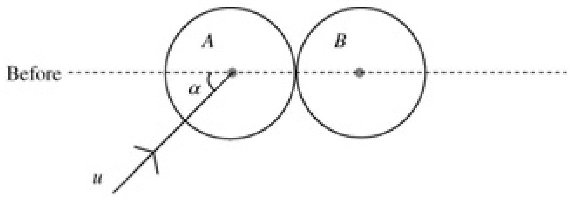
Question:

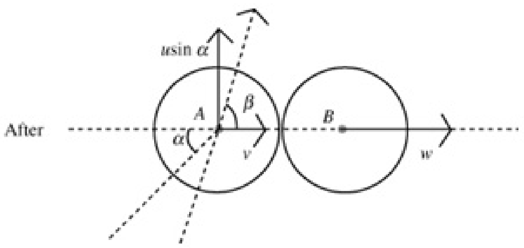
A smooth sphere A collides obliquely with an identical smooth sphere B. Just before the impact B is stationary and the velocity of A makes an angle of α with the lines of centres of the two spheres. The coefficient of restitution between the spheres is e $(e \neq 1)$. Immediately after the collision the velocity of A makes an angle of β with the line of centres.

a Show that $\tan\beta = \dfrac{2\tan\alpha}{1-e}$.

b Hence show that in the collision the direction of motion of A turns through an angle equal to $\tan^{-1}\left(\dfrac{(1+e)\tan\alpha}{2\tan^2\alpha+1-e}\right)$.

Solution:





Perpendicular to the line of centres, component of velocity of A is $u\sin\alpha$. Parallel to the line of centres:
conservation of momentum: $mu\cos\alpha = mv + mw$, $\quad u\cos\alpha = v + w$
law of restitution: $w - v = eu\cos\alpha$, so $2v = u\cos\alpha - eu\cos\alpha = u\cos\alpha(1-e)$

$$\Rightarrow \tan\beta = \frac{u\sin\alpha}{v} = \frac{2u\sin\alpha}{u\cos\alpha(1-e)} = \frac{2\tan\alpha}{1-e}$$

b The path of A has been deflected through an angle equal to $\beta - \alpha$.

$$\tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\alpha\tan\beta} = \frac{\frac{2\tan\alpha}{1-e} - \tan\alpha}{1+\tan\alpha\frac{2\tan\alpha}{1-e}} = \frac{2\tan\alpha - (1-e)\tan\alpha}{1-e+2\tan^2\alpha}$$

$$= \frac{(1+e)\tan\alpha}{2\tan^2\alpha + 1 - e}$$

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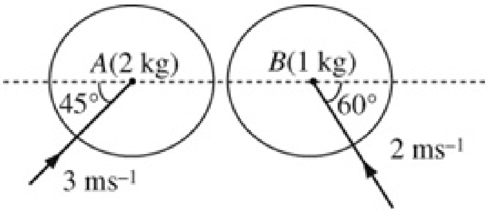
Elastic collisions in two dimensions
Exercise B, Question 7

Question:



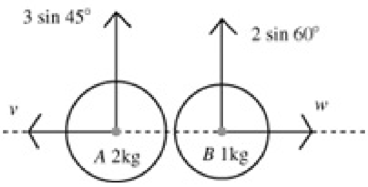
A small smooth sphere A of mass 2 kg collides with a small smooth sphere B of mass 1 kg. Just before the impact A is moving with a speed of 3 m s^{-1} in a direction at $45°$ to the line of centres and B is moving with speed 2 m s^{-1} at $60°$ to the line of centres, as shown in the diagram. The coefficient of restitution between the spheres is $\frac{\sqrt{2}}{3}$.

Find

a the kinetic energy lost in the impact,

b the magnitude of the impulse exerted by A on B.

Solution:



No change in the components of velocity perpendicular to the line of centres. Parallel to the line of centres:

conservation of momentum: $1\times 2\cos 60° - 2\times 3\cos 45° = 2v - w = 1 - 3\sqrt{2}$

law of restitution: $v + w = \frac{\sqrt{2}}{3}(3\cos 45° + 2\cos 60°)$

$$= \frac{\sqrt{2}}{3}\left(\frac{3\sqrt{2}}{2} + 1\right) = 1 + \frac{\sqrt{2}}{3}$$

Solving the simultaneous equations gives $3v = 2 - \frac{8\sqrt{2}}{3}, v = \frac{2}{3} - \frac{8\sqrt{2}}{9} \approx -0.590$ and

$w = 1 + \frac{\sqrt{2}}{3} - \frac{2}{3} + \frac{8\sqrt{2}}{9} = \frac{1}{3} + \frac{11\sqrt{2}}{9} \approx 2.06$

a K.E. lost in the impact

$$= \frac{1}{2}\times 2\times((3\cos 45°)^2 - 0.590^2) + \frac{1}{2}\times 1\times((2\cos 60°)^2 - 2.06^2) \approx 2.53 \text{ J}$$

b Impulse on $B = 1(w + 2\cos 60°) \approx 3.06$ Ns

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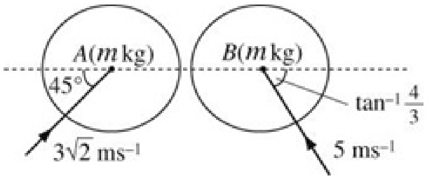
Elastic collisions in two dimensions
Exercise B, Question 8

Question:

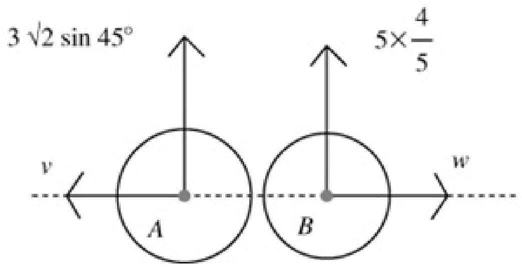
A small smooth sphere A collides with an identical small smooth sphere B. Just before the impact A is moving with a speed of $3\sqrt{2}$ m s^{-1} in a direction at 45° to the line of centres and B is moving with speed 5 m s^{-1} at $\tan^{-1}\frac{4}{3}$ to the line of centres, as shown in the diagram.



The coefficient of restitution between the spheres is $\frac{2}{3}$. Find

a the speeds of both spheres immediately after the impact,
b the fraction of the kinetic energy lost in the impact.

Solution:



After the collision the components of velocity perpendicular to the line of centres are 3 m s^{-1} and 4 m s^{-1}. (No change in this direction.)
Parallel to the line of centres:

conservation of momentum: $m\times 3\sqrt{2}\cos 45° - m\times 5\times\frac{3}{5} = mw - mv = 0$

law of restitution: $v + w = \frac{2}{3}\left(3\sqrt{2}\cos 45° + 5\times\frac{3}{5}\right) = 4$

so $v = w = 2$

a speed of $A = \sqrt{2^2+3^2} = \sqrt{13}$ m s^{-1}

speed of $B = \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$ m s^{-1}

b Total K.E. just before impact $= \frac{1}{2}\times m\times(3\sqrt{2})^2 + \frac{1}{2}\times m\times 5^2 = \frac{m\times 43}{2}$ J

Total K.E. just after impact $= \frac{1}{2}\times m\times(\sqrt{13})^2 + \frac{1}{2}\times m\times(2\sqrt{5})^2 = \frac{m\times 33}{2}$ J

Fraction of K.E. lost $= \frac{43-33}{43} = \frac{10}{43}$

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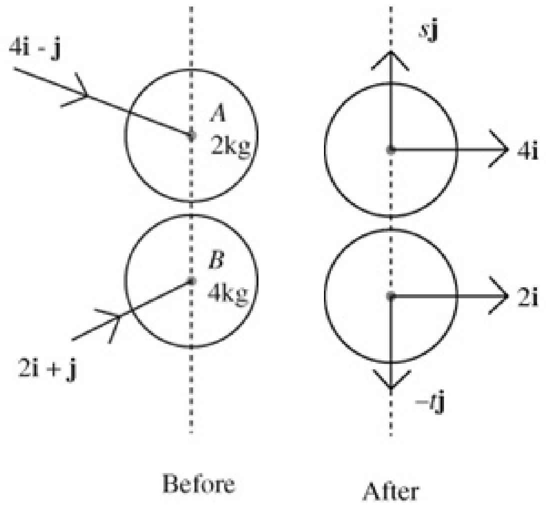
Elastic collisions in two dimensions
Exercise B, Question 9

Question:

A smooth sphere A of mass 2 kg is moving on a smooth horizontal surface with velocity $(4\mathbf{i}-\mathbf{j})$m s^{-1}. Another smooth sphere B of mass 4 kg and the same radius as A is moving on the same surface with velocity $(2\mathbf{i}+\mathbf{j})$m s^{-1}. The spheres collide when their line of centres is parallel to $\mathbf{j}$. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the velocities of both spheres after the impact.

Solution:



Line of centres parallel to $\mathbf{j} \Rightarrow$ no change in the components of velocity parallel to $\mathbf{i}$.

Conservation of momentum: $-2\times1+4\times1=2\times s-4\times t=2$

law of restitution: $s+t=\frac{1}{2}(1+1), s+t=1$

$$s-2t=1$$
$$3s=3$$
$$s=1, t=0$$

velocity of A is $4\mathbf{i}+\mathbf{j}$ m s^{-1}

velocity of B is $2\mathbf{i}$ m s^{-1}

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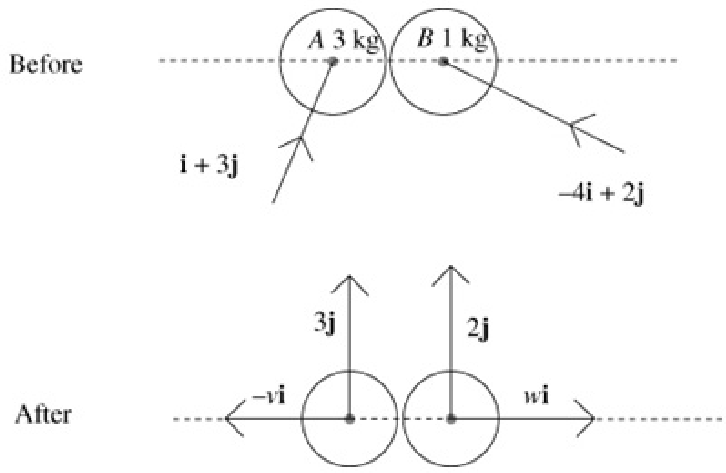
Elastic collisions in two dimensions
Exercise B, Question 10

Question:

A smooth sphere A of mass 3 kg is moving on a smooth horizontal surface with velocity $(\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$. Another smooth sphere B of mass 1 kg and the same radius as A is moving on the same surface with velocity $(-4\mathbf{i}+2\mathbf{j})\text{m s}^{-1}$. The spheres collide when their line of centres is parallel to $\mathbf{i}$. The coefficient of restitution between the spheres is $\frac{3}{4}$. Find the speeds of both spheres after the impact.

Solution:



Line of centres parallel to $\mathbf{i} \Rightarrow$ no change in the components of velocity parallel to $\mathbf{j}$

conservation of momentum: $3\times1-1\times4=1\times w-3\times v=-1$

law of restitution: $v+w=\frac{3}{4}(4+1), 4v+4w=15$

$$4w-12v=-4$$

$$16v=19$$

$$v=\frac{19}{16}, w=\frac{41}{16}$$

After the impact, speed of $A=\sqrt{3^2+\left(\frac{19}{16}\right)^2}\approx3.23\text{ m s}^{-1}$,

speed of $B=\sqrt{2^2+\left(\frac{41}{16}\right)^2}\approx3.25\text{ m s}^{-1}$

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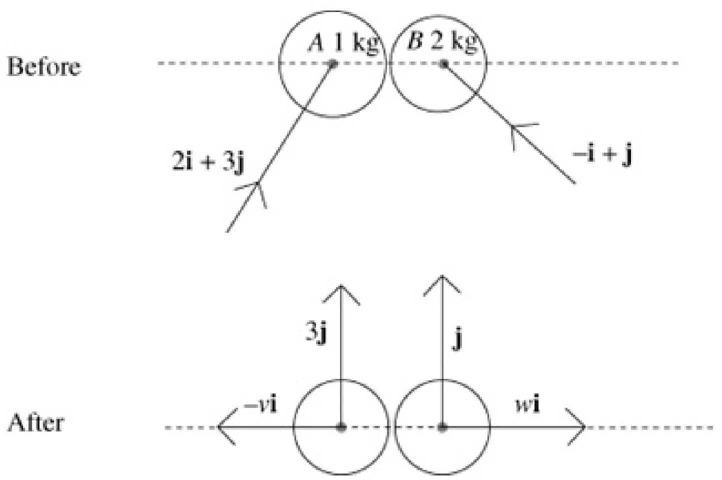
Elastic collisions in two dimensions
Exercise B, Question 11

Question:

A smooth sphere A of mass 1 kg is moving on a smooth horizontal surface with velocity $(2\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$. Another smooth sphere B of mass 2 kg and the same radius as A is moving on the same surface with velocity $(-\mathbf{i}+\mathbf{j})\text{ms}^{-1}$. The spheres collide when their line of centres is parallel to $\mathbf{i}$. The coefficient of restitution between the spheres is $\frac{3}{5}$. Find the kinetic energy lost in the impact.

Solution:



Line of centres parallel to $\mathbf{i} \Rightarrow$ no change in the components of velocity parallel to $\mathbf{j}$

conservation of momentum: $1\times2-2\times1=2\times w-1\times v=0$

law of restitution: $v+w=\frac{3}{5}(2+1)$

$2w-v=0, 3w=\frac{9}{5}, w=\frac{3}{5}$

$v=\frac{6}{5}$

$\text{K.E. lost}=\frac{1}{2}\times1\times\left(2^2-\left(\frac{6}{5}\right)^2\right)+\frac{1}{2}\times2\times\left(1^2-\left(\frac{3}{5}\right)^2\right)=\frac{48}{25}=1.92\text{ J}$

Components of velocity unchanged parallel to $\mathbf{j} \Rightarrow$ all K.E. lost parallel to $\mathbf{i}$.

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Elastic collisions in two dimensions
Exercise B, Question 12

Question:

Two small smooth spheres A and B have equal radii. The mass of A is m kg and the mass of B is $2m$ kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i}+5\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(3\mathbf{i}-\mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A *is* $(3\mathbf{i}+2\mathbf{j})\text{m s}^{-1}$. Find

a the velocity of B immediately after the collision,

b a unit vector parallel to the line of centres of the spheres at the instant of the collision.

Solution:

a Conservation of momentum $\Rightarrow m(2\mathbf{i}+5\mathbf{j})+2m(3\mathbf{i}-\mathbf{j})=m(3\mathbf{i}+2\mathbf{j})+2m\mathbf{v}$

$$2\mathbf{v}=\mathbf{i}(2+2\times 3-3)+\mathbf{j}(5-2\times 1-2)=5\mathbf{i}+\mathbf{j}$$

$$\mathbf{v}=\frac{5}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}$$

b Impulse on $A=m((3\mathbf{i}+2\mathbf{j})-(2\mathbf{i}+5\mathbf{j}))=m(\mathbf{i}-3\mathbf{j})$

$\Rightarrow$ line of centres parallel to $\frac{1}{\sqrt{10}}(\mathbf{i}-3\mathbf{j})$

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Elastic collisions in two dimensions
Exercise B, Question 13

Question:

Two small smooth spheres A and B have equal radii. The mass of A is $3m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(3\mathbf{i}-5\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(4\mathbf{i}+\mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(4\mathbf{i}-4\mathbf{j})\text{m s}^{-1}$. Find

a the speed of B immediately after the collision,

b the kinetic energy lost in the collision.

Solution:

a Conservation of momentum $\Rightarrow 3m(3\mathbf{i}-5\mathbf{j})+m(4\mathbf{i}+\mathbf{j})=3m(4\mathbf{i}-4\mathbf{j})+m\mathbf{v}$

$$\mathbf{v}=\mathbf{i}(3\times3+4-3\times4)+\mathbf{j}(-3\times5+1+3\times4)=\mathbf{i}-2\mathbf{j}$$

Speed of B is $\sqrt{1^2+2^2}=\sqrt{5}\text{ m s}^{-1}$

b $\text{K.E. lost}=\frac{3m}{2}((3^2+5^2)-(4^2+4^2))+\frac{m}{2}((4^2+1^2)-5)$

$$=\frac{m}{2}(3(34-32)+(17-5))=\frac{m}{2}(6+12)=9m\text{ J}$$

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Elastic collisions in two dimensions
Exercise B, Question 14

Question:

Two small smooth spheres A and B have equal radii. The mass of A is $2m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i}+5\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(2\mathbf{i}-2\mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i}+4\mathbf{j})\text{m s}^{-1}$. Find

a the velocity of B immediately after the collision,

b the coefficient of restitution between the two spheres.

Solution:

a Conservation of momentum $\Rightarrow 2m(2\mathbf{i}+5\mathbf{j})+m(2\mathbf{i}-2\mathbf{j})=2m(3\mathbf{i}+4\mathbf{j})+m\mathbf{v}$

$$\mathbf{v}=\mathbf{i}(2\times2+2-2\times3)+\mathbf{j}(2\times5-2-2\times4)=0$$

b B is brought to a halt in the collision $\Rightarrow$ the line of centres must be parallel to the original direction of motion of B, i.e. $\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})$

In this direction,

speed of A before $=((2\mathbf{i}+5\mathbf{j}).\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})=\frac{\sqrt{2}}{2}(2-5)=-3\frac{\sqrt{2}}{2}$

speed of A after $=(3\mathbf{i}+4\mathbf{j}).\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})=\frac{\sqrt{2}}{2}(3-4)=-\frac{\sqrt{2}}{2}$

speed of B before $=2\sqrt{2}$

speed of B after $=0$

Therefore the impact law gives $\dfrac{\frac{\sqrt{2}}{2}}{3\frac{\sqrt{2}}{2}+2\sqrt{2}}=e=\frac{1}{7}$

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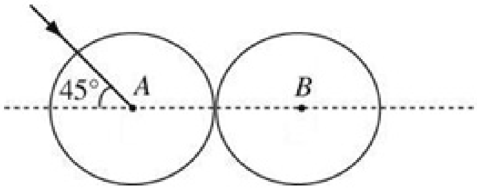
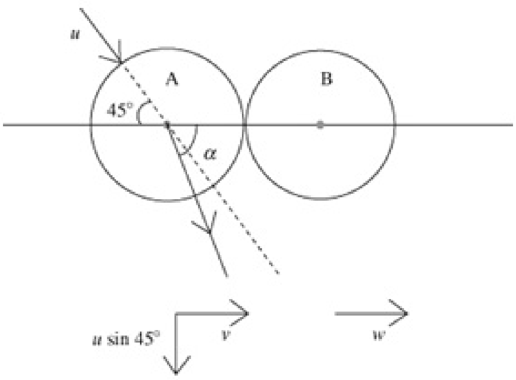
Elastic collisions in two dimensions
Exercise B, Question 15

Question:

A smooth uniform sphere A, moving on a smooth horizontal table, collides with an identical sphere B which is at rest on the table. When the spheres collide the line joining their centres makes an angle of $45°$ with the direction of motion of A, as shown in the diagram. The coefficient of restitution between the spheres is e. The direction of motion of A is deflected through an angle θ by the collision. Show that $\tan\theta = \frac{1+e}{3-e}$.



Solution:



Parallel to the line of centres, using conservation of momentum and the law of restitution gives $mu\cos 45° = mv + mw$ and $w - v = eu\cos 45°$

By subtracting

$$2v = u\cos 45°(1-e)$$

$$v = \frac{u\sqrt{2}(1-e)}{4}$$

$$\Rightarrow \tan\alpha = \frac{u\sin 45°}{\left(\frac{u\sqrt{2}(1-e)}{4}\right)} = \frac{2}{1-e}$$

$$\theta = \alpha - 45° \Rightarrow \tan\theta = \frac{\tan\alpha - \tan 45°}{1+\tan\alpha\tan 45°} = \frac{\frac{2}{1-e}-1}{1+\frac{2}{1-e}} = \frac{2-1+e}{1-e+2} = \frac{1+e}{3-e}$$

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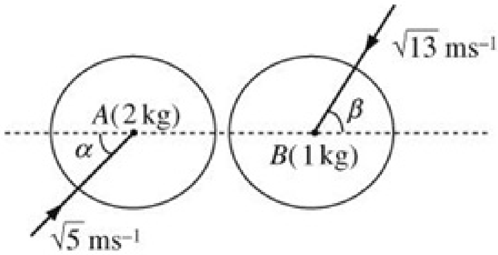
Elastic collisions in two dimensions
Exercise B, Question 16

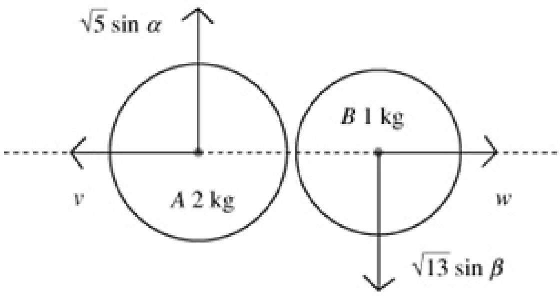
Question:



Two smooth uniform spheres A and B of equal radius have masses 2 kg and 1 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is $\sqrt{5}$ m s^{-1} and the speed of B is $\sqrt{13}$ m s^{-1}. When they collide the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B, where $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{3}{2}$, as shown in the diagram above. The coefficient of restitution between A and B is $\frac{1}{2}$.

Find the speed of each sphere after the collision.

Solution:



Before collision, components of velocity of A are 1 m s^{-1} perpendicular to the lines of centres and 2 m s^{-1} parallel to the line. The components of the velocity of B are 3 m s^{-1} perpendicular to the line, and 2 m s^{-1} parallel to it.

conservation of momentum: $2\times2-1\times2=1\times w-2\times v, 2=w-2v$

law of restitution: $w+v=e(2+2), w+v=4e=2$

Solving the simultaneous equations $\Rightarrow w=2, v=0$

$\Rightarrow$ speed of A is 1 m s^{-1} and speed of B is $\sqrt{3^2+2^2}=\sqrt{13}\text{ m s}^{-1}$

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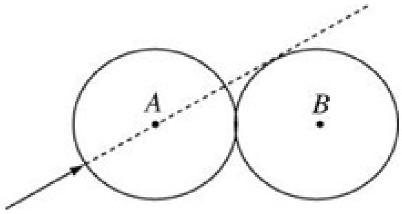
Elastic collisions in two dimensions
Exercise B, Question 17

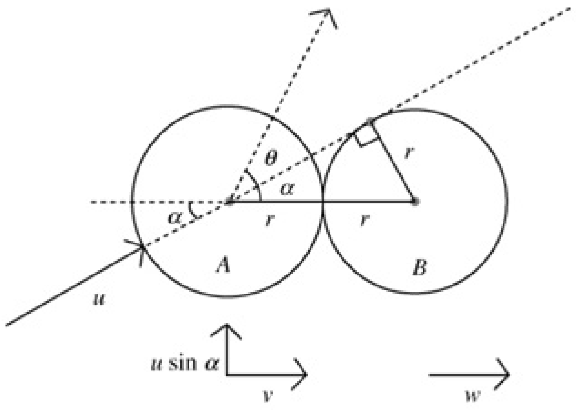
Question:



A smooth uniform sphere B is at rest on a smooth horizontal plane, when it is struck by an identical sphere A moving on the plane. Immediately before the impact, the line of motion of the centre of A is tangential to the sphere B, as shown in the diagram above. The coefficient of restitution between the spheres is $\frac{1}{2}$. The direction of motion of A is turned through an angle θ by the impact.

Show that $\tan\theta = \frac{3\sqrt{3}}{7}$.

Solution:



Tangent perpendicular to radius $\Rightarrow \sin\alpha = \frac{1}{2}$

Initial components of velocity of A are $u\cos\alpha$ parallel to the line of centres, and $u\sin\alpha$ perpendicular to the line of centres.

momentum $\Rightarrow mu\cos\alpha = mv + mw$

$$u\cos\alpha = v + w$$

impact $\Rightarrow w - v = eu\cos\alpha$

Subtracting gives

$$2v = u\cos\alpha - eu\cos\alpha \qquad v = \frac{u\cos\alpha\left(1-\frac{1}{2}\right)}{2} = \frac{u\times\frac{\sqrt{3}}{2}\times\frac{1}{2}}{2} = \frac{u\sqrt{3}}{8}$$

$$\Rightarrow \tan(\theta+\alpha) = \frac{u\sin\alpha}{v} = \frac{\left(\frac{u}{2}\right)}{\left(\frac{u\sqrt{3}}{8}\right)} = \frac{4}{\sqrt{3}}$$

$$\tan\theta = \frac{\tan(\theta+\alpha)-\tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha} = \frac{\frac{4}{\sqrt{3}}-\frac{1}{\sqrt{3}}}{1+\frac{4}{\sqrt{3}}\times\frac{1}{\sqrt{3}}} = \frac{\left(\frac{3}{\sqrt{3}}\right)}{\left(\frac{3+4}{3}\right)} = \frac{3\sqrt{3}}{7}$$

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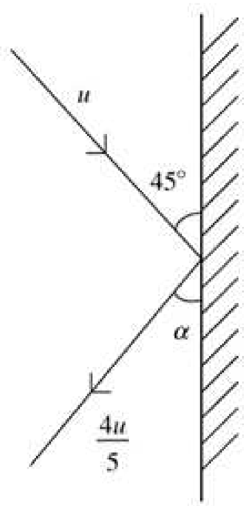
Elastic collisions in two dimensions
Exercise C, Question 1

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $45°$ with the wall. Immediately after the collision the speed of S is $\frac{4}{5}u$. Find the coefficient of restitution between S and the wall.

Solution:



R $\uparrow$: $\frac{4u}{5}\cos\alpha = u\cos 45°$

law of restitution $\leftrightarrow$: $\frac{4u}{5}\sin\alpha = eu\sin 45°$

squaring and adding:

$$\frac{16u^2}{25} = u^2\left(\frac{1}{2}+\frac{e^2}{2}\right)$$

$$\frac{32}{25} = 1+e^2$$

$$\frac{7}{25} = e^2, e = \frac{\sqrt{7}}{5}$$

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Elastic collisions in two dimensions
Exercise C, Question 2

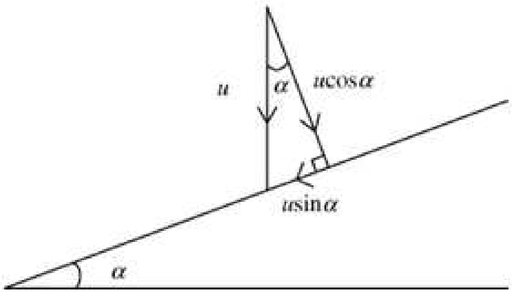
Question:

A small smooth ball of mass $\frac{1}{2}$ kg is falling vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{5}{12}$. Immediately before striking the plane the ball has speed 5.2 m s^{-1}. The coefficient of restitution between ball and plane is $\frac{1}{4}$. Find

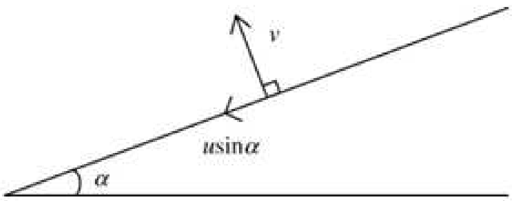
a the speed, to 3 significant figures, of the ball immediately after the impact,
b the magnitude of the impulse received by the ball as it strikes the plane.

Solution:

Before the collision:



After the collision:



a Considering the component of velocity parallel to the plane:

$u\sin\alpha = 5.2 \times \frac{5}{13} = 2$

Perpendicular to the plane:

$v = eu\cos\alpha = \frac{1}{4} \times 5.2 \times \frac{12}{13} = 1.2$

$\text{speed} = \sqrt{2^2 + 1.2^2} = \sqrt{5.44} = 2.33\text{ m s}^{-1}$

b $\text{Impulse} = \frac{1}{2}(1.2 - (-4.8)) = 3\text{ Ns}$

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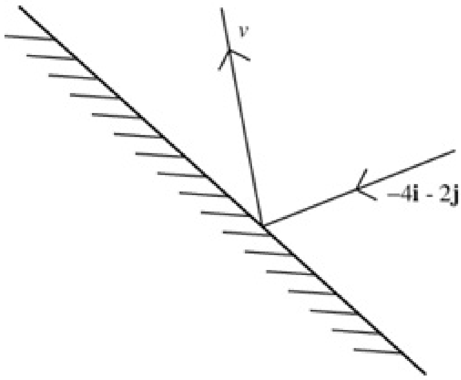
Elastic collisions in two dimensions
Exercise C, Question 3

Question:

A small smooth ball of mass 500 g is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the line $x + y = 3$. The velocity of the ball just before impact is $(-4\mathbf{i} - 2\mathbf{j})\text{ms}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{2}$. Find

a the velocity of the ball immediately after the impact,
b the kinetic energy lost as a result of the impact.

Solution:



a Suppose that $\mathbf{v} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a}$ is parallel to the wall and $\mathbf{b}$ is perpendicular to the wall.

$\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$ is a unit vector parallel to the wall and $\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$ is a unit vector perpendicular to the wall.

$$\nwarrow \mathbf{a} = \left[(-4\mathbf{i}-2\mathbf{j}).\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})\right]\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$$

$$= \frac{1}{\sqrt{2}}(4-2)\times\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j}) = (-\mathbf{i}+\mathbf{j})$$

$$\nearrow \mathbf{b} = -\frac{1}{2}[(-4\mathbf{i}-2\mathbf{j}).\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})]\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$$

$$= -\frac{1}{2}\times\frac{1}{\sqrt{2}}(-4-2)\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$$

$$= \frac{3}{2}(\mathbf{i}+\mathbf{j})$$

So $\mathbf{v} = (-\mathbf{i}+\mathbf{j}) + \frac{3}{2}(\mathbf{i}+\mathbf{j}) = \frac{1}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}$

b K.E. before impact $= \frac{1}{2}\times\frac{1}{2}\times(4^2+2^2) = 5$

K.E. after impact $= \frac{1}{2}\times\frac{1}{2}\times\left(\left(\frac{1}{2}\right)^2+\left(\frac{5}{2}\right)^2\right) = \frac{1}{4}\times\frac{26}{4} = \frac{13}{8}$

K.E. lost $= 5 - \frac{13}{8} = 3.375$ J

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Elastic collisions in two dimensions
Exercise C, Question 4

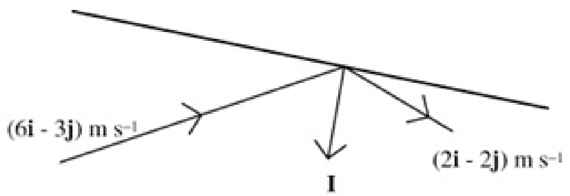
Question:

A small smooth sphere of mass m is moving with velocity $(6\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(2\mathbf{i}-2\mathbf{j})\text{m s}^{-1}$. Find

a the magnitude and direction of the impulse received by the sphere,

b the coefficient of restitution between the sphere and the wall.

Solution:



a $\mathbf{I} = m\mathbf{v} - m\mathbf{u}$

$= m\{(2\mathbf{i}-2\mathbf{j})-(6\mathbf{i}+3\mathbf{j})\}$

$= m(-4\mathbf{i}-5\mathbf{j})$

The impulse has magnitude $m\sqrt{16+25} = m\sqrt{41}$ Ns in the direction parallel to the unit vector $\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$.

b Component of $(6\mathbf{i}+3\mathbf{j})$ parallel to the impulse

$=[(6\mathbf{i}+3\mathbf{j})\cdot\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})]\times\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$

$=\frac{1}{\sqrt{41}}(-24-15)\times\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$

Component of $(2\mathbf{i}-2\mathbf{j})$ parallel to the impulse

$=[(2\mathbf{i}-2\mathbf{j})\cdot\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})]\times\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$

$=\frac{1}{\sqrt{41}}(-8+10)\times\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$

law of restitution

$\frac{2}{\sqrt{41}} = e\times\frac{39}{\sqrt{41}}$

$e = \frac{2}{39}$

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Elastic collisions in two dimensions
Exercise C, Question 5

Question:

Two small smooth spheres A and B have equal radii. The mass of A is $4m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(3\mathbf{i}-\mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i}+2\mathbf{j})\text{m s}^{-1}$. Find

a the velocity of B immediately after the collision,

b a unit vector parallel to the line of centres of the spheres at the instant of the collision.

Solution:

a Conservation of momentum $\Rightarrow$

$$4m(2\mathbf{i}+3\mathbf{j})+m(3\mathbf{i}-\mathbf{j})=4m(3\mathbf{i}+2\mathbf{j})+m\mathbf{v}$$

$$\mathbf{v}=\mathbf{i}(4\times2+1\times3-4\times3)+\mathbf{j}(4\times3-1\times1-4\times2)=-\mathbf{i}+3\mathbf{j}$$

b Impulse on $A=4m((3\mathbf{i}+2\mathbf{j})-(2\mathbf{i}+3\mathbf{j}))=4m(\mathbf{i}-\mathbf{j})$

$\Rightarrow \frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})$ is a unit vector parallel to the line of centres.

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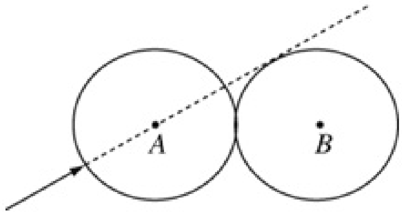
Elastic collisions in two dimensions
Exercise C, Question 6

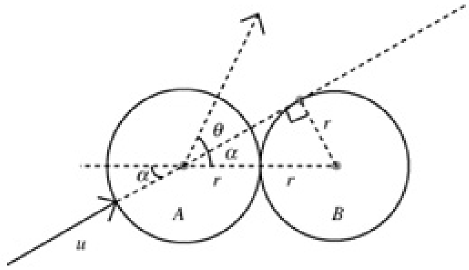
Question:



A smooth uniform sphere B is at rest on a smooth horizontal plane, when it is struck by an identical sphere A moving on the plane. Immediately before the impact, the line of motion of the centre of A is tangential to the sphere B, as shown in the diagram above. The coefficient of restitution between the spheres is $\frac{2}{3}$. The direction of motion of A is turned through an angle θ by the impact.

Show that $\theta = \tan^{-1}\frac{5\sqrt{3}}{9}$.

Solution:



Tangent perpendicular to radius

$\Rightarrow \sin\alpha = \frac{1}{2}$

Initial components of velocity of A are
$u\cos\alpha$ parallel to the line of centres, and
$u\sin\alpha$ perpendicular to the line of centres.

Momentum $\Rightarrow mu\cos\alpha = mv + mw$

$u\cos\alpha = v + w$

Impact $\Rightarrow w - v = eu\cos\alpha$

where v is the velocity of A along the line of centres and w the velocity of B along the line of centres immediately after the collision.

Subtracting gives $2v = u\cos\alpha - eu\cos\alpha,\ v = \dfrac{u\cos\alpha\left(1-\frac{2}{3}\right)}{2} = \dfrac{u\times\frac{\sqrt{3}}{2}\times\frac{1}{3}}{2} = \dfrac{u\sqrt{3}}{12}$

$$\Rightarrow \tan(\theta+\alpha) = \frac{u\sin\alpha}{v} = \frac{\left(\frac{u}{2}\right)}{\left(\frac{u\sqrt{3}}{12}\right)} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\tan\theta = \frac{\tan(\theta+\alpha)-\tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha} = \frac{2\sqrt{3}-\frac{1}{\sqrt{3}}}{1+2\sqrt{3}\times\frac{1}{\sqrt{3}}} = \frac{\left(\frac{6-1}{\sqrt{3}}\right)}{(1+2)} = \frac{5}{3\sqrt{3}} = \frac{5\sqrt{3}}{9}$$

$$\theta = \tan^{-1}\frac{5\sqrt{3}}{9}$$

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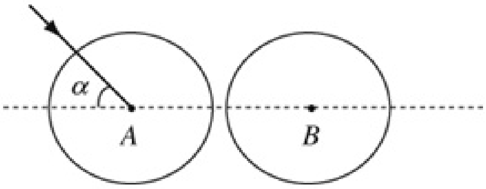
Elastic collisions in two dimensions
Exercise C, Question 7

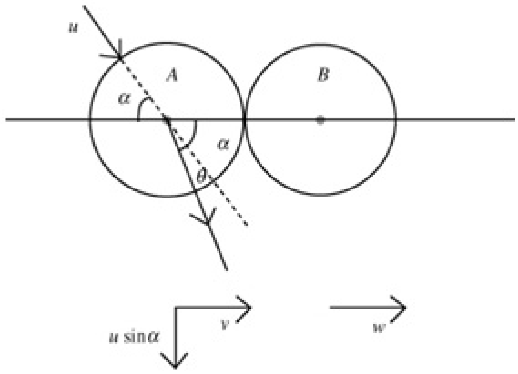
Question:



A smooth uniform sphere A, moving on a smooth horizontal table, collides with a second identical sphere B which is at rest on the table. When the spheres collide the line joining their centres makes an angle of α with the direction of motion of A, as shown in the diagram above. The direction of motion of A is deflected through an angle θ by the collision. Given that $\alpha = \tan^{-1}\frac{3}{4}$ and that the coefficient of restitution between the spheres is e,

show that $\tan\theta = \frac{6+6e}{17-8e}$.

Solution:



Parallel to the line of centres, using conservation of momentum and the impact law gives

$mu\cos\alpha = mv + mw$

and $w - v = eu\cos\alpha$

By subtracting,

$$2v = u\cos\alpha \times (1-e)$$

$$v = \frac{4u(1-e)}{10} = \frac{2u(1-e)}{5}$$

$$\Rightarrow \tan(\theta+\alpha) = \frac{u\sin\alpha}{\left(\frac{2u(1-e)}{5}\right)}$$

$$= \frac{3}{2(1-e)}$$

$$\tan\theta = \frac{\tan(\theta+\alpha) - \tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha} = \frac{\frac{3}{2(1-e)} - \frac{3}{4}}{1+\frac{3}{2(1-e)}\times\frac{3}{4}} = \frac{12-6(1-e)}{8(1-e)+9} = \frac{6+6e}{17-8e}$$

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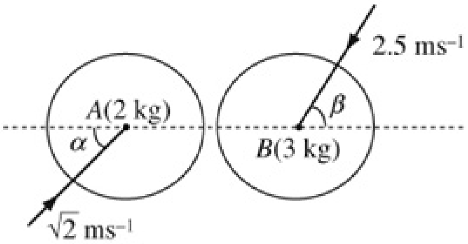
Elastic collisions in two dimensions
Exercise C, Question 8

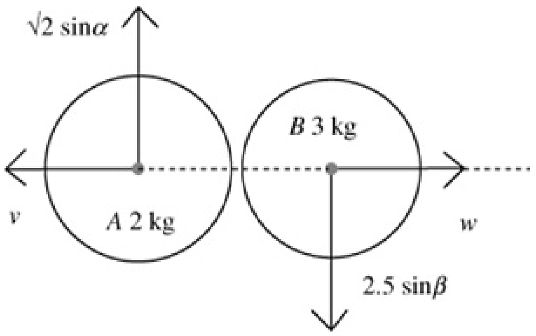
Question:



Two smooth uniform spheres A and B of equal radius have masses 2 kg and 3 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is $\sqrt{2}$ m s^{-1} and the speed of B is 2.5 m s^{-1}. When they collide the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B, where $\tan\alpha = 1$ and $\tan\beta = \frac{3}{4}$ as shown in the diagram. The coefficient of restitution between A and B is $\frac{2}{3}$.

Find the speed of each sphere after the collision.

Solution:



Before the collision, the components of the velocity of A are 1 m s^{-1} perpendicular to the line of centres and 1 m s^{-1} parallel to the line.

The components of the velocity of B are 1.5 m s^{-1} perpendicular to the line, and 2 m s^{-1} parallel to it.

Conservation of momentum: $2\times1-3\times2=3\times w-2\times v, -4=3w-2v$

Law of restitution: $w+v=e(1+2), w+v=3e=2$

Solving the simultaneous equations $-4=3w-2v$ and $4=2w+2v$

$\Rightarrow w=0, v=2$

$\Rightarrow$ speed of A is $\sqrt{1^2+2^2}=\sqrt{5}$ m s^{-1} and speed of B is 1.5 m s^{-1}

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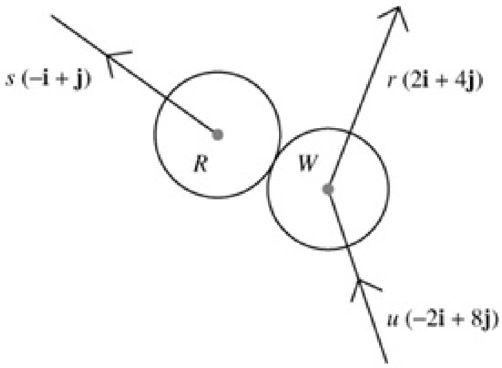
Elastic collisions in two dimensions
Exercise C, Question 9

Question:

A red ball is stationary on a rectangular billiard table $OABC$. It is then struck by a white ball of equal mass and equal radius moving with velocity $u(-2\mathbf{i}+8\mathbf{j})$ where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors parallel to OA and OC respectively. After the impact the velocity of the red ball is parallel to the vector $(-\mathbf{i}+\mathbf{j})$ and the velocity of the white ball is parallel to the vector $(2\mathbf{i}+4\mathbf{j})$. Prove that the coefficient of restitution between the two balls is $\frac{3}{5}$.

Solution:



Conservation of momentum:

$u(-2\mathbf{i}+8\mathbf{j}) = s(-\mathbf{i}+\mathbf{j}) + r(2\mathbf{i}+4\mathbf{j})$

$\Rightarrow -2u = -s + 2r$ and $8u = s + 4r$

Adding $\Rightarrow 6u = 6r, r = u, s = 4u$

Line of centres is parallel to $-\mathbf{i}+\mathbf{j}$ (as this is the direction of the impulse on the red ball).

In the direction of the line of centres

component of $(-2\mathbf{i}+8\mathbf{j})$ is $\dfrac{(-2\mathbf{i}+8\mathbf{j})\cdot(-\mathbf{i}+\mathbf{j})}{|(-\mathbf{i}+\mathbf{j})|} = \dfrac{10\sqrt{2}}{2} = 5\sqrt{2}$

component of $(2\mathbf{i}+4\mathbf{j})$ is $\dfrac{2\sqrt{2}}{2} = \sqrt{2}$ and component of $(-\mathbf{i}+\mathbf{j})$ is $\sqrt{2}$

so using law of restitution:

$4u\sqrt{2} - u\sqrt{2} = e \times 5u\sqrt{2}, 3\sqrt{2} = 5\sqrt{2}e$

$e = \dfrac{3}{5}$

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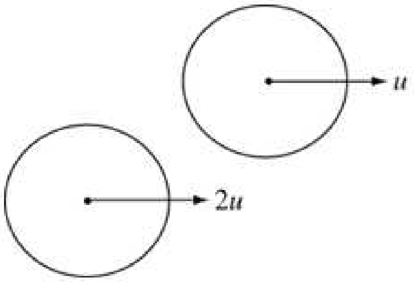
Elastic collisions in two dimensions
Exercise C, Question 10

Question:

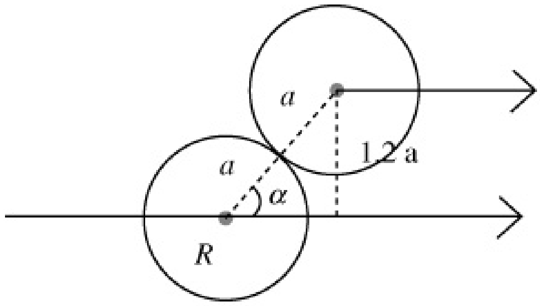


Two uniform spheres, each of mass m and radius a, collide when moving on a horizontal plane. Before the impact the spheres are moving with speeds $2u$ and u, as shown in the diagram.

The centres of the spheres are moving on parallel paths distance $\frac{6a}{5}$ apart.

The coefficient of restitution between the spheres is $\frac{3}{4}$. Find the speeds of the spheres just after the impact, and show that the angle between their paths is then equal to $\tan^{-1}\frac{14}{23}$.

Solution:



Before impact the balls are moving at angle α to the line of centres.

$\alpha = \sin^{-1}\frac{1.2}{2} = \sin^{-1}\frac{3}{5}$

momentum:

$2u \times \frac{4}{5} + u \times \frac{4}{5} = v + w = \frac{12u}{5}$

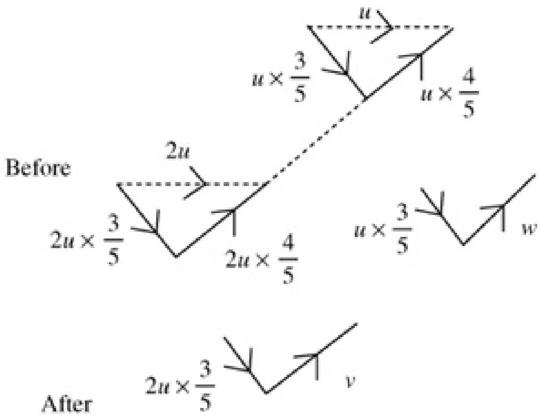
law of restitution:

$w - v = \frac{3}{4}\left(\frac{8u}{5} - \frac{4u}{5}\right) = \frac{3u}{5}$

Adding:

$2w = \frac{15u}{5} = 3u, w = \frac{3u}{2}$

$\Rightarrow v = \frac{12u}{5} - \frac{3u}{2} = \frac{9u}{10}$



Speeds are $u\sqrt{\frac{81}{100} + \frac{36}{25}} = u\sqrt{\frac{225}{100}} = \frac{3u}{2}$

and $u\sqrt{\frac{9}{4} + \frac{9}{25}} = u\sqrt{\frac{9 \times 29}{100}} = \frac{3\sqrt{29}}{10}u$

Directions relative to the line of centres are $\tan^{-1}\left(\frac{\frac{6}{5}}{\frac{9}{10}}\right) = \tan^{-1}\frac{4}{3}$ and

$\tan^{-1}\left(\frac{\frac{3}{5}}{\frac{3}{2}}\right) = \tan^{-1}\frac{2}{5}$, so the angle between the paths is

$\tan^{-1}\left(\frac{\frac{4}{3} - \frac{2}{5}}{1 + \frac{4}{3} \times \frac{2}{5}}\right) = \tan^{-1}\left(\frac{20-6}{15+8}\right) = \tan^{-1}\frac{14}{23}$

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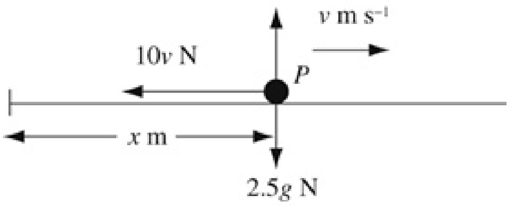
Resisted motion of a particle movig in a straight line
Exercise A, Question 1

Question:

A particle P of mass 2.5 kg moves in a straight horizontal line. When the speed of P is v m s^{-1}, the resultant force acting on P is a resistance of magnitude $10v$ N. Find the time P takes to slow down from 24 m s^{-1} to 6 m s^{-1}.

Solution:



R(→) $\mathbf{F} = m\mathbf{a}$

$$-10v = 2.5\frac{dv}{dt}$$

Separating the variables

$$\int 4\, dt = -\int \frac{1}{v}\, dv$$

$$4t = A - \ln v$$

When $t = 0,\ v = 24$

$0 = A - \ln 24 \Rightarrow A = \ln 24$

Hence

$4t = \ln 24 - \ln v$

$$t = \frac{1}{4}\ln\left(\frac{24}{v}\right)$$

When $v = 6$

$$t = \frac{1}{4}\ln 4\ (\approx 0.347)$$

P takes $\frac{1}{4}\ln 4$ s(= 0.347 s, 3 d.p.) to slow from 24 m s^{-1} to 6 m s^{-1}.

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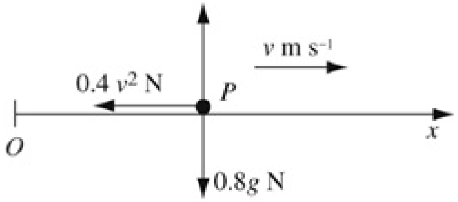
Resisted motion of a particle movig in a straight line
Exercise A, Question 2

Question:

A particle P of mass 0.8 kg is moving along the axis Ox in the direction of x-increasing. When the speed of P is v m s^{-1}, the resultant force acting on P is a resistance of magnitude $0.4v^2$ N. Initially P is at O and is moving with speed 12 m s^{-1}. Find the distance P moves before its speed is halved.

Solution:



R(→) $\quad \mathbf{F} = m\mathbf{a}$

$$-0.4v^2 = 0.8v\frac{dv}{dx}$$

Separating the variables

$$\int 1\,dx = -2\int \frac{1}{v}\,dv$$

$$x = A - 2\ln v$$

At $x = 0,\ v = 12$

$0 = A - 2\ln 12 \Rightarrow A = 2\ln 12$

Hence

$$x = 2\ln 12 - 2\ln v = 2\ln\left(\frac{12}{v}\right)$$

When $v = 6$

$x = 2\ln 2$

The distance P moves before its speed is halved is $2\ln 2$ m $= 1.39$ m (3 s.f.).

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Resisted motion of a particle movig in a straight line
Exercise A, Question 3

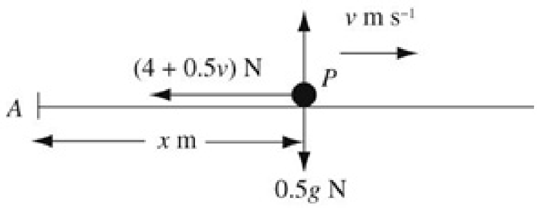
Question:

A particle P of mass 0.5 kg moves in a straight horizontal line against a resistance of magnitude $(4+0.5v)$N, where v m s^{-1} is the speed of P at time t seconds. When $t=0, P$ is at a point A moving with speed 12 m s^{-1}. The particle P comes to rest at the point B. Find

a the time P takes to move from A to B,
b the distance AB.

Solution:

a



R(→) $\mathbf{F} = m\mathbf{a}$

$$-(4+0.5v) = 0.5\frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt = -\int \frac{1}{8+v}\,dv$$

$$t = A - \ln(8+v)$$

When $t=0,\ v=12$

$$0 = A - \ln 20 \Rightarrow A = \ln 20$$

Hence

$$t = \ln 20 - \ln(8+v) = \ln\left(\frac{20}{8+v}\right)$$

When $v=0$

$$t = \ln\left(\frac{20}{8}\right) = \ln 2.5$$

The time taken for P to move from A to B is $\ln 2.5\text{ s} = 0.916\text{ s}$ (3 d.p.).

b R(→) $\mathbf{F} = m\mathbf{a}$

$$-(4+0.5v) = 0.5v\frac{dv}{dx}$$

Separating the variables

$$\int 1\,dx = -\int \frac{v}{8+v}\,dv$$

$$\frac{v}{8+v} = \frac{8+v-8}{8+v} = 1 - \frac{8}{8+v}$$

Hence

$$\int 1\,dx = -\int\left(1 - \frac{8}{8+v}\right)dv$$

$$x = A - v + 8\ln(8+v)$$

At $x=0, v=12$

$$0 = A - 12 + 8\ln 20 \Rightarrow A = 12 - 8\ln 20$$

Hence $x = 12 - v - (8\ln 20 - 8\ln(8+v))$

$$= 12 - v - 8\ln\left(\frac{20}{8+v}\right)$$

When $v=0$

$$x = 12 - 8\ln 2.5$$

$$AB = (12 - 8\ln 2.5)\text{m} = 4.67\text{ m} \quad (3\text{ s.f.})$$

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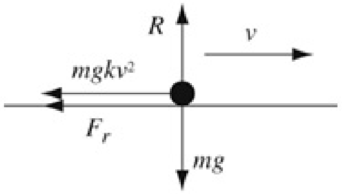
Resisted motion of a particle movig in a straight line
Exercise A, Question 4

Question:

A particle of mass m is projected along a rough horizontal plane with velocity u m s^{-1}. The coefficient of friction between the particle and the plane is μ. When the particle is moving with speed v m s^{-1}, it is also subject to an air resistance of magnitude $kmgv^2$, where k is a constant. Find the distance the particle moves before coming to rest.

Solution:



R(↑) $R = mg$

As friction is limiting

$$F_r = \mu R = \mu mg$$

R(→) $\mathbf{F} = m\mathbf{a}$

$$-F_r - kmgv^2 = ma$$

$$-\mu \not{m} g - k \not{m} g v^2 = \not{m} v \frac{dv}{dx}$$

Separating the variables

$$\int g \, dx = -\int \frac{v}{\mu + kv^2} \, dv$$

$$gx = A - \frac{1}{2k} \ln(\mu + kv^2)$$

At $x = 0, v = u$

$$0 = A - \frac{1}{2k} \ln(\mu + ku^2) \Rightarrow A = \frac{1}{2k} \ln(\mu + ku^2)$$

Hence

$$x = \frac{1}{2kg} (\ln(\mu + ku^2) - \ln(\mu + kv^2)) = \frac{1}{2kg} \ln\left(\frac{\mu + ku^2}{\mu + kv^2}\right)$$

When $v = 0$

$$x = \frac{1}{2kg} \ln\left(\frac{\mu + ku^2}{\mu}\right)$$

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Resisted motion of a particle movig in a straight line
Exercise A, Question 5

Question:

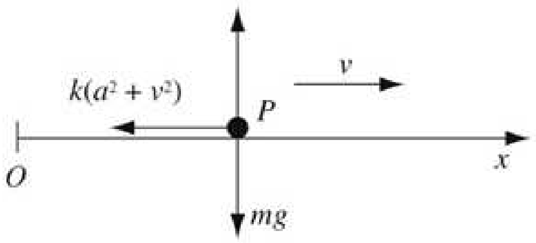
A particle P of mass m is moving along the axis Ox in the direction of x-increasing. At time t seconds, the velocity of P is v. The only force acting on P is a resistance of magnitude $k(a^2+v^2)$. At time $t=0$, P is at O and its speed is U. At time $t=T, v=\frac{1}{2}U$

a Show that $T=\frac{m}{ak}\left[\arctan\left(\frac{U}{a}\right)-\arctan\left(\frac{U}{2a}\right)\right]$.

b Find the distance travelled by P as its speed is reduced from U to $\frac{1}{2}U$.

Solution:

a



R($\rightarrow$) $\quad$ $\mathbf{F} = m\mathbf{a}$

$$-k(a^2 + v^2) = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

Separating the variables

$$\int 1\,\mathrm{d}t = -\frac{m}{k}\int \frac{1}{a^2 + v^2}\,\mathrm{d}v$$

$$t = A - \frac{m}{ak}\arctan\left(\frac{v}{a}\right)$$

When $t = 0, v = U$

$$0 = A - \frac{m}{ak}\arctan\left(\frac{U}{a}\right) \Rightarrow A = \frac{m}{ak}\arctan\left(\frac{U}{a}\right)$$

Hence

$$t = \frac{m}{ak}\arctan\left(\frac{U}{a}\right) - \frac{m}{ak}\arctan\left(\frac{v}{a}\right)$$

When $t = T, v = \frac{1}{2}U$

$$T = \frac{m}{ak}\arctan\left(\frac{U}{a}\right) - \frac{m}{ak}\arctan\left(\frac{\frac{1}{2}U}{a}\right)$$

$$T = \frac{m}{ak}\left[\arctan\left(\frac{U}{a}\right) - \arctan\left(\frac{U}{2a}\right)\right], \quad \text{as required}$$

b $\quad$ R($\rightarrow$) $\quad$ $\mathbf{F} = m\mathbf{a}$

$$-k(a^2 + v^2) = mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$\int 1\,\mathrm{d}x = -\frac{m}{k}\int \frac{v}{a^2 + v^2}\,\mathrm{d}v$$

$$x = A - \frac{m}{2k}\ln(a^2 + v^2)$$

When $x = 0, v = U$

$$0 = A - \frac{m}{2k}\ln(a^2 + U^2) \Rightarrow A = \frac{m}{2k}\ln(a^2 + U^2)$$

Hence

$$x = \frac{m}{2k}\ln(a^2 + U^2) - \frac{m}{2k}\ln(a^2 + v^2) = \frac{m}{2k}\ln\left(\frac{a^2 + U^2}{a^2 + v^2}\right)$$

When $v = \frac{1}{2}U$

$$x = \frac{m}{2k}\ln\left(\frac{a^2 + U^2}{a^2 + \frac{1}{4}U^2}\right) = \frac{m}{2k}\ln\left(\frac{4a^2 + 4U^2}{4a^2 + U^2}\right)$$

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Resisted motion of a particle movig in a straight line
Exercise A, Question 6

Question:

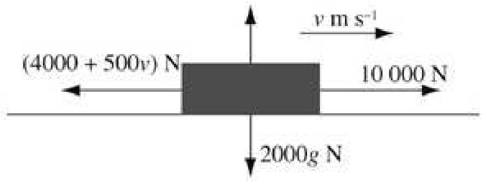
A lorry of mass 2000 kg travels along a straight horizontal road. The engine of the lorry produces a constant driving force of magnitude 10 000 N. At time t seconds, the speed of the lorry is v m s^{-1}. As the lorry moves, the total resistance to the motion of the lorry is of magnitude $(4000+500v)$N. The lorry starts from rest. Find

a v in terms of t,

b the terminal speed of the lorry.

Solution:

a



R(→) $\mathbf{F} = m\mathbf{a}$

$$10\,000-(4000+500v) = 2000a$$

$$6000-500v = 2000\frac{dv}{dt}$$

Dividing throughout by 500

$$12-v = 4\frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt = 4\int \frac{1}{12-v}\,dv$$

$$t = A-4\ln(12-v)$$

$$\ln(12-v) = B-\frac{t}{4}, \text{ where } B=\frac{1}{4}A$$

$$12-v = e^{B-\frac{t}{4}} = e^{B}e^{-\frac{t}{4}} = Ce^{-\frac{t}{4}}, \text{ where } C=e^{B}$$

Hence

$$v = 12-Ce^{-\frac{t}{4}}$$

When $t=0, v=0$

$$0=12-C \Rightarrow C=12$$

Hence

$$v=12\left(1-e^{-\frac{t}{4}}\right)$$

b As $t\to\infty, e^{-\frac{t}{4}}\to 0$ and $v\to 12$

The terminal speed of the lorry is 12 m s^{-1}.

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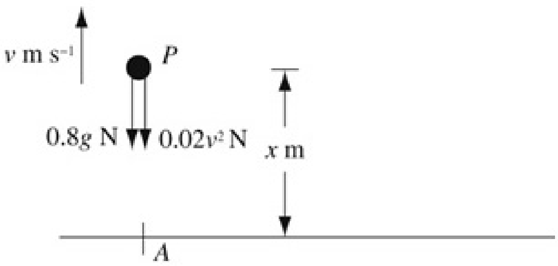
Resisted motion of a particle movig in a straight line
Exercise B, Question 1

Question:

A particle P of mass 0.8 kg is projected vertically upwards with velocity 30 m s^{-1} from a point A on horizontal ground. Air resistance is modelled as a force of magnitude $0.02v^2\text{N}$, where $v\text{ m s}^{-1}$ is the velocity of P.
Find the greatest height above A attained by P.

Solution:



$\text{R}(\uparrow)\quad \mathbf{F} = m\mathbf{a}$

$-0.8g - 0.02v^2 = 0.8a$

$-7.84 - 0.02v^2 = 0.8v\dfrac{dv}{dx}$

Separating the variables

$\int 1\,dx = -0.8\int \dfrac{v}{7.84 + 0.02v^2}\,dv$

$x = A - \dfrac{0.8}{0.04}\ln(7.84 + 0.02v^2)$

At $x = 0,\ v = 30$

$0 = A - 20\ln(7.84 + 18) \Rightarrow A = 20\ln 25.84$

Hence

$x = 20\ln 25.84 - 20\ln(7.84 + 0.02v^2)$

$= 20\ln\left(\dfrac{25.84}{7.84 + 0.02v^2}\right)$

At the greatest height, $v = 0$

$x = 20\ln\left(\dfrac{25.84}{7.84}\right) \approx 23.9$

The greatest height above A attained by P is 23.9 m (3 s.f).

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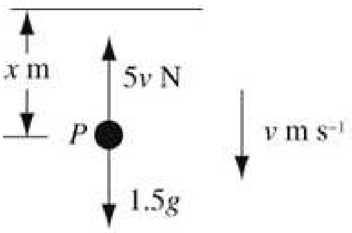
Resisted motion of a particle movig in a straight line
Exercise B, Question 2

Question:

A particle P of mass 1.5 kg is released from rest at time $t = 0$ and falls vertically through a liquid. The liquid resists the motion of P with a force of magnitude $5v$ N where v m s^{-1} is the speed of P at time t seconds.

Find the value of t when the speed of P is 2 m s^{-1}.

Solution:



R(↓) $\mathbf{F} = m\mathbf{a}$

$1.5g - 5v = 1.5a$

$14.7 - 5v = 1.5\frac{dv}{dt}$

Separating the variables

$\int 1\,dt = 1.5\int \frac{1}{14.7-5v}\,dv$

$t = A - \frac{1.5}{5}\ln(14.7-5v)$

When $t = 0$, $v = 0$

$0 = A - 0.3\ln 14.7 \Rightarrow A = 0.3\ln 14.7$

Hence

$t = 0.3\ln 14.7 - 0.3\ln(14.7-5v)$

$= 0.3\ln\left(\frac{14.7}{14.7-5v}\right)$

When $v = 2$

$t = 0.3\ln\left(\frac{14.7}{14.7-10}\right) = 0.342$ s (3 s.f.)

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Resisted motion of a particle movig in a straight line
Exercise B, Question 3

Question:

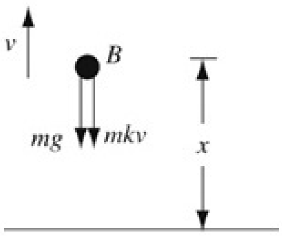
A small ball B of mass m is projected upwards from horizontal ground with speed u. Air resistance is modelled as a force of magnitude mkv, where v m s^{-1} is the velocity of P at time t seconds.

a Show that the greatest height above the ground reached by B is $\frac{u}{k}-\frac{g}{k^2}\ln\left(1+\frac{ku}{g}\right)$.

b Find the time taken to reach this height.

Solution:

a



$$R(\uparrow) \quad \mathbf{F} = m\mathbf{a}$$

$$-mg - mkv = ma$$

$$-g - kv = v\frac{dv}{dx}$$

Separating the variables

$$\int 1\, dx = -\int \frac{v}{g + kv} dv$$

$$= -\frac{1}{k}\int \frac{g + kv - g}{g + kv} dv$$

$$= -\frac{1}{k}\int \left(1 - \frac{g}{g + kv}\right) dv$$

$$x = A - \frac{1}{k}\left[v - \frac{g}{k}\ln(g + kv)\right] = A - \frac{v}{k} + \frac{g}{k^2}\ln(g + kv)$$

At $x = 0, v = u$

$$0 = A - \frac{u}{k} + \frac{g}{k^2}\ln(g + ku) \Rightarrow A = \frac{u}{k} - \frac{g}{k^2}\ln(g + ku)$$

Hence

$$x = \frac{u}{k} - \frac{v}{k} - \left[\frac{g}{k^2}\ln(g + ku) - \frac{g}{k^2}\ln(g + kv)\right]$$

$$= \frac{1}{k}(u - v) - \frac{g}{k^2}\ln\left(\frac{g + ku}{g + kv}\right)$$

At the greatest height, $v = 0$

$$x = \frac{u}{k} - \frac{g}{k^2}\ln\left(\frac{g + ku}{g}\right) = \frac{u}{k} - \frac{g}{k^2}\ln\left(1 + \frac{ku}{g}\right)$$, as required

b $-g - kv = \frac{dv}{dt}$

Separating the variables

$$\int 1\, dt = -\int \frac{1}{g + kv}\, dt$$

$$t = B - \frac{1}{k}\ln(g + kv)$$

When $t = 0, v = u$

$$0 = B - \frac{1}{k}\ln(g + ku) \Rightarrow B = \frac{1}{k}\ln(g + ku)$$

Hence

$$t = \frac{1}{k}\ln(g + ku) - \frac{1}{k}\ln(g + kv) = \frac{1}{k}\ln\left(\frac{g + ku}{g + kv}\right)$$

At the greatest height, $v = 0$

$$t = \frac{1}{k}\ln\left(\frac{g + ku}{g}\right) = \frac{1}{k}\ln\left(1 + \frac{ku}{g}\right)$$

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Resisted motion of a particle movig in a straight line
Exercise B, Question 4

Question:

A parachutist of mass 60 kg falls vertically from rest from a fixed balloon. For the first 3 s of her motion, her fall is resisted by air resistance of magnitude $20v$N where $v\text{ m s}^{-1}$ is her velocity.

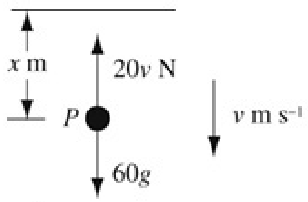
a Find the velocity of the parachutist after 3 s.

After 3 s, her parachute opens and her further motion is resisted by a force of magnitude $(20v+60v^2)$N.

b Find the terminal speed of the parachutist.

Solution:

a



R(↓) $\mathbf{F} = m\mathbf{a}$

$$60g - 20v = 60a$$

$$588 - 20v = 60\frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt = 60\int \frac{1}{588-20v}\,dv$$

$$t = A - \frac{60}{20}\ln(588-20v)$$

When $t = 0, v = 0$

$0 = A - 3\ln 588 \Rightarrow A = 3\ln 588$

Hence

$$t = 3\ln 588 - 3\ln(588-20v) = 3\ln\left(\frac{588}{588-20v}\right)$$

When $t = 3$

$$3 = 3\ln\left(\frac{588}{588-20v}\right) \Rightarrow \ln\left(\frac{588}{588-20v}\right) = 1$$

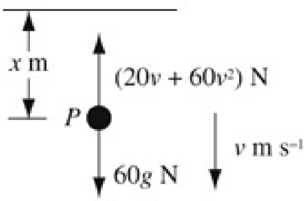
$$\frac{588}{588-20v} = e$$

$$588 - 20v = 588e^{-1}$$

$$v = \frac{588}{20}(1-e^{-1}) \approx 18.6$$

The velocity of the parachutist after 3 s is 18.6 m s^{-1} (3 s.f.)

b



R(↓) $\mathbf{F} = m\mathbf{a}$

$60g - (20v + 60v^2) = 60a$

At the terminal speed $a = 0$

$588 - 20v - 60v^2 = 0$

$60v^2 + 20v - 588 = 0$

Only the positive root need be considered

$$v = \frac{-20 + \sqrt{(20^2 + 4 \times 60 \times 588)}}{120} \approx 2.97$$

The terminal speed of the parachutist is 2.97 m s^{-1} (3 s.f.)

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Resisted motion of a particle movig in a straight line
Exercise B, Question 5

Question:

A particle P of mass m is projected vertically upwards with speed u from a point A on horizontal ground. The particle P is subject to air resistance of magnitude $mgkv$, where v is the speed of P and k is a positive constant.

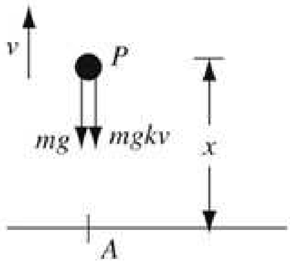
a Find the greatest height above A reached by P.

Assuming P has not reached the ground,

b find an expression for the speed of the particle t seconds after it has reached its greatest height.

Solution:

a



R(↑) $\mathbf{F} = m\mathbf{a}$

$$-mg - mgkv = ma$$

$$-g - gkv = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$-\int g\ \mathrm{d}x = \int \frac{v}{1+kv}\mathrm{d}v$$

$$= \frac{1}{k}\int \frac{1+kv-1}{1+kv}\mathrm{d}v$$

$$= \frac{1}{k}\int \left(1-\frac{1}{1+kv}\right)\mathrm{d}v$$

$$-gx = \frac{1}{k}\left[v-\frac{1}{k}\ln(1+kv)\right]+A$$

At $x=0, v=u$

$$0 = \frac{1}{k}\left[u-\frac{1}{k}\ln(1+ku)\right]+A \Rightarrow A = -\frac{u}{k}+\frac{1}{k^2}\ln(1+ku)$$

Hence

$$-gx = \frac{v}{k}-\frac{1}{k^2}\ln(1+kv)-\frac{u}{k}+\frac{1}{k^2}\ln(1+ku)$$

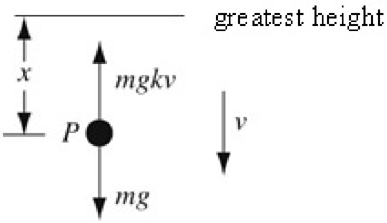
$$x = \frac{1}{gk}(u-v)-\frac{1}{gk^2}\ln\left(\frac{1+ku}{1+kv}\right)$$

At the greatest height $v=0$

$$x = \frac{u}{gk}-\frac{1}{gk^2}\ln(1+ku) = \frac{1}{gk^2}(ku-\ln(1+ku))$$

The greatest height above A reached by P is $\frac{1}{gk^2}(ku-\ln(1+ku))$.

b



R(↓) $\mathbf{F} = m\mathbf{a}$

$mg - mgkv = ma$

$$g(1 - kv) = \frac{dv}{dt}$$

Separating the variables

$$\int g\ dt = \int \frac{1}{1 - kv}\ dv$$

$$gt = A - \frac{1}{k}\ln(1 - kv)$$

When $t = 0, v = 0$

$0 = A - \ln 1 \Rightarrow A = 0$

Hence

$$gt = -\frac{1}{k}\ln(1 - kv)$$

$$-kgt = \ln(1 - kv)$$

$$1 - kv = e^{-kgt}$$

$$v = \frac{1}{k}(1 - e^{-kgt})$$

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Resisted motion of a particle movig in a straight line
Exercise B, Question 6

Question:

A particle of mass m is projected vertically upwards from a point A on horizontal ground with speed u. The particle reaches its greatest height above the ground at the point B.

a Ignoring air resistance, find the distance AB.

Instead of ignoring air resistance, it is modelled as a resisting force of magnitude mkv^2, where v m s^{-1} is the velocity of the particle and k is a positive constant. Using this model find

b the distance AB,

c the work done by air resistance against the motion of the particle as it moves from A to B.

Solution:

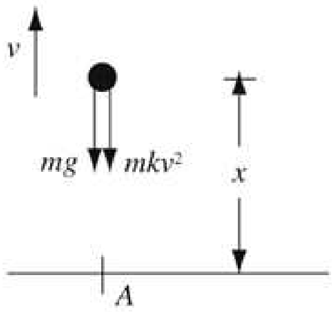
a $v^2 = u^2 + 2as$

At the greatest height, $v = 0$

$$0 = u^2 - 2g \times AB$$

$$AB = \frac{u^2}{2g}$$

b



$$\text{R}(\uparrow) \quad \mathbf{F} = m\mathbf{a}$$

$$-mg - mkv^2 = ma$$

$$-g - kv^2 = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\int 1\,\mathrm{d}x = -\int \frac{v}{g + kv^2}\,\mathrm{d}v$$

$$x = A - \frac{1}{2k}\ln(g + kv^2)$$

At $x = 0, v = u$

$$0 = A - \frac{1}{2k}\ln(g + ku^2) \Rightarrow A = \frac{1}{2k}\ln(g + ku^2)$$

Hence

$$x = \frac{1}{2k}\ln(g + ku^2) - \frac{1}{2k}\ln(g + kv^2) = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g + kv^2}\right)$$

At the greatest height $v = 0$ and $x = AB$

$$AB = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right) = \frac{1}{2k}\ln\left(1 + \frac{ku^2}{g}\right)$$

c The work done by air resistance is the difference between the potential energies of the particle at the greatest heights in parts **a** and **b** and is given by

$$mg \times \frac{u^2}{2g} - mg \times \frac{1}{2k}\ln\left(1 + \frac{ku^2}{g}\right)$$

$$= mg\left(\frac{u^2}{2g} - \frac{1}{2k}\ln\left(1 + \frac{ku^2}{g}\right)\right)$$

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Resisted motion of a particle movig in a straight line
Exercise B, Question 7

Question:

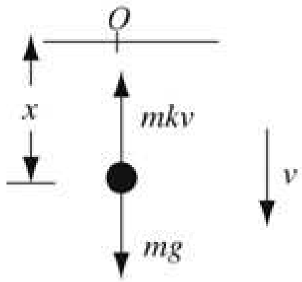
A particle P of mass m is projected vertically downwards from a fixed point O with speed $\frac{g}{2k}$, where k is a constant. At time t seconds after projection, the displacement of P from O is x and the velocity of P is v. The particle P is subject to a resistance of magnitude mkv.

a Show that $v = \frac{g}{2k}(2 - e^{-kt})$.

b Find x when $t = \frac{\ln 4}{k}$.

Solution:

a



$$R(\downarrow) \quad \mathbf{F} = m\mathbf{a}$$

$$mg - mkv = ma$$

$$g - kv = \frac{dv}{dt}$$

Separating the variables

$$\int 1\, dt = \int \frac{1}{g - kv}\, dv$$

$$t = A - \frac{1}{k}\ln(g - kv)$$

$$kt = kA - \ln(g - kv)$$

$$\ln(g - kv) = kA - kt$$

$$g - kv = e^{kA - kt} = Be^{-kt}, \quad \text{where} \quad B = e^{kA}$$

$$kv = g - Be^{-kt}$$

When $t = 0, v = \frac{g}{2k}$

$$k \times \frac{g}{2k} = g - B \Rightarrow B = g - \frac{g}{2} = \frac{g}{2}$$

Hence

$$kv = g - \frac{g}{2}e^{-kt} = \frac{g}{2}(2 - e^{-kt})$$

$$v = \frac{g}{2k}(2 - e^{-kt}), \text{ as required}$$

b From part **a**

$v = \frac{dx}{dt} = \frac{g}{2k}(2 - e^{-kt})$

$x = \int \frac{g}{2k}(2 - e^{-kt})\,dt$

$= \frac{g}{2k}\left(2t + \frac{1}{k}e^{-kt}\right) + B$

When $t = 0, x = 0$

$0 = \frac{g}{2k^2} + B \Rightarrow B = -\frac{g}{2k^2}$

Hence

$x = \frac{g}{k}t + \frac{g}{2k^2}(e^{-kt} - 1)$

When $t = \frac{\ln 4}{k}$

$x = \frac{g}{k^2}\ln 4 + \frac{g}{2k^2}(e^{-\ln 4} - 1) = \frac{2g}{k^2}\ln 2 + \frac{g}{2k^2}\left(\frac{1}{4} - 1\right)$

$= \frac{2g}{k^2}\ln 2 - \frac{3g}{8k^2}$

$= \frac{g}{8k^2}(16\ln 2 - 3)$

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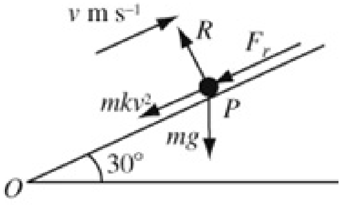
Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise B, Question 8

Question:

A particle P of mass m is projected with speed U up a rough plane inclined at an angle $30°$ to the horizontal. The coefficient of friction between P and the plane is $\frac{\sqrt{3}}{4}$. The particle P is subject to an air resistance of magnitude mkv^2, where v is the speed of P and k is a positive constant.
Find the distance P moves before coming to rest.

Solution:



R(↖) $R = mg\cos 30°$

Friction is limiting

$$F_r = \mu R = \frac{\sqrt{3}}{4} mg\cos 30° = \frac{\sqrt{3}}{4} mg \times \frac{\sqrt{3}}{2} = \frac{3}{8} mg$$

$$\text{R}(\nearrow) \quad \mathbf{F} = m\mathbf{a}$$

$$-F_r - mg\sin 30° - mkv^2 = ma$$

$$-\frac{3}{8}mg - \frac{1}{2}mg - mkv^2 = mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

Dividing throughout by m and multiplying throughout by 8

$$-7g - 8kv^2 = 8v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$\int 1\,\mathrm{d}x = -\int \frac{8v}{7g + 8kv^2}\,\mathrm{d}v$$

$$x = A - \frac{1}{2k}\ln(7g + 8kv^2)$$

At $x = 0, v = U$

$$0 = A - \frac{1}{2k}\ln(7g + 8kU^2) \Rightarrow A = \frac{1}{2k}\ln(7g + 8kU^2)$$

Hence

$$x = \frac{1}{2k}\ln(7g + 8kU^2) - \frac{1}{2k}\ln(7g + 8kv^2)$$

$$= \frac{1}{2k}\ln\left(\frac{7g + 8kU^2}{7g + 8kv^2}\right)$$

When $v = 0$

$$x = \frac{1}{2k}\ln\left(\frac{7g + 8kU^2}{7g}\right) = \frac{1}{2k}\ln\left(1 + \frac{8kU^2}{7g}\right)$$

The distance P moves before coming to rest is $\frac{1}{2k}\ln\left(1 + \frac{8kU^2}{7g}\right)$.

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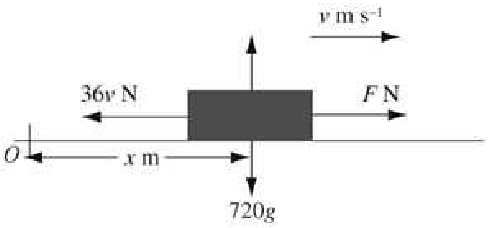
Resisted motion of a particle movig in a straight line
Exercise C, Question 1

Question:

A car of mass 720 kg is moving along a straight horizontal road with the engine of the car working at 30 kW. At time $t = 0$, the car passes a point A moving with speed 12 m s^{-1}. The total resistance to the motion of the car is $36v$ N, where $v\text{ m s}^{-1}$ is the speed of the car at time t seconds.
Find the time the car takes to double its speed.

Solution:



$30\text{ kW} = 30\ 000\text{W}$

Let the tractive force generated by the engine be F N.

$$P = Fv$$

$$30\ 000 = Fv \Rightarrow F = \frac{30\ 000}{v}$$

$$\text{R}(\rightarrow)\quad \mathbf{F} = ma$$

$$F - 36v = 720a$$

$$\frac{30\ 000}{v} - 36v = 720\frac{dv}{dt}$$

$$30\ 000 - 36v^2 = 720v\frac{dv}{dt}$$

Separating the variables

$$\int 1\, dt = \int \frac{720v}{30\ 000 - 36v^2}\, dv$$

$$t = A - 10\ln(30\ 000 - 36v^2)$$

When $t = 0, v = 12$

$$0 = A - 10\ln(30\ 000 - 36\times 12^2) \Rightarrow A = 10\ln 24\ 816$$

Hence

$$t = 10\ln 24\ 816 - 10\ln(30\ 000 - 36v^2) = 10\ln\left(\frac{24\ 816}{30\ 000 - 36v^2}\right)$$

When $v = 24$

$$t = 10\ln\left(\frac{24\ 816}{30\ 000 - 36\times 24^2}\right) = 10\ln\left(\frac{24\ 816}{9264}\right) \approx 9.85$$

The time the car takes to double its speed is 9.85s (3 s.f.)

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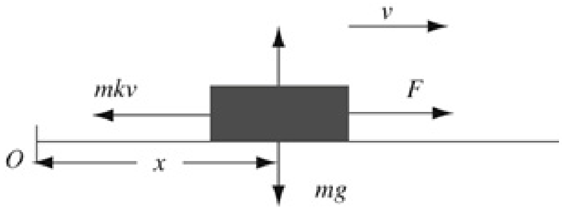
Resisted motion of a particle movig in a straight line
Exercise C, Question 2

Question:

A train of mass m is moving along a straight horizontal track with its engine working at a constant rate of $16mkU^2$, where k and U are constants. The resistance to the motion of the train has magnitude mkv, where v is the speed of the train.
Find the time the train takes to increase its speed from U to $3U$.

Solution:



Let the tractive force generated by the engine be F.

$$P = Fv$$
$$16mkU^2 = Fv$$
$$F = \frac{16mkU^2}{v}$$

R$(\rightarrow)$ $\quad \mathbf{F} = m\mathbf{a}$

$$\frac{16mkU^2}{v} - mkv = ma$$
$$\frac{16kU^2}{v} - kv = \frac{\mathrm{d}v}{\mathrm{d}t}$$
$$k(16U^2 - v^2) = v\frac{\mathrm{d}v}{\mathrm{d}t}$$

Separating the variables

$$\int k \,\mathrm{d}t = \int \frac{v}{16U^2 - v^2}\,\mathrm{d}v$$
$$kt = A - \frac{1}{2}\ln(16U^2 - v^2)$$

Let $t = 0$ when $v = U$

$$0 = A - \frac{1}{2}\ln(16U^2 - U^2) \Rightarrow A = \frac{1}{2}\ln(15U^2)$$

Hence

$$kt = \frac{1}{2}\ln(15U^2) - \frac{1}{2}\ln(16U^2 - v^2)$$
$$t = \frac{1}{2k}\ln\left(\frac{15U^2}{16U^2 - v^2}\right)$$

When $v = 3U$

$$t = \frac{1}{2k}\ln\left(\frac{15U^2}{16U^2 - 9U^2}\right) = \frac{1}{2k}\ln\left(\frac{15}{7}\right)$$

The time the train takes to increase its speed from U to $3U$ is $\frac{1}{2k}\ln\left(\frac{15}{7}\right)$.

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise C, Question 3

Question:

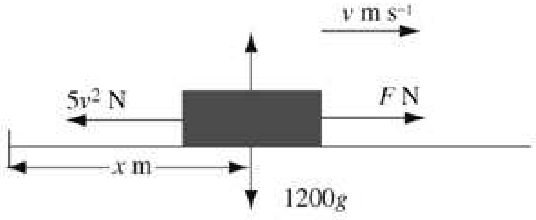
A van of mass 1200 kg is moving along a horizontal road with its engine working at a constant rate of 40 kW. The resistance to motion of the van is of magnitude of $5v^2$ N, where v m s^{-1} is the speed of the van. Find

a the terminal speed of the van,

b the distance the van travels while increasing its speed from 10 m s^{-1} to 15 m s^{-1}.

Solution:

a



40 kW = 40 000 W

Let the tractive force generated by the engine be F N.

$$P = Fv$$

$$40\,000 = Fv \Rightarrow F = \frac{40\,000}{v}$$

R(→) $\quad \mathbf{F} = m\mathbf{a}$

$$\frac{40\,000}{v} - 5v^2 = 1200a \quad *$$

At the terminal speed $a = 0$

$$\frac{40\,000}{v} - 5v^2 = 0 \Rightarrow v^3 = 8000 \Rightarrow v = 20$$

The terminal speed of the van is 20 m s^{-1}.

b Equation * can be written

$$\frac{40\,000}{v} - 5v^2 = 1200v\frac{dv}{dx}$$

Dividing throughout by 5 and multiplying throughout by v

$$8000 - v^3 = 240v^2\frac{dv}{dx}$$

Separating the variables

$$\int 1\,dx = 240\int \frac{v^2}{8000 - v^3}\,dv$$

$$x = A - \frac{240}{3}\ln(8000 - v^3)$$

Let $x = 0$ when $v = 10$

$$0 = A - 80\ln(8000 - 1000) \Rightarrow A = 80\ln 7000$$

Hence

$$x = 80\ln 7000 - 80\ln(8000 - v^3) = 80\ln\left(\frac{7000}{8000 - v^3}\right)$$

When $v = 15$

$$x = 80\ln\left(\frac{7000}{8000 - 15^3}\right) = 80\ln\left(\frac{7000}{4625}\right) \approx 33.2$$

The distance the van travels while increasing its speed from 10 m s^{-1} to 15 m s^{-1} is 33.2 m (3 s.f.)

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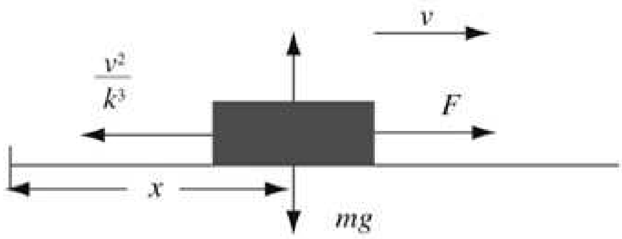
Resisted motion of a particle movig in a straight line
Exercise C, Question 4

Question:

A car of mass m is moving along a straight horizontal road with its engine working at a constant rate D^3. The resistance to the motion of the car is of magnitude $\frac{v^2}{k^3}$, where v is the speed of the car and k is a positive constant.

Find the distance travelled by the car as its speed increases from $\frac{kD}{4}$ to $\frac{kD}{2}$.

Solution:



Let the tractive force generated by the engine be F.

$$P = Fv$$

$$D^3 = Fv \Rightarrow F = \frac{D^3}{v}$$

$R(\rightarrow)$ $\quad \mathbf{F} = m\mathbf{a}$

$$\frac{D^3}{v} - \frac{v^2}{k^3} = mv\frac{dv}{dx}$$

Multiplying throughout by k^3v

$$k^3D^3 - v^3 = mk^3v^2\frac{dv}{dx}$$

Separating the variables

$$\int 1\,dx = mk^3\int\frac{v^2}{k^3D^3 - v^3}\,dv$$

$$x = A - \frac{mk^3}{3}\ln(k^3D^3 - v^3)$$

Let $x = 0$ when $v = \frac{kD}{4}$

$$0 = A - \frac{mk^3}{3}\ln\left(k^3D^3 - \frac{k^3D^3}{64}\right) \Rightarrow A = \frac{mk^3}{3}\ln\left(\frac{63k^3D^3}{64}\right)$$

Hence

$$x = \frac{mk^3}{3}\left(\ln\left(\frac{63k^3D^3}{64}\right) - \ln(k^3D^3 - v^3)\right)$$

When $v = \frac{kD}{2}$

$$x = \frac{mk^3}{3}\left(\ln\left(\frac{63k^3D^3}{64}\right) - \ln\left(k^3D^3 - \frac{k^3D^3}{8}\right)\right)$$

$$= \frac{mk^3}{3}\left(\ln\left(\frac{63k^3D^3}{64}\right) - \ln\left(\frac{7k^3D^3}{8}\right)\right)$$

$$= \frac{mk^3}{3}\ln\left(\frac{63k^3D^3}{64} \times \frac{8}{7k^3D^3}\right) = \frac{mk^3}{3}\ln\left(\frac{9}{8}\right)$$

The distance travelled by the car as its speed increases from $\frac{kD}{4}$ to $\frac{kD}{2}$ is

$$\frac{mk^3}{3}\ln\left(\frac{9}{8}\right)$$

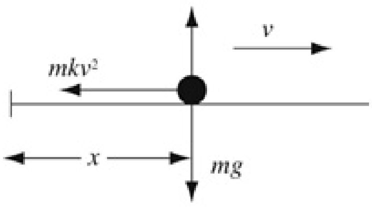
Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise D, Question 1

Question:

A particle of mass m moves in a straight line on a smooth horizontal plane in a medium which exerts a resistance of magnitude mkv^2, where v is the speed of the particle and k is a positive constant. At time $t = 0$ the particle has speed U.
Find, in terms of k and U, the time at which the particle's speed is $\frac{3}{4}U$. **[E]**

Solution:



$$\mathrm{R}(\rightarrow) \quad \mathbf{F} = m\mathbf{a}$$
$$-mkv^2 = ma$$
$$-kv^2 = \frac{\mathrm{d}v}{\mathrm{d}t}$$

Separating the variables

$$\int k \,\mathrm{d}t = -\int v^{-2}\,\mathrm{d}v$$
$$kt = -\frac{v^{-1}}{-1} + A = \frac{1}{v} + A$$

At $t = 0, v = U$

$$0 = \frac{1}{U} + A \Rightarrow A = -\frac{1}{U}$$

Hence

$$t = \frac{1}{k}\left(\frac{1}{v} - \frac{1}{U}\right)$$

When $v = \frac{3}{4}U$

$$t = \frac{1}{k}\left(\frac{1}{\frac{3}{4}U} - \frac{1}{U}\right) = \frac{1}{k}\left(\frac{4}{3U} - \frac{1}{U}\right) = \frac{1}{3kU}$$

The time at which the particle's speed is $\frac{3}{4}U$ is $\frac{1}{3kU}$.

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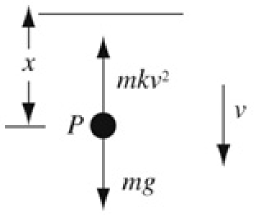
Resisted motion of a particle movig in a straight line
Exercise D, Question 2

Question:

A small pebble of mass m is placed in a viscous liquid and sinks vertically from rest through the liquid. When the speed of the particle is v the magnitude of the resistance due to the liquid is modelled as mkv^2, where k is a positive constant.
Find the speed of the pebble after it has fallen a distance D through the liquid. **[E]**

Solution:



R(↓) $\mathbf{F} = m\mathbf{a}$

$$mg - mkv^2 = ma$$

$$g - kv^2 = v\frac{dv}{dx}$$

Separating the variables

$$\int 1\,dx = \int \frac{v}{g-kv^2}\,dv$$

$$x = A - \frac{1}{2k}\ln(g-kv^2)$$

When $x = 0, v = 0$

$$0 = A - \frac{1}{2k}\ln g \Rightarrow A = \frac{1}{2k}\ln g$$

$$x = \frac{1}{2k}\ln g - \frac{1}{2k}\ln(g-kv^2) = \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$$

$$\ln\left(\frac{g}{g-kv^2}\right) = 2kx$$

$$\frac{g}{g-kv^2} = e^{2kx}$$

$$g - kv^2 = g\,e^{-2kx}$$

$$v^2 = \frac{g}{k}(1-e^{-2kx})$$

When $x = D$

$$v^2 = \frac{g}{k}(1-e^{-2kD})$$

$$v = \left(\frac{g}{k}\right)^{\frac{1}{2}}(1-e^{-2kD})^{\frac{1}{2}}$$

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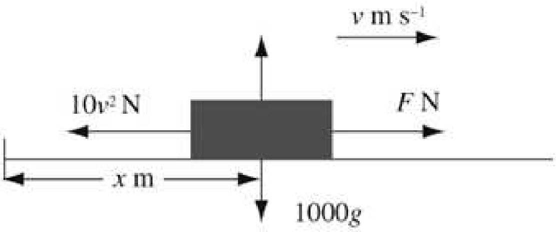
Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise D, Question 3

Question:

A car of mass 1000 kg is driven by an engine which generates a constant power of 12 kW. The only resistance to the car's motion is air resistance of magnitude $10v^2$ N, where v m s^{-1} is the speed of the car.
Find the distance travelled as the car's speed increases from 5 m s^{-1} to 10 m s^{-1}. **[E]**

Solution:



12 kW = 12 000 W

$P = Fv$

$F = \dfrac{12\,000}{v}$

R(→) $\quad F = ma$

$F - 10v^2 = 1000a$

$\dfrac{12\,000}{v} - 10v^2 = 1000v\dfrac{dv}{dx}$

Dividing throughout by 10 and multiplying throughout by v

$1200 - v^3 = 100v^2\dfrac{dv}{dx}$

Separating the variables

$\int 1\,dx = 100\int \dfrac{v^2}{1200 - v^3}\,dv$

$x = A - \dfrac{100}{3}\ln(1200 - v^3)$

Let $x = 0$ when $v = 5$

$0 = A - \dfrac{100}{3}\ln(1200 - 125) \Rightarrow A = \dfrac{100}{3}\ln 1075$

Hence

$x = \dfrac{100}{3}\ln 1075 - \dfrac{100}{3}\ln(1200 - v^3) = \dfrac{100}{3}\ln\left(\dfrac{1075}{1200 - v^3}\right)$

When $v = 10$

$x = \dfrac{100}{3}\ln\left(\dfrac{1075}{1200 - 10^3}\right) = \dfrac{100}{3}\ln\left(\dfrac{1075}{200}\right) \approx 56.1$

The distance travelled as the car's speed increases from 5 m s^{-1} to 10 m s^{-1} is 56.1 m (3 s.f.).

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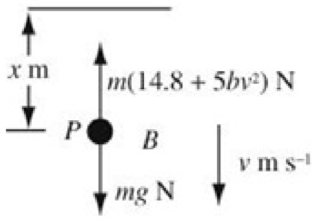
Resisted motion of a particle movig in a straight line
Exercise D, Question 4

Question:

A bullet B, of mass m kg, is fired vertically downwards into a block of wood W which is fixed in the ground. The bullet enters W with speed U m s^{-1} and W offers a resistance of magnitude $m(14.8+5bv^2)$ N, where v m s^{-1} is the speed of B and b is a positive constant. The path of B in W remains vertical until B comes to rest after travelling a distance d metres into W.
Find d in terms of b and U. **[E]**

Solution:



R(↓) $\mathbf{F} = m\mathbf{a}$

$$mg - m(14.8+5bv^2) = ma$$

$$9.8 - 14.8 - 5bv^2 = v\frac{dv}{dx}$$

$$-5(1+bv^2) = v\frac{dv}{dx}$$

Separating the variables

$$\int 1\,dx = -\frac{1}{5}\int \frac{v}{1+bv^2}\,dv$$

$$x = A - \frac{1}{10b}\ln(1+bv^2)$$

At $x=0, v=U$

$$0 = A - \frac{1}{10b}\ln(1+bU^2) \Rightarrow A = \frac{1}{10b}\ln(1+bU^2)$$

$$x = \frac{1}{10b}\ln(1+bU^2) - \frac{1}{10b}\ln(1+bv^2) = \frac{1}{10b}\ln\left(\frac{1+bU^2}{1+bv^2}\right)$$

When $v=0, x=d$

$$d = \frac{1}{10b}\ln(1+bU^2)$$

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Resisted motion of a particle movig in a straight line
Exercise D, Question 5

Question:

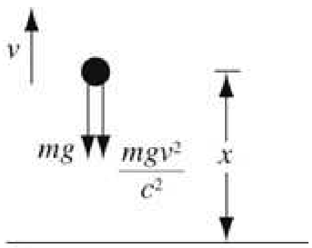
A particle of mass m is projected vertically upwards, with speed V, in a medium which exerts a resisting force of magnitude $\frac{mgv^2}{c^2}$, where v is the speed of the particle and c is a positive constant.

a Show that the greatest height attained above the point of projection is $\frac{c^2}{2g}\ln\left(1+\frac{V^2}{c^2}\right)$.

b Find an expression, in terms of V, c and g, for the time to reach this height. **[E]**

Solution:

a



R(↑) **F** = ma

$$-mg-\frac{mgv^2}{c^2}=ma$$

$$-g\left(\frac{c^2+v^2}{c^2}\right)=v\frac{dv}{dx}$$

$$\int g\,dx=-c^2\int\frac{v}{c^2+v^2}\,dv$$

$$gx=A-\frac{c^2}{2}\ln(c^2+v^2)$$

At $x=0, v=V$

$$0=A-\frac{c^2}{2}\ln(c^2+V^2)\Rightarrow A=\frac{c^2}{2}\ln(c^2+V^2)$$

Hence

$$gx=\frac{c^2}{2}\ln(c^2+V^2)-\frac{c^2}{2}\ln(c^2+v^2)=\frac{c^2}{2}\ln\left(\frac{c^2+V^2}{c^2+v^2}\right)$$

$$x=\frac{c^2}{2g}\ln\left(\frac{c^2+V^2}{c^2+v^2}\right)$$

At the greatest height $v=0$

$$x=\frac{c^2}{2g}\ln\left(\frac{c^2+V^2}{c^2}\right)=\frac{c^2}{2g}\ln\left(1+\frac{V^2}{c^2}\right)$$, as required.

b $\text{R}(\uparrow)$ $\mathbf{F} = m\mathbf{a}$

$$-mg - \frac{mgv^2}{c^2} = ma$$

$$-g\left(\frac{c^2+v^2}{c^2}\right) = \frac{\mathrm{d}v}{\mathrm{d}t}$$

Separating the variables

$$\frac{g}{c^2}\int 1\,\mathrm{d}t = -\int \frac{1}{c^2+v^2}\,\mathrm{d}v$$

$$\frac{gt}{c^2} = A - \frac{1}{c}\arctan\left(\frac{v}{c}\right)$$

When $t = 0, v = V$

$$0 = A - \frac{1}{c}\arctan\left(\frac{V}{c}\right) \Rightarrow A = \frac{1}{c}\arctan\left(\frac{V}{c}\right)$$

Hence

$$\frac{gt}{c^2} = \frac{1}{c}\arctan\left(\frac{V}{c}\right) - \frac{1}{c}\arctan\left(\frac{v}{c}\right)$$

At the greatest height $v = 0$

$$\frac{gt}{c^2} = \frac{1}{c}\arctan\left(\frac{V}{c}\right) \Rightarrow t = \frac{c}{g}\arctan\left(\frac{V}{c}\right)$$

The time taken to reach the greatest height is $\frac{c}{g}\arctan\left(\frac{V}{c}\right)$.

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Resisted motion of a particle movig in a straight line
Exercise D, Question 6

Question:

A particle is projected vertically upwards with speed U in a medium in which the resistance is proportional to the square of the speed. Given that U is also the speed for which the resistance offered by the medium is equal to the weight of the particle show that

a the time of ascent is $\frac{\pi U}{4g}$,

b the distance ascended is $\frac{U^2}{2g}\ln 2$. **[E]**

Solution:

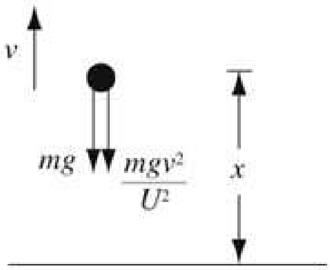
a Let the mass of the particle be m.

Let the resistance be kv^2, where k is a constant of proportionality.

If U is the speed for which the resistance is equal to the weight of the particle thei

$kU^2 = mg \Rightarrow k = \frac{mg}{U^2}$

Hence the resistance is $\frac{mgv^2}{U^2}$.



R(↑) $\quad \mathbf{F} = m\mathbf{a}$

$-mg - \frac{mgv^2}{U^2} = ma$

$-\frac{g(U^2+v^2)}{U^2} = \frac{dv}{dt}$ *

Separating the variables

$\int g\ dt = -U^2 \int \frac{1}{U^2+v^2}\,dv$

$gt = A - U^2 \times \frac{1}{U}\arctan\left(\frac{v}{U}\right)$

When $t = 0, v = U$

$0 = A - U\arctan 1 \Rightarrow A = U\arctan 1 = \frac{\pi U}{4}$

Hence

$gt = \frac{\pi U}{4} - U\arctan\left(\frac{v}{U}\right)$

$t = \frac{\pi U}{4g} - \frac{U}{g}\arctan\left(\frac{v}{U}\right)$

Let the time of ascent be T.

When $t = T, v = 0$

$T = \frac{\pi U}{4g} - \frac{U}{g}\arctan 0$

$= \frac{\pi U}{4g}$, as required

b Equation * in part **a** can be written as

$$-\frac{g(U^2+v^2)}{U^2}=v\frac{dv}{dx}$$

Separating the variables

Equation * in part **a** can be written as

$$-\frac{g(U^2+v^2)}{U^2}=v\frac{dv}{dx}$$

Separating the variables

$$\int g\ dx=-U^2\int\frac{v}{U^2+v^2}\ dv$$

$$gx=B-\frac{U^2}{2}\ln(U^2+v^2)$$

When $x=0, v=U$

$$0=B-\frac{U^2}{2}\ln(2U^2)\Rightarrow B=\frac{U^2}{2}\ln(2U^2)$$

Hence

$$gx=\frac{U^2}{2}\ln(2U^2)-\frac{U^2}{2}\ln(U^2+v^2)$$

$$x=\frac{U^2}{2g}\ln\left(\frac{2U^2}{U^2+v^2}\right)$$

Let the total distance ascended be H.

When $h=H, v=0$

$$H=\frac{U^2}{2g}\ln\left(\frac{2U^2}{U^2}\right)=\frac{U^2}{2g}\ln 2\text{, as required}$$

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Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise D, Question 7

Question:

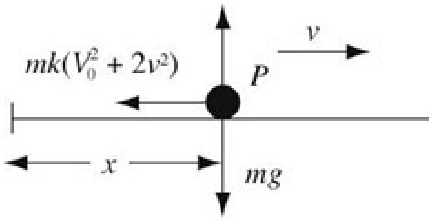
At time t, a particle P, of mass m, moving in a straight line has speed v. The only force acting is a resistance of magnitude $mk(V_0^2+2v^2)$, where k is a positive constant and V_0 is the speed of P when $t=0$.

a Show that, as v reduces from V_0 to $\frac{1}{2}V_0$, P travels a distance $\frac{\ln 2}{4k}$.

b Express the time P takes to cover this distance in the form $\frac{\lambda}{kV_0}$, giving the value of λ to two decimal places. **[E]**

Solution:

a



R(→) $\mathbf{F} = m\mathbf{a}$

$$-mk(V_0^2 + 2v^2) = ma$$

$$-k(V_0^2 + 2v^2) = v\frac{dv}{dx} \quad *$$

Separating the variables

$$\int k \, dx = -\int \frac{v}{V_0^2 + 2v^2} dv$$

$$kx = A - \frac{1}{4}\ln(V_0^2 + 2v^2)$$

At $x = 0, v = V_0$

$$0 = A - \frac{1}{4}\ln(V_0^2 + 2V_0^2) \Rightarrow A = \frac{1}{4}\ln(3V_0^2)$$

Hence

$$kx = \frac{1}{4}\ln(3V_0^2) - \frac{1}{4}\ln(V_0^2 + 2v^2)$$

$$x = \frac{1}{4k}\ln\left(\frac{3V_0^2}{V_0^2 + 2v^2}\right)$$

When $v = \frac{1}{2}V_0$

$$x = \frac{1}{4k}\ln\left(\frac{3V_0^2}{V_0^2 + \frac{1}{2}V_0^2}\right) = \frac{1}{4k}\ln\left(\frac{3V_0^2}{\frac{3}{2}V_0^2}\right)$$

$$= \frac{\ln 2}{4k}, \text{ as required}$$

b Equation * can be written as

$$-k(V_0^2+2v^2)=\frac{dv}{dt}$$

Separating the variables

$$\int k\ dt=-\int\frac{1}{V_0^2+2v^2}\,dv=-\frac{1}{2}\int\frac{1}{\left(\frac{V_0}{\sqrt{2}}\right)^2+v^2}\,dv$$

$$kt=B-\frac{1}{2}\times\frac{1}{\left(\frac{V_0}{\sqrt{2}}\right)}\arctan\frac{v}{\left(\frac{V_0}{\sqrt{2}}\right)}$$

When $t=0, v=V_0$

$$0=B-\frac{\sqrt{2}}{2V_0}\arctan\left(\frac{\sqrt{2}V_0}{V_0}\right)\Rightarrow B=\frac{\sqrt{2}}{2V_0}\arctan\sqrt{2}$$

Hence

$$t=\frac{\sqrt{2}}{2kV_0}\left(\arctan\sqrt{2}-\arctan\left(\frac{\sqrt{2}v}{V_0}\right)\right)$$

$$v=\frac{1}{2}V_0$$

$$t=\frac{\sqrt{2}}{2kV_0}\left(\arctan\sqrt{2}-\arctan\left(\frac{\sqrt{2}\times\frac{1}{2}V_0}{V_0}\right)\right)$$

$$=\frac{1}{kV_0}\left[\frac{\sqrt{2}}{2}\left(\arctan\sqrt{2}-\arctan\left(\frac{\sqrt{2}}{2}\right)\right)\right]$$

This has the form $\frac{\lambda}{kV_0}$, as required, where

$$\lambda=\frac{\sqrt{2}}{2}\left(\arctan\sqrt{2}-\arctan\left(\frac{\sqrt{2}}{2}\right)\right)\approx 0.24\text{ (2 d.p.)}$$

Solutionbank M4
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Resisted motion of a particle movig in a straight line
Exercise D, Question 8

Question:

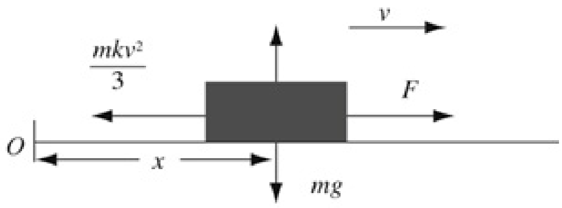
A car of mass m is moving along a straight horizontal road. When displacement of the car from a fixed point O is x, its speed is v. The resistance to the motion of the car has magnitude $\frac{mkv^2}{3}$, where k is a positive constant. The engine of the car is working at a constant rate P.

a Show that $3mv^2 \frac{dv}{dx} = 3P - mkv^3$.

When $t = 0$, the speed of the car is half of its limiting speed.

b Find x in terms of m, k, P and v.

Solution:



a $P = Fv \Rightarrow F = \dfrac{P}{v}$

$R(\rightarrow) \qquad \mathbf{F} = m\mathbf{a}$

$$F - \frac{mkv^2}{3} = ma$$

$$\frac{P}{v} - \frac{mkv^2}{3} = mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

Multiplying throughout by $3v$

$$3P - mkv^3 = 3mv^2\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$3mv^2\frac{\mathrm{d}v}{\mathrm{d}x} = 3P - mkv^3\text{, as required}$$

b The limiting speed is given by $a = v\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$

$$0 = 3P - mkv^3 \Rightarrow v^3 = \frac{3P}{mk} \Rightarrow v = \left(\frac{3P}{mk}\right)^{\frac{1}{3}}$$

Separating the variables in the answer to part a

$$\int 1\,\mathrm{d}x = \int \frac{3mv^2}{3P - mkv^3}\,\mathrm{d}v$$

$$x = A - \frac{1}{k}\ln(3P - mkv^3)$$

When $x = 0, v = \dfrac{1}{2}\left(\dfrac{3P}{mk}\right)^{\frac{1}{3}} \Rightarrow v^3 = \dfrac{3P}{8mk}$

$$0 = A - \frac{1}{k}\ln\left(3P - \frac{3P}{8}\right) \Rightarrow A = \frac{1}{k}\ln\left(\frac{21P}{8}\right)$$

Hence

$$x = \frac{1}{k}\ln\left(\frac{21P}{8}\right) - \frac{1}{k}\ln(3P - mkv^3)$$

$$= \frac{1}{k}\ln\left(\frac{21P}{8(3P - mkv^3)}\right)$$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise A, Question 1

Question:

A particle P is moving in a straight line. At time t, the displacement of P from a fixed point on the line is x. The motion of the particle is modelled by the differential equation $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+8x=0$

When $t=0$ P is at rest at the point where $x=2$.

a Find x as a function of t.

b Calculate the value of x when $t=\frac{\pi}{3}$.

c State whether the motion is heavily, critically or lightly damped.

Solution:

a $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+8x=0$

Auxiliary equation: $m^2+4m+8=0$ ← Solve the equation using the methods of book FP2 chapter 5.

$m=\frac{-4\pm\sqrt{(16-32)}}{2}$

$m=-2\pm 2\text{i}$

General solution:

$x=e^{-2t}(A\cos 2t+B\sin 2t)$

$t=0, x=2 \Rightarrow 2=A$ ← Use the initial conditions given in the question to obtain values for A and B.

$\dot{x}=-2e^{-2t}(A\cos 2t+B\sin 2t)+e^{-2t}(-2A\sin 2t+2B\cos 2t)$

$t=0, \dot{x}=0 \Rightarrow 0=-2A+2B$

$B=A$

$\therefore x=2e^{-2t}(\cos 2t+\sin 2t)$

b $t=\frac{\pi}{3}$ $\quad x=2e^{-\frac{2\pi}{3}}\left(\cos\frac{2\pi}{3}+\sin\frac{2\pi}{3}\right)$

$x=0.09014\ldots$

$\therefore x=0.0901 \quad (3 \text{ s.f.})$

c Lightly damped

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Damped and forced harmonic motion
Exercise A, Question 2

Question:

A particle P is moving in a straight line. At time t, the displacement of P from a fixed point on the line is x. The motion of the particle is modelled by the differential equation $\frac{d^2x}{dt^2}+8\frac{dx}{dt}+12x=0$

When $t=0$ P is at rest at the point where $x=4$.

Find x as a function of t.

Solution:

$$\frac{d^2x}{dt^2}+8\frac{dx}{dt}+12x=0$$

Auxiliary equation: $m^2+8m+12=0$ ← Solve the equation using the methods of book FP2 chapter 5.

$(m+6)(m+2)=0$

$m=-6$ or -2

General solution:

$x=Ae^{-6t}+Be^{-2t}$

$t=0, x=4 \Rightarrow 4=A+B$ ① ← Use the information given in the question to obtain values for A and B.

$\dot{x}=-6Ae^{-6t}-2Be^{-2t}$

$t=0, \dot{x}=0 \quad 0=-6A-2B$

$0=3A+B$ ②

$\therefore 2A=-4$ ← Solve equations ① and ② simultaneously.

$A=-2, B=6$

$\therefore x=6e^{-2t}-2e^{-6t}$

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Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise A, Question 3

Question:

A particle P is moving in a straight line. At time t, the displacement of P from a fixed point on the line is x. The motion of the particle is modelled by the differential equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+6x=0$

When $t=0$ P is at rest at the point where $x=1$.

a Find x as a function of t.

The smallest value of $t, t>0$, for which P is instantaneously at rest is T.

b Find the value of T.

Solution:

a $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+6x=0$

Auxiliary equation: $m^2+2m+6=0$ ← Solve the equation using the methods of book FP2 Chapter 5.

$m=\frac{-2\pm\sqrt{(4-24)}}{2}$

$m=-1\pm i\sqrt{5}$

General solution:

$x=e^{-t}(A\cos\sqrt{5}t+B\sin\sqrt{5}t)$

$t=0 \quad x=1\Rightarrow 1=A$ ← Use the initial conditions given in the question to obtain values for A and B.

$\dot{x}=-e^{-t}(A\cos\sqrt{5}t+B\sin\sqrt{5}t)$
$+e^{-t}(-A\sqrt{5}\sin\sqrt{5}t+B\sqrt{5}\cos\sqrt{5}t)$

$t=0 \quad \dot{x}=0\Rightarrow 0=-A+B\sqrt{5}$

$B=\frac{1}{\sqrt{5}}$

$\therefore x=e^{-t}\left(\cos\sqrt{5}t+\frac{1}{\sqrt{5}}\sin\sqrt{5}t\right)$

b $\dot{x}=0 \quad t=T$

$\Rightarrow 0=-e^{-T}\left(\cos\sqrt{5}T+\frac{1}{\sqrt{5}}\sin\sqrt{5}T\right)+e^{-T}\left(-\sqrt{5}\sin\sqrt{5}T+\frac{1}{\sqrt{5}}\sqrt{5}\cos\sqrt{5}T\right)$

$e^{-T}\neq 0$

$\therefore -\cos\sqrt{5}T-\frac{1}{\sqrt{5}}\sin\sqrt{5}T-\sqrt{5}\sin\sqrt{5}T+\cos\sqrt{5}T=0$

$\sin\sqrt{5}T=0$

$\sqrt{5}T=0,\pi,\ldots$

$T=\frac{\pi}{\sqrt{5}},\ldots$ ← $T>0$

$\therefore$ Smallest value of T is $\frac{\pi}{\sqrt{5}}$ or 1.40^c (3 s.f.)

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Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise A, Question 4

Question:

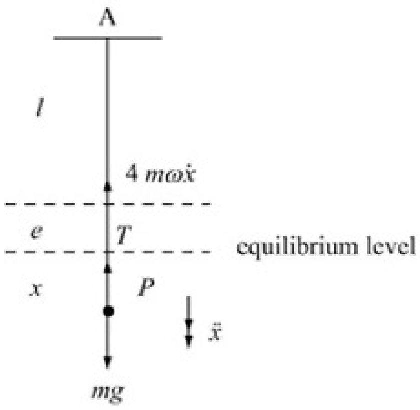
A particle P of mass m is attached to one end of a light elastic spring of natural length l and modulus of elasticity $4m\omega^2 l$, where ω is a positive constant. The other end of the spring is attached to a fixed point A and P hangs in equilibrium vertically below A. At time $t = 0$, P is projected vertically downwards with speed u. A resistance of magnitude $4m\omega v$, where v is the speed of P, acts on P. The displacement of P downwards from its equilibrium position at time t is x.

a Show that $\frac{d^2x}{dt^2} + 4\omega\frac{dx}{dt} + 4\omega^2 x = 0$

b Find an expression for x in terms of u, t and ω.

c Find the time at which P comes to instantaneous rest.

Solution:



$\lambda = 4m\omega^2 l$

a In equilibrium: $R(\uparrow)\ T = mg$

Hooke's Law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{4m\omega^2 e}{l}$$

$\therefore 4m\omega^2 e = mg$ ①

When extension is $(e + x)$

$$T = \frac{\lambda(e + x)}{l} = \frac{4m\omega^2 l(e + x)}{l}$$

$F = ma$:

$$mg - T - 4m\omega\dot{x} = m\ddot{x}$$

$$mg - 4m\omega^2(e + x) - 4m\omega\dot{x} = m\ddot{x}$$

$$mg - mg - 4m\omega^2 x - 4m\omega\dot{x} = m\ddot{x}$$

> Use 1.

$$\ddot{x} + 4\omega\dot{x} + 4\omega^2 x = 0$$

or $\dfrac{d^2x}{dt^2} + 4\omega\dfrac{dx}{dt} + 4\omega^2 x = 0$

b Auxiliary equation: $m^2 + 4\omega m + 4\omega^2 = 0$

> Now solve the differential equation using the methods of book FP2 chapter 5

$(m + 2\omega)^2 = 0$

$m = -2\omega$ (twice)

General solution: $x = (A + Bt)e^{-2\omega t}$

$t = 0,\ x = 0 \Rightarrow 0 = A$

$\dot{x} = Be^{-2\omega t} - 2\omega Bte^{-2\omega t}$

$t = 0,\ \dot{x} = u \Rightarrow u = B$

$\therefore x = ute^{-2\omega t}$

c $\dot{x} = ue^{-2\omega t} - 2\omega ut\,e^{-2\omega t}$

$= u\,e^{-2\omega t}(1 - 2\omega t)$

$\dot{x} = 0 \quad 1 - 2\omega t = 0$

> $e^{-2\omega t} \neq 0$

$$t = \frac{1}{2\omega}$$

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Damped and forced harmonic motion
Exercise A, Question 5

Question:

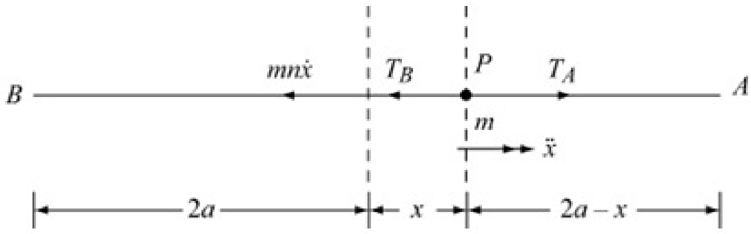
A particle P of mass m is attached to the mid-point of a light elastic string AB of natural length $2a$ and modulus of elasticity mg. The ends A and B of the string are attached to fixed points on a smooth horizontal table with $AB = 4a$. The particle is released from rest at the point C where A, C and B lie in a straight line and $AC = \frac{3}{2}a$. At time t the displacement of P from its equilibrium position is x. The particle is subject to a resisting force of magnitude mnv where v is the speed of P and $n = \sqrt{\frac{2g}{a}}$.

a Show that $\frac{d^2x}{dt^2} + n\frac{dx}{dt} + n^2x = 0$.

b Find an expression for x in terms of a, n and t.

Solution:

a



$n = \sqrt{\frac{2g}{a}}$ $\lambda = mg$

$l = a$

Consider each half string separately.

$T = \frac{\lambda x}{l}$

Use Hooke's Law.

$T_A = \frac{mg(a-x)}{a}$ $T_B = \frac{mg(a+x)}{a}$

$F = ma$

$$T_A - T_B - mn\dot{x} = m\ddot{x}$$

$$\frac{mg(a-x)}{a} - \frac{mg(a+x)}{a} - mn\dot{x} = m\ddot{x}$$

$$-\frac{2gx}{a} - n\dot{x} = \ddot{x}$$

$$\ddot{x} + n\dot{x} + \frac{2g}{a}x = 0$$

or $$\frac{d^2x}{dt^2} + n\frac{dx}{dt} + n^2x = 0$$

From the question, $n^2 = \frac{2g}{a}$

b Auxiliary equation: $m^2 + nm + n^2 = 0$

Now solve the differential equation using the methods of book FP2 chapter 5

$$m = \frac{-n \pm \sqrt{(n^2 - 4n^2)}}{2}$$

$$m = \frac{-n \pm in\sqrt{3}}{2}$$

General solution:

$$x = e^{-\frac{1}{2}nt}\left(A\cos\frac{n\sqrt{3}}{2}t + B\sin\frac{n\sqrt{3}}{2}t\right)$$

$t = 0$ $x = \frac{1}{2}a \Rightarrow \frac{1}{2}a = A$

Use the initial conditions given in the question to obtain values for A and B.

$$\dot{x} = -\frac{1}{2}ne^{-\frac{1}{2}nt}\left(A\cos\frac{n\sqrt{3}}{2}t + B\sin\frac{n\sqrt{3}}{2}t\right)$$
$$+e^{-\frac{1}{2}nt}\left(\frac{-n\sqrt{3}}{2}A\sin\frac{n\sqrt{3}}{2}t + \frac{n\sqrt{3}}{2}B\cos\frac{n\sqrt{3}}{2}t\right)$$

$t = 0 \;\; \dot{x} = 0 \;\; \Rightarrow 0 = -\frac{1}{2}nA + \frac{n\sqrt{3}}{2}B$

$B = \frac{A}{\sqrt{3}} = \frac{a}{2\sqrt{3}}$

$\therefore x = e^{-\frac{1}{2}nt}\left(\frac{1}{2}a\cos\frac{n\sqrt{3}}{2}t + \frac{a}{2\sqrt{3}}\sin\frac{n\sqrt{3}}{2}t\right)$

or $x = \frac{a}{2}e^{-\frac{1}{2}nt}\left(\cos\frac{n\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{n\sqrt{3}}{2}t\right)$

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Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion

Exercise B, Question 1

Question:

A particle P is attached to end A of a light elastic spring AB. The end B of the spring is oscillating. At time t the displacement of P from a fixed point is x. When $t=0, x=0$ and $\frac{dx}{dt}=\frac{k}{5}$ where k is a constant. Given that x satisfies the differential equation $\frac{d^2x}{dt^2}+9x=k\cos t$, find x as a function of t.

Solution:

$$\frac{d^2x}{dt^2}+9x=k\cos t$$

Solve the equation using the methods of book FP2 chapter 5.

Auxiliary equation:

$$m^2+9=0$$
$$m=\pm 3\text{i}$$

Complementary function:

$$x=A\cos 3t+B\sin 3t$$

Particular integral:

try $x=p\cos t+q\sin t$

$$\dot{x}=-p\sin t+q\cos t$$
$$\ddot{x}=-p\cos t-q\sin t$$

$$-p\cos t-q\sin t+9(p\cos t+q\sin t)=k\cos t$$

Substitute the above results in the differential equation.

$$-p+9p=k$$
$$p=\frac{k}{8}$$

Equating coefficients of $\cos t$ …

$$-q+9q=0\ q=0$$

… and of $\sin t$.

Complete solution:

$$x=A\cos 3t+B\sin 3t+\frac{k}{8}\cos t$$

$$t=0\ x=0\Rightarrow 0=A+\frac{k}{8}\quad A=-\frac{k}{8}$$

$$\dot{x}=-3A\sin 3t+3B\cos 3t-\frac{k}{8}\sin t$$

$$t=0,\ \dot{x}=\frac{k}{5}\Rightarrow \frac{k}{5}=3B\quad B=\frac{k}{15}$$

Use the initial conditions given in the question to obtain values of A and B.

$$\therefore x=-\frac{k}{8}\cos 3t+\frac{k}{15}\sin 3t+\frac{k}{8}\cos t$$

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Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise B, Question 2

Question:

A particle P of mass m lies at rest on a horizontal table attached to end A of a light elastic spring AB of natural length l and modulus of elasticity $9m\omega^2 l$. At time $t = 0, AB = l$. The end B of the spring is now moved along the table in the direction AB with constant speed V. The resistance to motion of P has magnitude $6m\omega v$, where v is the speed of P and ω is a constant. At time t the extension of the spring is x and the displacement of P from its initial position is y.

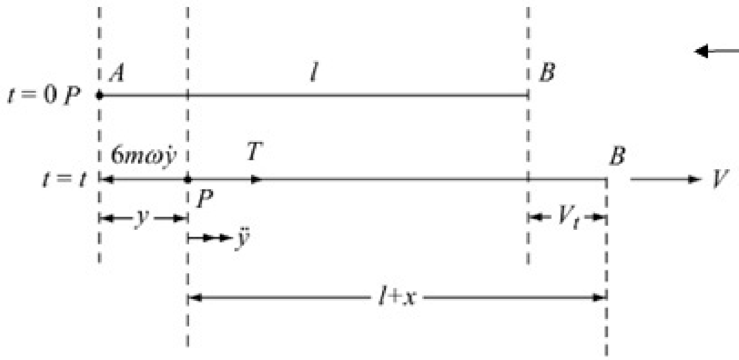
Show that

a $x + y = Vt$,

b $\frac{d^2x}{dt^2} + 6\omega\frac{dx}{dt} + 9\omega^2 x = 6\omega V$.

c Find an expression for x in terms of t, ω and V.

Solution:



Draw diagrams to show the initial situation and the situation at time t. Mark all distances clearly.

$\lambda = 9m w^2 l$

a $y + (l + x) = l + Vt$

$x + y = Vt$ ①

Use the diagrams to form this equation.

b Hooke's law: $T = \dfrac{\lambda x}{l} = \dfrac{9m\omega^2 l}{l} x$

$T = 9m\omega^2 x$

$F = ma$: $T - 6m\omega\dot{y} = m\ddot{y}$

$9m\omega^2 x - 6m\omega\dot{y} = m\ddot{y}$

The displacement of P from its initial position is y, not x.

From ①

$\dot{x} + \dot{y} = V$

$\ddot{x} + \ddot{y} = 0$

Use ① to obtain $\dot{y}$ and $\ddot{y}$ in terms of $\dot{x}$ and $\ddot{x}$

$\therefore 9m\omega^2 x - 6m\omega(V - \dot{x}) = m(-\ddot{x})$

$\ddot{x} + 6\omega\dot{x} + 9\omega^2 x = 6\omega V$

or $\dfrac{d^2x}{dt^2} + 6\omega\dfrac{dx}{dt} + 9\omega^2 x = 6\omega V$

c Auxiliary equation: $m^2 + 6m\omega + 9\omega^2 = 0$

Solve the equation using the methods of book FP2 Chapter 5.

$(m + 3\omega)^2 = 0$

$m = -3\omega$ (twice)

Complementary function:

$x = (A + Bt)e^{-3\omega t}$

Particular integral: try $x = k$

$\dot{x} = \ddot{x} = 0$

$9\omega^2 k = 6\omega V$

$k = \dfrac{2V}{3\omega}$

$\therefore$ Complete solution:

$x = (A + Bt)e^{-3\omega t} + \dfrac{2V}{3\omega}$

$t = 0,\ x = 0 \Rightarrow 0 = A + \dfrac{2V}{3\omega}$

Use the initial conditions given in the question to obtain expressions for A and B.

$A = -\dfrac{2V}{3\omega}$

$\dot{x} = Be^{-3\omega t} - 3\omega(A + Bt)e^{-3\omega t}$

$t = 0,\ \dot{x} = 0 \Rightarrow 0 = B - 3\omega A$

$B = 3\omega A = -2V$

$\therefore x = \left(-\dfrac{2V}{3\omega} - 2Vt\right)e^{-3\omega t} + \dfrac{2V}{3\omega}$

or $x = \dfrac{2V}{3\omega}(1 - e^{-3\omega t} - 3\omega t\, e^{-3\omega t})$

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Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise B, Question 3

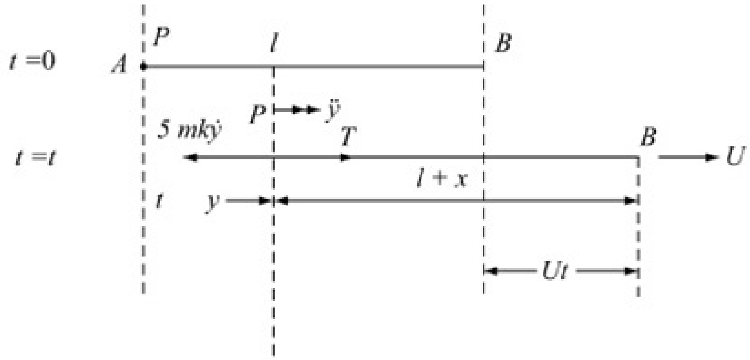
Question:

A particle P of mass m is attached to end A of a light elastic spring AB of natural length l and modulus of elasticity $6mk^2l$. Initially the spring and the particle lie at rest on a horizontal surface with $AB = l$. The end B of the spring is then moved in a straight line in the direction AB with constant speed U. As P moves on the surface it is subject to a resistance of magnitude $5mkv$ where v is the speed of P. At time $t, t > 0$, the extension of the spring is x.

a Show that $\frac{d^2x}{dt^2} + 5k\frac{dx}{dt} + 6k^2x = 5kU$.

b Find an expression for x in terms of t.

Solution:



Draw diagrams to show the initial situation and the situation at time t.
Let the distance moved by P from its initial position be y.
Mark all distances clearly.

a $y+(l+x)=l+Ut$

$y+x=Ut$ ①

Obtain a connection between y and the extension x.

Hooke's law: $T=\frac{\lambda x}{l}=\frac{6mk^2lx}{l}$

$T=6mk^2x$

$F=ma \quad T-5mk\dot{y}=m\ddot{y}$

Using ① $\dot{y}+\dot{x}=U$

$\ddot{y}+\ddot{x}=0$

Use ① to obtain $\dot{y}$ and $\ddot{y}$ in terms of $\dot{x}$ and $\ddot{x}$.

$\therefore 6mk^2x-5mk(U-\dot{x})=m(-\ddot{x})$

$\ddot{x}+5k\dot{x}+6k^2x=5kU$

or $\frac{d^2x}{dt^2}+5k\frac{dx}{dt}+6k^2x=5kU$

b Auxiliary equation: $m^2+5km+6k^2=0$

Now solve the equation using the methods of book FP2 Chapter 5.

$(m+3k)(m+2k)=0$

$m=-3k \text{ or } -2k$

Complementary function:

$x=Ae^{-3kt}+Be^{-2kt}$

Particular integral: try $x=a$

$\dot{x}=\ddot{x}=0$

$\therefore 6k^2a=5kU$

$a=\frac{5U}{6k}$

Complete Solution: $x=Ae^{-3kt}+Be^{-2kt}+\frac{5U}{6k}$

$t=0, x=0 \Rightarrow 0=A+B+\frac{5U}{6k}$ ①

$\dot{x}=-3kAe^{-3kt}-2kBe^{-2kt}$

$t=0, \dot{x}=0 \Rightarrow 0=-3kA-2kB$

$3A+2B=0$ ②

$\therefore 2A-3A+\frac{5U}{3k}=0$

Solve ① and ② simultaneously.

$A=\frac{5U}{3k}, B=-\frac{5U}{2k}$

$\therefore x=\frac{5U}{3k}e^{-3kt}-\frac{5U}{2k}e^{-2kt}+\frac{5U}{6k}$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise B, Question 4

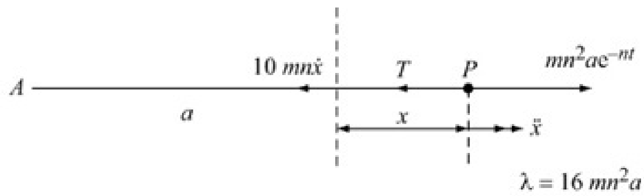
Question:

A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity $16mn^2a$. The other end of the string is attached to a fixed point A on the horizontal table on which P lies. At time $t = 0$, P is at rest on the table with $AP = a$. A force of magnitude mn^2ae^{-nt}, $t \geq 0$, acting in the direction AP is applied to P. The motion of P is opposed by a resistance of magnitude $10mnv$, where v is the speed of P. At time t, $t > 0$, the extension of the string is x.

a Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 10n\frac{\mathrm{d}x}{\mathrm{d}t} + 16n^2x = n^2ae^{-nt}$.

b Find an expression for x in terms of t.

Solution:



a Hooke's law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{16mn^2a}{a}x = 16mn^2x$

$F = ma$

$mn^2ae^{-nt} - T - 10mn\dot{x} = m\ddot{x}$

$\ddot{x} + 10n\dot{x} + 16n^2x = n^2ae^{-nt}$

or $\dfrac{d^2x}{dt^2} + 10n\dfrac{dx}{dt} + 16n^2x = n^2ae^{-nt}$

b Auxiliary equation: $m^2 + 10mn + 16n^2 = 0$

$(m+2n)(m+8n) = 0$

$m = -8n, m = -2n$

Solve the equation using the methods of book FP2 Chapter 5.

$\therefore$ Complementary function:

$x = Ae^{-8nt} + Be^{-2nt}$

Particular integral: try $x = ke^{-nt}$

$\dot{x} = -nke^{-nt}$

$\ddot{x} = n^2ke^{-nt}$

$\therefore n^2ke^{-nt} - 10n^2ke^{-nt} + 16n^2ke^{-nt} = n^2ae^{-nt}$

$7n^2ke^{-nt} = n^2ae^{-nt}$

$k = \dfrac{a}{7}$

Complete solution:

$x = Ae^{-8nt} + Be^{-2nt} + \dfrac{a}{7}e^{-nt}$

$t = 0, x = 0 \Rightarrow 0 = A + B + \dfrac{a}{7}$ ①

Use the initial conditions given in the question to obtain values for A and B.

$\dot{x} = -8nAe^{-8nt} - 2nBe^{-2nt} - \dfrac{an}{7}e^{-nt}$

$t = 0, \dot{x} = 0 \quad 0 = -8nA - 2nB - \dfrac{an}{7}$

$8A + 2B + \dfrac{a}{7} = 0$ ②

Solve equations ① and ② simultaneously.

$6A - \dfrac{a}{7} = 0$

$A = \dfrac{a}{42}$

$B = -\dfrac{a}{42} - \dfrac{a}{7} = -\dfrac{a}{6}$

$\therefore x = \dfrac{a}{42}e^{-8nt} - \dfrac{a}{6}e^{-2nt} + \dfrac{a}{7}e^{-nt}$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise B, Question 5

Question:

A particle P of mass 0.5 kg is attached to end A of a light elastic string AB of natural length 0.8 m and modulus of elasticity 5 N. The particle and string lie on a smooth horizontal plane with $AB = 0.8$ m. At time $t = 0$ a variable force F N is applied to the end B of the string which then moves with a constant speed 5 m s^{-1} in the direction AB. The particle moves along the plane and is subject to air resistance of magnitude $0.5v$ newtons, where $v \text{ m s}^{-1}$ is the speed of P. At time t seconds the displacement of P from its initial position is y metres and the extension of the string is x metres.
Show that, while the string is taut,

a $x + y = 5t$,

b $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 12.5x = 5.$

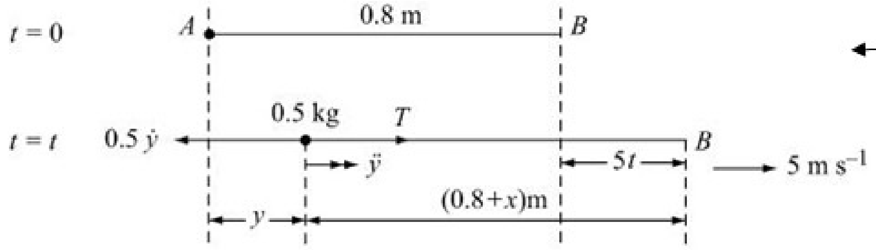
Find

c an expression for x in terms of t,

d the exact distance travelled by P in the first π seconds,

e the exact value of F when $t = \pi$.

Solution:



Draw diagrams to show all the distances at $t = 0$ and at time t clearly.

a $y + (0.8 + x) = 0.8 + 5t$

$x + y = 5t$ ①

Use the diagrams to form this equation

b Hooke's law: $T = \frac{\lambda x}{l} = \frac{5x}{0.8} = 6.25x$

$F = ma \quad T - 0.5\dot{y} = 0.5\ddot{y}$

$6.25x - 0.5\dot{y} = 0.5\ddot{y}$

$\dot{x} + \dot{y} = 5$

$\ddot{x} + \ddot{y} = 0$

Use equation ①.

$\therefore 6.25x - 0.5(5 - \dot{x}) = 0.5(-\ddot{x})$

$\ddot{x} + \dot{x} + 12.5x = 5$

or $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 12.5x = 5$

c Auxiliary equation: $m^2 + m + 12.5 = 0$

Now solve the equation using the methods of book FP2 Chapter 6.

$m = \frac{-1 \pm \sqrt{(1-50)}}{2}$

$m = \frac{-1 \pm 7i}{2}$

Complementary function is

$x = e^{-\frac{1}{2}t}\left(A\cos\frac{7}{2}t + B\sin\frac{7}{2}t\right)$

Particular integral: try $x = k$

$\dot{x} = \ddot{x} = 0$

$12.5k = 5$

$k = \frac{5}{12.5} = \frac{2}{5}$

General solution:

$x = e^{-\frac{1}{2}t}\left(A\cos\frac{7}{2}t + B\sin\frac{7}{2}t\right) + \frac{2}{5}$

$t = 0,\ x = 0 \Rightarrow 0 = A + \frac{2}{5} \Rightarrow A = -\frac{2}{5}$

Use the initial conditions given in the question to obtain values for A and B.

$\dot{x} = -\frac{1}{2}e^{-\frac{1}{2}t}\left(A\cos\frac{7}{2}t + B\sin\frac{7}{2}t\right)$

$$+\,e^{-\frac{1}{2}t}\left(-\frac{7}{2}A\sin\frac{7}{2}t+\frac{7}{2}B\cos\frac{7}{2}t\right)$$

$$t=0,\ \dot{x}=0\Rightarrow 0=-\frac{1}{2}A+\frac{7}{2}B$$

$$B=\frac{A}{7}=-\frac{2}{35}$$

$$\therefore x=e^{-\frac{1}{2}t}\left(-\frac{2}{5}\cos\frac{7}{2}t-\frac{2}{35}\sin\frac{7}{2}t\right)+\frac{2}{5}$$

d $t=\pi,\ x=\frac{2}{5}-e^{-\pi/2}\times\left(\frac{-2}{35}\right)(-1)$

x is the extension of the string, not the distance travelled by P.

$$y=5t-x$$

$$=5\pi-\frac{2}{5}+\frac{2}{35}e^{-\frac{\pi}{2}}$$

e $F=T=\frac{25x}{4}$

End B is moving at a constant speed, so it is in equilibrium.

$$t=\pi\ \ F=\frac{25}{4}\left(\frac{2}{5}-\frac{2}{35}e^{-\frac{\pi}{2}}\right)$$

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Damped and forced harmonic motion
Exercise C, Question 1

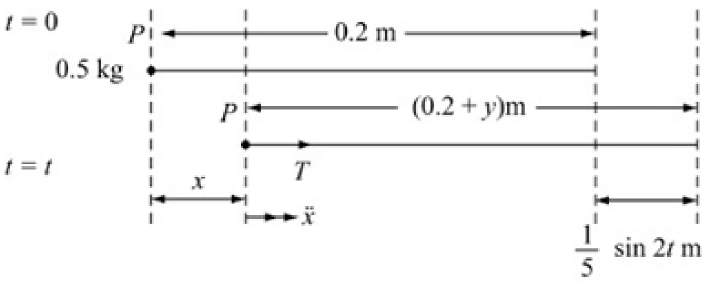
Question:

A particle P of mass 0.5 kg is free to move horizontally inside a smooth cylindrical tube. The particle is attached to one end of a light elastic spring of natural length 0.2 m and modulus of elasticity 5 N. At time $t = 0$ the system is at rest with the spring at its natural length. The other end of the spring is then forced to oscillate with simple harmonic motion so that at time t seconds, $t > 0$, its displacement from its initial position is $\frac{1}{5}\sin 2t$ metres and the displacement of P from its initial position is x metres.

a Show that $\frac{d^2x}{dt^2} + 50x = 10\sin 2t$.

b Find an expression for x in terms of t.

Solution:



Let the extension of the spring be y m and draw diagrams showing the situation when $t = 0$ and at time t seconds.

a Hooke's law: $T = \frac{\lambda x}{l} = \frac{5y}{0.2} = 25y$

$F = ma$:

$T = 0.5\ddot{x}$

$25y = 0.5\ddot{x}$

From the diagrams:

$$0.2 + \frac{1}{5}\sin 2t = (0.2 + y) + x$$

Use the lengths shown in the diagrams to form this equation.

$$x + y = \frac{1}{5}\sin 2t$$

$$\therefore 25\left(\frac{1}{5}\sin 2t - x\right) = 0.5\ddot{x}$$

$$\ddot{x} + 50x = 10\sin 2t$$

$$\text{or } \frac{d^2x}{dt^2} + 50x = 10\sin 2t$$

b Auxiliary equation: $m^2 + 50 = 0$

$$m = \pm 5i\sqrt{2}$$

Now solve the differential equation using the methods of book FP2 Chapter 5.

Complementary function:

$x = A\cos 5\sqrt{2}t + B\sin 5\sqrt{2}t$

Particular integral:

Try : $x = P\cos 2t + Q\sin 2t$

$\dot{x} = -2P\sin 2t + 2Q\cos 2t$

$\ddot{x} = -4P\cos 2t - 4Q\sin 2t$

$\therefore -4P\cos 2t - 4Q\sin 2t + 50(P\cos 2t + Q\sin 2t)$

$= 10\sin 2t$

$\Rightarrow 46Q = 10 \quad Q = \frac{10}{46} = \frac{5}{23}$

$P = 0$

Equate coefficients of $\sin 2t$ and $\cos 2t$.

$\therefore$ Complete solution is

$$x = A\cos 5\sqrt{2}t + B\sin 5\sqrt{2}t + \frac{5}{23}\sin 2t$$

Use the initial conditions given in the question to obtain values for A and B.

$t = 0, x = 0 \Rightarrow 0 = A$

$$\dot{x} = 5\sqrt{2}B\cos 5\sqrt{2}t + \frac{10}{23}\cos 2t$$

$t = 0, \dot{x} = 0 \quad 0 = 5\sqrt{2}B + \frac{10}{23}$

$$B = -\frac{\sqrt{2}}{23}$$

$$\therefore x = \frac{5}{23}\sin 2t - \frac{\sqrt{2}}{23}\sin 5\sqrt{2}t$$

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Damped and forced harmonic motion
Exercise C, Question 2

Question:

A particle P of mass m is moving in a straight line. At time t the displacement of P from a fixed point O of the line is x. Given that x satisfies the differential equation $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + n^2x = 0$ where k and n are positive constants with $k < n$,

a find an expression for x in terms of k, n and t.

b Write down the period of the motion.

Solution:

a $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + n^2x = 0$ ← Solve the equation using the methods of book FP2 Chapter 5.

Auxiliary equation: $m^2 + 2km + n^2 = 0$

$$m = \frac{-2k \pm \sqrt{(4k^2 - 4n^2)}}{2}$$

$$m = -k \pm \sqrt{(k^2 - n^2)}$$

$0 < k < n \Rightarrow k^2 - n^2 < 0$

$$\therefore m = -k \pm i\sqrt{(n^2 - k^2)}$$

General solution:

$$x = e^{-kt}(A\cos\sqrt{(n^2 - k^2)}t + B\sin\sqrt{(n^2 - k^2)}t)$$

b Period $= \frac{2\pi}{\sqrt{(n^2 - k^2)}}$

[You can write the general solution in its alternative form $x = A'e^{-kt}\cos(\omega t + \varepsilon)$ where $\omega = \sqrt{(n^2 - k^2)}$ if you prefer.]

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Damped and forced harmonic motion
Exercise C, Question 3

Question:

A particle P of mass m is attached to one end of light elastic spring of natural length l and modulus of elasticity $2mk^2l$. The other end of the spring is attached to a fixed point A and P is hanging in equilibrium with AP vertical.

a Find the length of the spring.

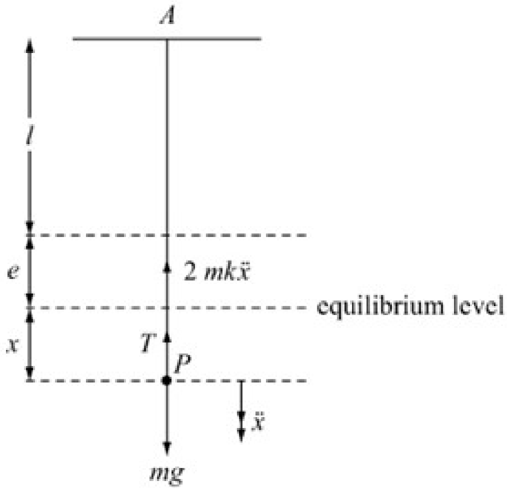
The particle is now projected vertically downwards from its equilibrium position with speed U. A resistance of magnitude $2mkv$, where v is the speed of P, acts on P. At time $t, t > 0$, the displacement of P from its equilibrium position is x.

b Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + 2k^2x = 0$.

c Show that P is instantaneously at rest when $kt = (n + \frac{1}{4})\pi$, where $n \in \mathbb{N}$

d Sketch the graph of x against t.

Solution:



$\lambda = 2mk^2 l$

a In equilibrium: $R(\uparrow) T = mg$

Hooke's law $T = \frac{\lambda x}{l} = 2mk^2 e$

$\therefore mg = 2mk^2 e$ ①

$e = \frac{g}{2k^2}$

The length of the spring is $l + \frac{g}{2k^2}$

b Hooke's law: $T = \frac{2mk^2 l}{l}(x + e)$

$F = ma : mg - T - 2mk\dot{x} = m\ddot{x}$

$g - 2k^2(x + e) - 2k\dot{x} = m\ddot{x}$

$g - 2k^2 x - g - 2k\dot{x} = \ddot{x}$

$\ddot{x} + 2k\dot{x} + 2k^2 x = 0$

or $\frac{d^2 x}{dt^2} + 2k\frac{dx}{dt} + 2k^2 x = 0$

From ① $2k^2 e = g$

c Auxiliary equation: $m^2 + 2km + 2k^2 = 0$

$m = \frac{-2k \pm \sqrt{(4k^2 - 8k^2)}}{2}$

$m = \frac{-2k \pm \sqrt{(-4k^2)}}{2}$

$m = -k \pm k\text{i}$

An expression for x must be found in order to answer parts **c** and **d**. Use the methods of book FP2 chapter 5 to solve the differential equation.

General solution:

$$x = e^{-kt}(A\cos kt + B\sin kt)$$

$t = 0,\ x = 0 \Rightarrow 0 = A$

$$\dot{x} = -ke^{-kt}B\sin kt + Be^{-kt}k\cos kt$$

$t = 0,\ \dot{x} = U \Rightarrow U = Bk$

$$B = \frac{U}{k}$$

$$\therefore x = e^{-kt}\frac{U}{k}\sin kt$$

$$\dot{x} = -ke^{-kt}\frac{U}{k}\sin kt + \frac{U}{k}e^{-kt}k\cos kt$$

$$\dot{x} = 0 \quad Ue^{-kt}(\sin kt - \cos kt) = 0$$

$$\sin kt = \cos kt$$

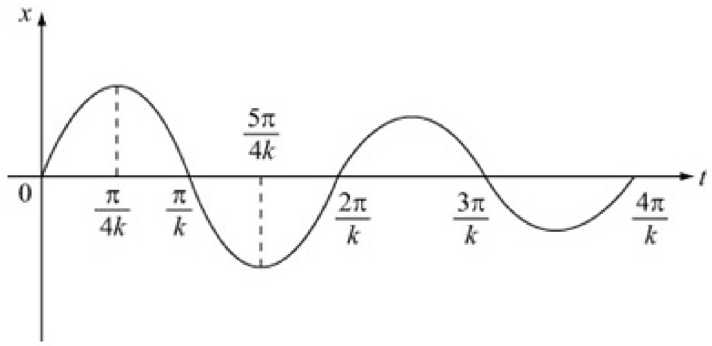
$$\tan kt = 1$$

$$kt = \frac{\pi}{4} + n\pi$$

$$kt = \left(n + \frac{1}{4}\right)\pi, n \in \mathbb{N}$$

Use the initial conditions given in the question to obtain expressions for A and B.

d



From **c**, the maxima and minima occur when $kt = \left(n + \frac{1}{4}\right)\pi$.
Multiplying $\sin kt$ by e^{-kt} causes the amplitude to decrease as t increases.

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Damped and forced harmonic motion
Exercise C, Question 4

Question:

A particle P of mass m is attached to one end of light elastic spring of natural length l and modulus of elasticity mn^2l. The other end of the spring is attached to the roof of a stationary lift. The particle is hanging in equilibrium with the spring vertical. At time $t = 0$ the lift starts to move vertically upwards with constant speed U. At time $t, t > 0$, the displacement of P from its initial position is x.
By considering the extension in the spring,

a show that $\frac{d^2x}{dt^2} + n^2x = n^2Ut$,

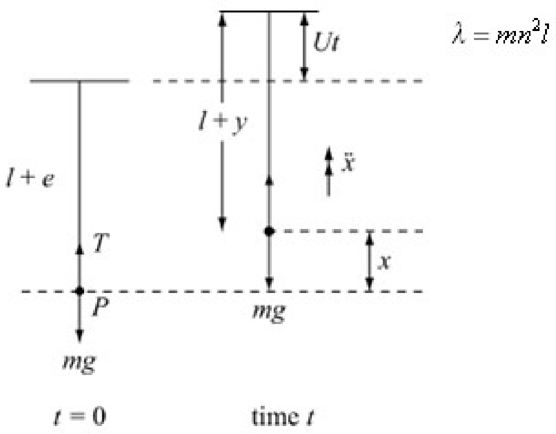
b find an expression for x in terms of t and n.

At time $t = T$, the particle is instantaneously at rest. Find

c the smallest value of T,

d the displacement of P from its initial position at this time.

Solution:



a Let the extension in the spring at time t be y.

$l+y+x=l+e+Ut$

$y+x=e+Ut$ ①

Use the distances on the diagrams to form this equation.

When $t=0$, P is in equilibrium

R$(\uparrow)$ $T=Mg$

Hooke's Law: $T=\frac{\lambda x}{l}=\frac{mn^2 l}{l}\times e$

$\therefore mn^2 e=mg$ ②

At time t:

Hooke's law: $T=\frac{mn^2 l}{l}y$

$F=ma$

$T-mg=m\ddot{x}$

$mn^2 y-mg=m\ddot{x}$

Using ①:

$mn^2(e+Ut-x)-mg=m\ddot{x}$

From ① $y=e+Ut-x$

Using ②:

$mg+mn^2Ut-mn^2x-mg=m\ddot{x}$

$\ddot{x}+n^2x=n^2Ut$

or $\frac{d^2x}{dt^2}+n^2x=n^2Ut$

From ② $mn^2e=mg$

b Auxiliary equation:

$m^2 + n^2 = 0$

$m = \pm in$

Solve the differential equation using the methods of book FP2 Chapter 5.

Complementary function:

$x = A\cos nt + B\sin nt$

Particular integral:

try $x = Ct + D$

$\dot{x} = C$

$\ddot{x} = 0$

$\therefore n^2(Ct + D) = n^2Ut$

$C = U \quad D = 0$

Complete solution:

$x = A\cos nt + B\sin nt + Ut$

$t = 0,\ x = 0 \Rightarrow A = 0$

$\dot{x} = Bn\cos nt + U$

Use the initial conditions given in the question to obtain expressions for A and B.

$t = 0,\ \dot{x} = 0 \Rightarrow 0 = Bn + U$

$B = -\frac{U}{n}$

$\therefore x = Ut - \frac{U}{n}\sin nt$

c $\dot{x} = Bn\cos ut + U$

$\dot{x} = U - U\cos nt$

From **b**.

$\dot{x} = 0 \quad 0 = 1 - \cos nt$

$\cos nt = 1$

$nt = 0, 2\pi, \ldots$

$\therefore$ Smallest T is $\frac{2\pi}{n}$

d $t = \frac{2\pi}{n} \Rightarrow x = U \times \frac{2\pi}{n} - \frac{U}{n}\sin 2\pi$

$x = \frac{2U\pi}{n} - 0$

P has moved a distance $\frac{2U\pi}{n}$ when it first comes to rest.

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Damped and forced harmonic motion
Exercise C, Question 5

Question:

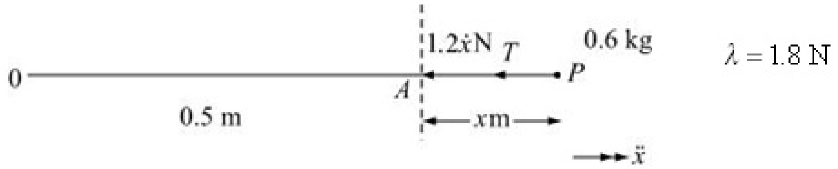
A particle P of mass 0.6 kg is attached to one end of light elastic spring of natural length 0.5 m and modulus of elasticity 1.8 N. The other end of the spring is attached to a fixed point O of the horizontal table on which P lies. At time $t = 0, P$ is at the point A, where $OA = 0.5\text{ m}$. The particle is then projected in the direction OA with speed 6 m s^{-1}. The particle is subject to a resistance of magnitude $1.2v$ N, where $v\text{ m s}^{-1}$ is the speed of P. At time t seconds the extension in the spring is x metres.

a Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 6x = 0$.

b Find x in terms of t.

c Find the value of t the first time P comes to instantaneous rest.

Solution:



a Hooke's Law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{1.8x}{0.5} = 3.6x$$

$F = ma$:

$$T + 1.2\dot{x} = -0.6\,\ddot{x}$$
$$3.6x + 1.2\dot{x} = -0.6\ddot{x}$$
$$\ddot{x} + 2\dot{x} + 6x = 0$$

or $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 6x = 0$

b Auxiliary equation: $m^2 + 2m + 6 = 0$

$$m = \frac{-2 \pm \sqrt{(4-24)}}{2}$$

$$m = -1 \pm i\sqrt{5}$$

Solve the differential equation using the methods of book FP2 Chapter 5.

General solution:

$$x = e^{-t}\left(A\cos\sqrt{5}t + B\sin\sqrt{5}t\right)$$

$t = 0,\ x = 0 \Rightarrow 0 = A$

$$\dot{x} = -e^{-t}B\sin\sqrt{5}t + e^{-t}\sqrt{5}B\cos\sqrt{5}t$$

$t = 0,\ \dot{x} = 6 \Rightarrow 6 = \sqrt{5}B$

$$B = \frac{6}{\sqrt{5}}$$

$$\therefore x = \frac{6}{\sqrt{5}}e^{-t}\sin\sqrt{5}t$$

Use the initial conditions given in the question to obtain values for A and B.

c $\dot{x} = -\frac{6}{\sqrt{5}}e^{-t}\sin\sqrt{5}t + 6e^{-t}\cos\sqrt{5}t$

$6e^{-t} \neq 0$

$$\dot{x} = 0 \Rightarrow \frac{1}{\sqrt{5}}\sin\sqrt{5}t = \cos\sqrt{5}t$$

$\tan\sqrt{5}t = \sqrt{5}$

$$t = \frac{1}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

$$t = 0.5144\ldots$$

P first comes to instantaneous rest when $t = 0.514$ s (3 s.f.)

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Damped and forced harmonic motion
Exercise C, Question 6

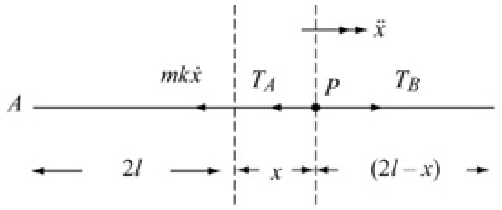
Question:

A particle P of mass m is attached to one end of each of two identical elastic strings of natural length l and modulus of elasticity $2mg$. The free ends of the strings are fixed at points A and B on a smooth horizontal plane where $AB = 4l$. At time $t = 0, P$ is at rest at its equilibrium position. The particle is then projected along the line AB with speed U and moves in a straight line. At time t the displacement of P from its equilibrium position is x. A resistance of magnitude mkv, where v is the speed of P and $k = \sqrt{\frac{g}{l}}$, acts on P. Both strings remain taut throughout the motion.

a Show that $\frac{d^2x}{dt^2} + k\frac{dx}{dt} + 4k^2x = 0$.

b Find an expression for x in terms of U, k, and t.

Solution:



$\lambda = 2mg$

a Hooke's law: $T = \frac{\lambda x}{l}$

$T_A = \frac{2mg(l+x)}{l}, T_B = \frac{2mg(l-x)}{l}$

$F = ma$:

$$T_B - T_A - mk\dot{x} = m\ddot{x}$$

$$\frac{2mg(l-x)}{l} - \frac{2mg(l+x)}{l} - mk\dot{x} = m\ddot{x}$$

$$-4\frac{mgx}{l} - mk\dot{x} = m\ddot{x}$$

$$\ddot{x} + k\dot{x} + \frac{4gx}{l} = 0$$

$$\text{or} \quad \frac{d^2x}{dt^2} + k\frac{dx}{dt} + 4k^2x = 0$$

$$\text{where } k = \sqrt{\frac{g}{l}}$$

b Auxiliary equation: $m^2 + km + 4k^2 = 0$

$$m = \frac{-k \pm \sqrt{(k^2 - 16k^2)}}{2}$$

$$m = \frac{-k \pm ik\sqrt{15}}{2}$$

General solution:

$$x = e^{-\frac{kt}{2}}(A\cos k\frac{\sqrt{15}}{2}t + B\sin k\frac{\sqrt{15}}{2}t)$$

$$t = 0, x = 0 \Rightarrow A = 0$$

$$\dot{x} = -\frac{k}{2}e^{-\frac{kt}{2}}B\sin\frac{k\sqrt{15}}{2}t + e^{-\frac{kt}{2}}\frac{k\sqrt{15}}{2}B\cos\frac{k\sqrt{15}}{2}t$$

$$t = 0, \dot{x} = U \Rightarrow U = \frac{Bk\sqrt{15}}{2}$$

$$B = \frac{2U}{k\sqrt{15}}$$

$$\therefore x = \frac{U}{k\sqrt{15}}e^{\frac{-kt}{2}}\sin\frac{k\sqrt{15}}{2}t$$

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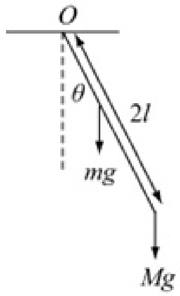
Stability
Exercise A, Question 1

Question:

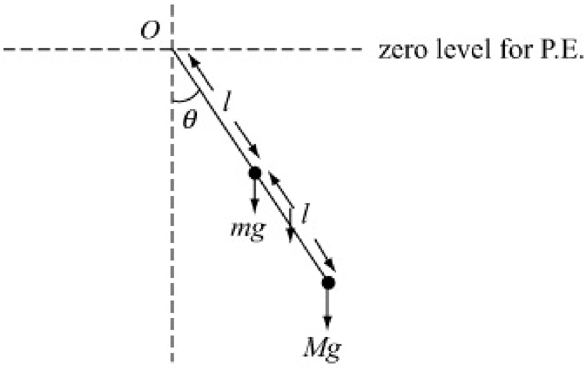
A pendulum is modelled as a uniform rod of mass m and length $2l$ attached to a particle of mass M. The pendulum is smoothly hinged at one end to a fixed point O, as shown in the figure.



a Express the potential energy of the system in terms of θ, the angle which the pendulum makes with the vertical through O.

b Show that there are two positions of equilibrium and determine whether they are stable or unstable.

Solution:



a Take the horizontal level through O as the zero level for potential energy – as O is fixed.

P.E. for rod $= -mgl\cos\theta$

P.E. for particle $= -Mg2l\cos\theta$

$\therefore V = -mgl\cos\theta - 2Mgl\cos\theta$

b $\frac{dV}{d\theta} = mgl\sin\theta + 2Mgl\sin\theta$

Put $\frac{dV}{d\theta} = 0$. Then $\sin\theta = 0 \Rightarrow \theta = 0$ or π

$\frac{d^2V}{d\theta^2} = mgl\cos\theta + 2Mgl\cos\theta$

when $\theta = 0, \frac{d^2V}{d\theta^2} = mgl + 2Mgl > 0$ $\therefore$ Equilibrium is stable at the point of minimum potential energy, when $\theta = 0$.

When $\theta = \pi, \frac{d^2V}{d\theta^2} = -mgl - 2Mgl < 0$ $\therefore$ Equilibrium is unstable when $\theta = \pi$. (This is a point of maximum potential energy.)

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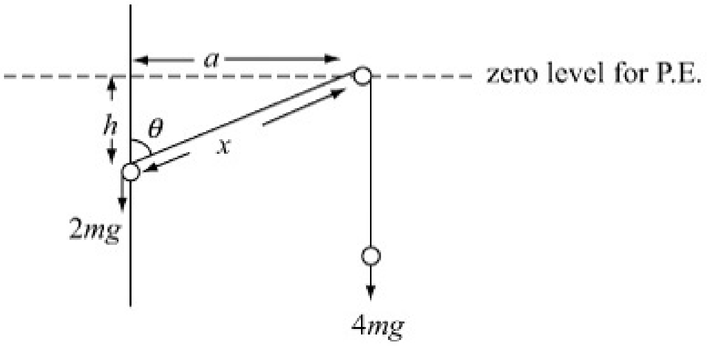
Stability
Exercise A, Question 2

Question:

A small smooth pulley is fixed at a distance a from a fixed smooth vertical wire. A ring of mass $2m$ is free to slide on the wire. It is attached to one end of a string which passes over the pulley and carries a load of mass $4m$ hanging from the other end. The angle between the sloping part of the string and the vertical is θ.

By expressing the potential energy in terms of θ find how far the ring is below the pulley in the equilibrium position and determine whether the equilibrium is stable or unstable.

Solution:



Take the horizontal level through the pulley as the zero level for potential energy as the pulley is fixed.

P.E. for ring $= -2mgh$

But $\tan\theta = \frac{a}{h}$, so $h = \frac{a}{\tan\theta}$ or $a\cot\theta$ ①

So P.E. for ring $= -2mga\cot\theta$

P.E. for load $= -4mg(l-x)$, where l is the length of the string.

But $\sin\theta = \frac{a}{x}$, so $x = \frac{a}{\sin\theta}$ or $a\operatorname{cosec}\theta$

$\therefore$P.E. for load $= -4mg(l - a\operatorname{cosec}\theta)$

$\therefore$Total P.E. for system $V = -2mga\cot\theta + 4mga\operatorname{cosec}\theta + k$ where k is constant.

For equilibrium $\frac{dV}{d\theta} = 0$

But $\frac{dV}{d\theta} = 2mga\operatorname{cosec}^2\theta - 4mga\operatorname{cosec}\theta\cot\theta$ ②

when $\frac{dV}{d\theta} = 0$, $\operatorname{cosec}\theta = 0$ or $\cot\theta = \frac{1}{2}\operatorname{cosec}\theta$

But $\operatorname{cosec}\theta \neq 0$, for any value of θ

So $\frac{\cos\theta}{\sin\theta} = \frac{1}{2\sin\theta}$

$\therefore \cos\theta = \frac{1}{2}$ and $\theta = \frac{\pi}{3}$.

But $h = a\cot\theta = \frac{a}{\sqrt{3}}$ (from ①)

i.e. the ring is a distance $\frac{a}{\sqrt{3}}$ below the pulley in the equilibrium position.

Differentiate equation ②

$$\frac{d^2V}{d\theta^2} = -4mga\operatorname{cosec}^2\theta\cot\theta + 4mga\operatorname{cosec}^3\theta + 4mga\operatorname{cosec}\theta\cot^2\theta$$

Substitute $\theta = \frac{\pi}{3}$, then as $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$ and $\operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}$

Then $\frac{d^2V}{d\theta^2} = \frac{-16mga}{3\sqrt{3}} + \frac{32mga}{3\sqrt{3}} + \frac{8mga}{3\sqrt{3}} > 0$

$\therefore$ There is a position of stable equilibrium when $h = \frac{a}{\sqrt{3}}$.

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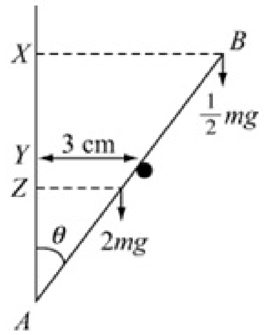
Stability
Exercise A, Question 3

Question:

The diagram shows a uniform rod AB of length 40 cm and mass $2m$ resting with its end A in contact with a smooth vertical wall. The rod is supported by a smooth horizontal rod which is fixed parallel to the wall and a distance 3 cm from the wall as shown in the figure. A particle of mass $\frac{1}{2}m$ is attached to the rod at B.

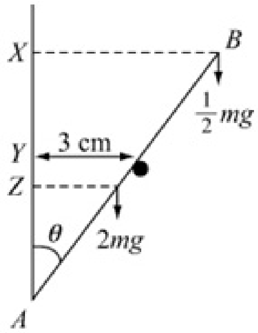


a Show that when AB makes an angle θ with the vertical the potential energy is given by
$V = 0.6mg\cos\theta - 0.075mg\cot\theta + \text{constant}$.

b Find any positions of equilibrium and establish whether they are stable or unstable.

Solution:

a Take the horizontal level through the support rod as the zero level for potential energy – as the support rod is fixed



The P.E. for particle $= \frac{1}{2}mg \times XY$

But $XY = AX - AY$

$= 0.4\cos\theta - \dfrac{0.03}{\tan\theta}$

$\therefore$ P.E. for particle $= 0.2mg\cos\theta - 0.015mg\cot\theta$

The P.E. for the rod $= -2mg \times YZ$

But $YZ = AY - AZ$

$= 0.03\cot\theta - 0.2\cos\theta$

$\therefore$ P.E. for rod $= -0.06mg\cot\theta + 0.4mg\cos\theta$

$\therefore$ Total P.E. $= V = 0.2mg\cos\theta - 0.015mg\cot\theta - 0.06mg\cot\theta + 0.4mg\cos\theta$

$= 0.6mg\cos\theta - 0.075mg\cot\theta$

b $\dfrac{dV}{d\theta} = -0.6mg\sin\theta + 0.075mg\operatorname{cosec}^2\theta$

Put $\dfrac{dV}{d\theta} = 0$. Then

$0.6\sin\theta = \dfrac{0.075}{\sin^2\theta}$

$\therefore \sin^3\theta = \dfrac{0.075}{0.6}$

$= \dfrac{1}{8}$

$\therefore \sin\theta = \dfrac{1}{2}$

So $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{5\pi}{6}$ correspond to positions of equilibrium.

$\dfrac{d^2V}{d\theta^2} = -0.6mg\cos\theta - 0.15mg\operatorname{cosec}^2\theta\cot\theta$

when $\theta = \dfrac{\pi}{6}, \dfrac{d^2V}{d\theta^2} = -\dfrac{9\sqrt{3}}{10}mg < 0$ so unstable

when $\theta = \dfrac{5\pi}{6}, \dfrac{d^2V}{d\theta^2} = \dfrac{9\sqrt{3}}{10}mg > 0$ so stable

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

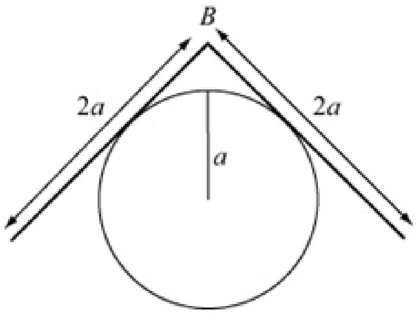
Stability

Exercise A, Question 4

Question:

Two uniform smooth heavy rods, each of mass M and length $2a$, are smoothly jointed together at B. They are placed symmetrically in a vertical plane, over a fixed sphere of radius a as shown.

a Show that when the rods make an angle θ with the horizontal the potential energy V is given by $V = 2Mga(\sec\theta - \sin\theta) + \text{constant}$.

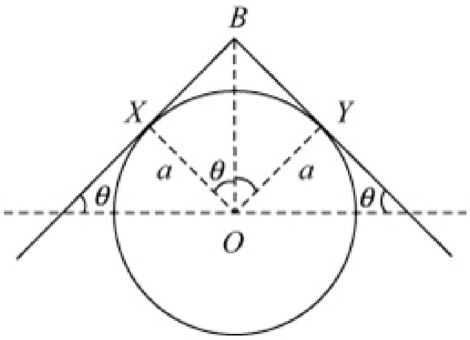


Hint: use the horizontal plane through the centre of the sphere as the zero level for the potential energy.

b Show that the rods are in equilibrium if $\cos^3\theta = \sin\theta$ and verify that $\theta = 0.60$ is accurate as a solution to 2 s.f.

Solution:

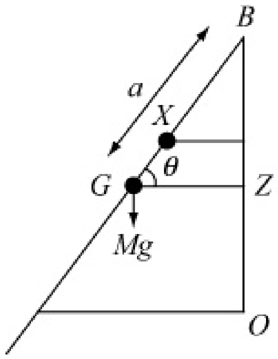
a



Take the horizontal level through the centre of the sphere O as the zero level for potential energy. Let the rods touch the sphere at points X and Y.

From geometry $X\hat{O}B = Y\hat{O}B = \theta$. ($O\hat{X}B = O\hat{Y}B = 90°$ angle between tangent and radius.)

$\therefore BO = \dfrac{a}{\cos\theta} = a\sec\theta$



Consider one of the rods. Let its mid-point be G.
Then potential energy of rod $= Mg \times OZ$.

But $OZ = OB - BZ$

$= a\sec\theta - a\sin\theta$

$\therefore$P.E. of rod $= Mg(a\sec\theta - a\sin\theta)$

As there are two symmetric rods in the system

$V = 2Mg\left(a\sec\theta - a\sin\theta\right)$

[The constant here is zero but if you chose the base of the sphere as the zero level for P.E. then you would have a constant $2Mga$.]

b For equilibrium $\dfrac{dV}{d\theta} = 0$

But $\dfrac{dV}{d\theta} = +2Mga\sec\theta\tan\theta - 2Mga\cos\theta$

$\therefore 2Mga \times \dfrac{1}{\cos\theta} \times \dfrac{\sin\theta}{\cos\theta} = 2Mga\cos\theta$

$\therefore \sin\theta = \cos^3\theta$

If 0.60 is accurate to 2 s.f. there should be a sign change when substituting 0.595 and 0.605 into $f(\theta) = \sin\theta - \cos^3\theta$

$f(0.595) = -7.46 \times 10^{-3} < 0$

$f(0.605) = 0.012 > 0$

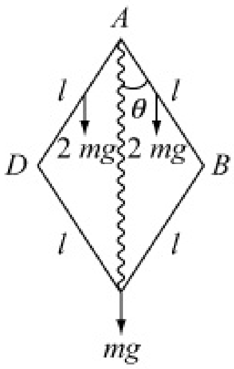
Sign change $\therefore$ 0.60 is a solution accurate to 2 s.f.

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Stability
Exercise A, Question 5

Question:

Four light rods each of length l are freely hinged at their ends to form a rhombus $ABCD$ which is suspended from point A.
A light spring of natural length l and modulus of elasticity $10mg$ connects the points A and C.
A particle of mass m is attached at point C and the rods AB and AD each carry a particle of mass $2m$ at their mid-points. C moves freely in a vertical line through A and the angle between AB and the downward vertical is θ.



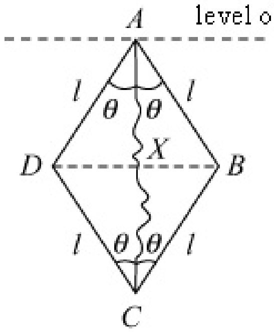
a Show that the potential energy of the system V is given by
$V = mgl(20\cos^2\theta - 24\cos\theta) + \text{constant}$.

b Find the values of θ which correspond to positions of equilibrium.

c Determine whether these values correspond to stable or to unstable equilibrium.

Solution:

a



Take the horizontal through A as the zero level for potential energy.
Use symmetry to mark all the equal angles in the figure.
Let the diagonals meet at X.
From the isosceles $\Delta ADC, \Delta ADX$ is right-angled

$\therefore AX = l\cos\theta \Rightarrow AC = 2l\cos\theta$

$\therefore$ Extension x of the elastic string $AC = 2l\cos\theta - l$

The total P.E. of the system is V where

$$V = -2mg\frac{l}{2}\cos\theta - 2mg\frac{l}{2}\cos\theta - mg(AC) + \frac{1}{2}\lambda\frac{x^2}{l}$$

i.e. $V = -mgl\cos\theta - mgl\cos\theta - 2mgl\cos\theta + 5mg\dfrac{(2l\cos\theta - l)^2}{l}$

$$= -4mgl\cos\theta + 5mgl(4\cos^2\theta - 4\cos\theta + 1)$$

$$= mgl(20\cos^2\theta - 24\cos\theta) + \text{constant}$$

b $\dfrac{dV}{d\theta} = mgl\left[-40\cos\theta\sin\theta + 24\sin\theta\right]$

Put $\dfrac{dV}{d\theta} = 0$, then $8(3\sin\theta - 5\sin\theta\cos\theta) = 0$

i.e. $8\sin\theta(3 - 5\cos\theta) = 0$

$\therefore \sin\theta = 0$ or $\cos\theta = \dfrac{3}{5}$

$\therefore \theta = 0$ or 0.93 radians (2 s.f.)

c As $\dfrac{dV}{d\theta} = mgl\left[-20\sin 2\theta + 24\sin\theta\right]$

$$\frac{d^2V}{d\theta^2} = mgl\left[-40\cos 2\theta + 24\cos\theta\right]$$

when $\theta = 0, \dfrac{d^2V}{d\theta^2} = -16mgl < 0 \quad \therefore$ unstable equilibrium

when $\theta = 0.93^c\ \dfrac{d^2V}{d\theta^2} = mgl\left[-40\times\dfrac{-7}{25} + 24\times\dfrac{3}{5}\right]$

$$= \frac{128}{5}mgl > 0 \ \therefore \text{stable equilibrium.}$$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Stability
Exercise A, Question 6

Question:

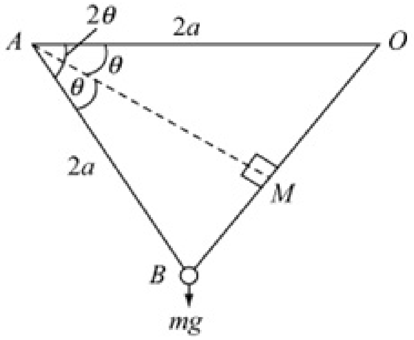
A light rod AB of length $2a$ can turn freely in a vertical plane about a smooth fixed hinge at A. A particle of mass m is attached at point B. One end of a light elastic string, of natural length $\frac{3}{2}a$ and modulus of elasticity $mg\sqrt{3}$ is also attached to the rod at B. The other end of the string is attached to a fixed point O at the same horizontal level as A. Given that $OA = 2a$ and that the angle between AB and the horizontal is 2θ,

a show that, provided the string remains taut, the potential energy of the system is given by $V = -2mga(\sin 2\theta + \frac{4}{3}\sqrt{3}\cos 2\theta + 2\sqrt{3}\sin\theta) + \text{constant}$.

b Verify that there is a position of equilibrium in which $\theta = \frac{\pi}{6}$ and determine the stability of this equilibrium.

Solution:

a



……level of zero potential energy.

$$\text{P.E. of particle} = -mg \times 2a \sin 2\theta$$

$$\text{P.E. of string} = \frac{1}{2} mg\sqrt{3}\frac{x^2}{\frac{3}{2}a} = \frac{1}{3} mg\sqrt{3}\frac{x^2}{a}$$

But from the isosceles triangle OAB length $OB = 2 \times BM = 2 \times 2a \sin\theta$

$\therefore$ Extension $x = 4a\sin\theta - \frac{3a}{2}$

$$\therefore \text{Total P.E., } V = -2mga \sin 2\theta + \frac{1}{3} mg\sqrt{3}a\left[4\sin\theta - \frac{3}{2}\right]^2$$

$$\text{i.e. } V = -2mga\left[\sin 2\theta - \frac{1}{6}\sqrt{3}\left(16\sin^2\theta - 12\sin\theta + \frac{9}{4}\right)\right]$$

$$= -2mga\left[\sin 2\theta - \frac{1}{6}\sqrt{3}\left(8 - 8\cos 2\theta - 12\sin\theta + \frac{9}{4}\right)\right]$$

$$= -2mga\left[\sin 2\theta + \frac{4}{3}\sqrt{3}\cos 2\theta + 2\sqrt{3}\sin\theta\right] + \text{constant}$$

b $\frac{dV}{d\theta} = -2mga\left[2\cos 2\theta - \frac{8}{3}\sqrt{3}\sin 2\theta + 2\sqrt{3}\cos\theta\right]$

When $\theta = \frac{\pi}{6}, \frac{dV}{d\theta} = -2mga\left[2\cos\frac{\pi}{3} - \frac{8}{3}\sqrt{3}\sin\frac{\pi}{3} + 2\sqrt{3}\cos\frac{\pi}{6}\right]$

$$= -2mga\,[1 - 4 + 3] = 0$$

This confirms that $\theta = \frac{\pi}{6}$ gives a position of equilibrium.

$\frac{d^2V}{d\theta^2} = -2mga\left[-4\sin 2\theta - \frac{16}{3}\sqrt{3}\cos 2\theta - 2\sqrt{3}\sin\theta\right]$

when $\theta = \frac{\pi}{6}, \frac{d^2V}{d\theta^2} = -2mga\left[-2\sqrt{3} - \frac{8}{3}\sqrt{3} - \sqrt{3}\right] = \frac{34}{3}\sqrt{3}mga > 0$

$\therefore$ this is a position of stable equilibrium.

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Stability

Exercise A, Question 7

Question:

A small bead B of mass km can slide on a smooth vertical circular wire with centre O and radius a which is fixed in a vertical plane. B is attached to one end of a light elastic string of natural length $\frac{3}{2}a$ and modulus of elasticity $12mg$. The other end of the string is attached to a fixed point A which is vertically above the centre point O of the circular wire.

The angle between the string AB and the downward vertical at A is θ.

a Show that the potential energy V of the system is given by

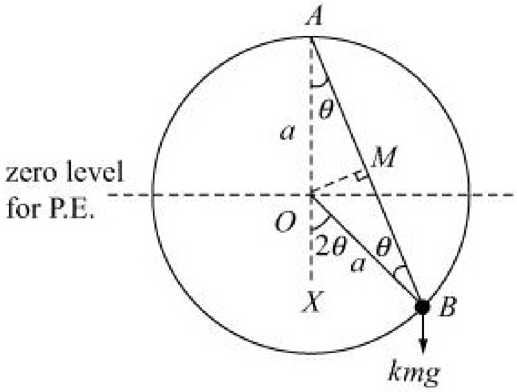
$V = 2mga((8-k)\cos^2\theta - 12\cos\theta) + \text{constant}$.

b Find the restrictions on k if there is only one point of equilibrium, where $\theta = 0$.

c Subject to these restrictions, determine the stability of this equilibrium.

Solution:

a



Note $B\hat{O}X = 2\theta$
(angle at centre $= 2\times$ angle at circumference)
As ΔAOB is isosceles
$O\hat{B}A = O\hat{A}B = \theta$
Also $AB = 2\times AM$, where M is the mid-point of AB.
So $AB = 2\times a\cos\theta = 2a\cos\theta$
Let the extension in the string be x.
Then $x = 2a\cos\theta - \dfrac{3a}{2}$

The potential energy of the bead $B = -kmga\cos 2\theta$

The potential energy of the string $= \dfrac{1}{2}\times 12mg\dfrac{x^2}{3\frac{a}{2}} = 4mg\dfrac{a^2}{a}\left(2\cos\theta - \dfrac{3}{2}\right)^2$

$\therefore$ Total potential energy $V = -kmga\cos 2\theta + 4mga\left(4\cos^2\theta - 6\cos\theta + \dfrac{9}{4}\right)$

i.e. $V = -kmga(2\cos^2\theta - 1) + 16mga\cos^2\theta - 24mga\cos\theta + 9mga$
$= 2mga((8-k)\cos^2\theta - 12\cos\theta) + \text{constant}$

b $\dfrac{dV}{d\theta} = 2mga\left[-2(8-k)\cos\theta\sin\theta + 12\sin\theta\right]$

Put $\dfrac{dV}{d\theta} = 0 \;\therefore 4mga\sin\theta\left[6-(8-k)\cos\theta\right] = 0$

$\therefore \sin\theta = 0$ *or* $\cos\theta = \dfrac{6}{8-k}$

Only one point of equilibrium if $\dfrac{6}{8-k} \geq 1$ i.e. $2 \leq k < 8$

c $\dfrac{d^2V}{d\theta^2} = 2mga\left[-2(8-k)\cos 2\theta + 12\cos\theta\right]$

When $\theta = 0$

$\dfrac{d^2V}{d\theta^2} = 2mga\left[-2(8-k)+12\right]$
$= 2mga\left[2k-4\right] \geq 0$ as $k \geq 2$

$\therefore$ Equilibrium is stable.

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Edexcel AS and A Level Modular Mathematics

Stability
Exercise A, Question 8

Question:

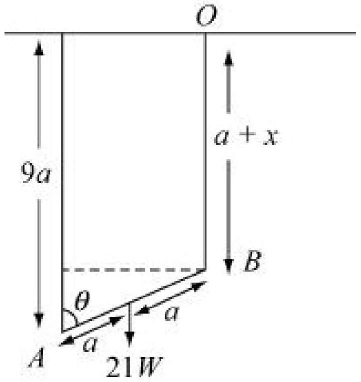
A uniform rod AB of length $2a$ and weight $21W$ is freely pivoted to a fixed support at A. A light elastic string of natural length a and modulus $\frac{3}{2}W$ has one end attached to B and the other to a small ring which is free to slide on a smooth horizontal straight wire passing through a point at a height $9a$ above A.

a Show that when the rod makes an angle θ with the upward vertical at A and the string is vertical, the potential energy of the system is $V = 3Wa\cos\theta(\cos\theta - 1) + \text{constant}$.

b Find the positions of equilibrium and determine whether they are stable or unstable.

Solution:

a



zero level for P.E.

P.E. of rod $= -21W\left(9a - a\cos\theta\right)$

P.E. of string $= \frac{1}{2} \times \frac{3}{2} W \frac{x^2}{a}$

where $a + x = 9a - 2a\cos\theta$

$\therefore x = 8a - 2a\cos\theta$

So total P.E. $V = -21W\left(9a - a\cos\theta\right) + \frac{3W}{4a} a^2 \left(8 - 2\cos\theta\right)^2$

i.e. $V = -189Wa + 21Wa\cos\theta + \frac{3}{4}Wa(64 - 32\cos\theta + 4\cos^2\theta)$

$= 3Wa\cos^2\theta - 24Wa\cos\theta + 21Wa\cos\theta + 48Wa - 189Wa$

$\therefore V = 3Wa\cos\theta(\cos\theta - 1) + \text{constant}$

b $\frac{dV}{d\theta} = 3Wa\cos\theta(-\sin\theta) - 3Wa\sin\theta\left(\cos\theta - 1\right)$

Put $\frac{dV}{d\theta} = 0$, then $-6Wa\cos\theta\sin\theta + 3Wa\sin\theta = 0$

$\therefore \sin\theta\left(1 - 2\cos\theta\right) = 0$

i.e. $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$

$\therefore \theta = 0, \pi$ or $\frac{\pi}{3}$

$\frac{d^2V}{d\theta^2} = -6Wa\cos 2\theta + 3Wa\cos\theta$

When $\theta = 0$ $\frac{d^2V}{d\theta^2} = -3Wa < 0$ $\therefore$ unstable

$\theta = \pi$ $\frac{d^2V}{d\theta^2} = -9Wa < 0$ $\therefore$ unstable

$\theta = \frac{\pi}{3}$ $\frac{d^2V}{d\theta^2} = 3Wa + \frac{3Wa}{2} > 0$ $\therefore$ stable

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Edexcel AS and A Level Modular Mathematics

Stability
Exercise A, Question 9

Question:

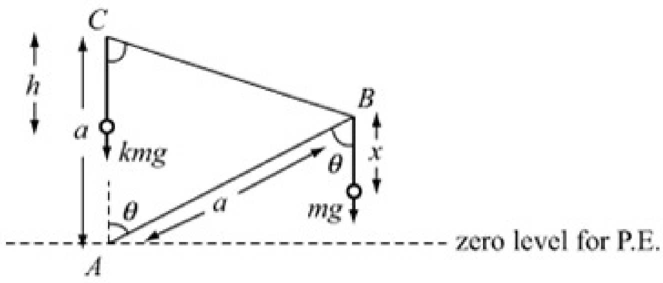
A light rod AB can freely turn in a vertical plane about a smooth hinge at A and carries a mass m hanging from B. A light string of length $2a$ fastened to the rod at B passes over a smooth peg at a point C vertically above A and carries a mass km at its free end. If $AC = AB = a$,

a find the range of values of k for which equilibrium is possible with the rod inclined to the vertical.

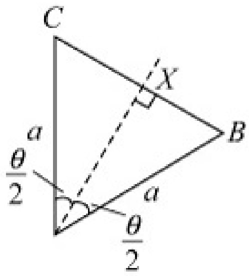
b Given that equilibrium is possible with the rod horizontal find the value of k.

c If the rod is slightly disturbed when horizontal and in equilibrium, determine whether it will return to the horizontal position or not. **[E]**

Solution:



a ΔABC is isosceles and $CB = 2 \times CX$ where



$CX \;\; = a\sin\frac{\theta}{2}$

$\therefore CB = 2a\sin\frac{\theta}{2}$

As the string has length $2a$, $h = 2a - 2a\sin\frac{\theta}{2}$

$\therefore$ Total P.E. $V = +kmg\left(a - \left(2a - 2a\sin\frac{\theta}{2}\right)\right) + mg(a\cos\theta - x)$

where x is constant.

$\therefore V = 2kmga\sin\frac{\theta}{2} + mga\cos\theta + \text{constant}$

For equilibrium, $\frac{dV}{d\theta} = 0$

$$\frac{dV}{d\theta} = kmga\cos\frac{\theta}{2} - mga\sin\theta$$

$$= kmga\cos\frac{\theta}{2} - 2mga\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= mga\cos\frac{\theta}{2}\left(k - 2\sin\frac{\theta}{2}\right)$$

$\therefore$ Equilibrium when $\cos\frac{\theta}{2} = 0$ or when $\sin\frac{\theta}{2} = \frac{k}{2}$ when $\cos\frac{\theta}{2} = 0, \theta = \pi$,

i.e. not inclined to the vertical.

$\therefore \sin\frac{\theta}{2} = \frac{k}{2}$ must have a solution

As $0 < \sin\frac{\theta}{2} < 1$

$\therefore 0 < k < 2$

b When the rod is horizontal $\theta = \frac{\pi}{2}$.

$\therefore k - 2\sin\frac{\pi}{4} = 0$ for equilibrium

i.e. $k = \sqrt{2}$

c $\frac{d^2V}{d\theta^2} = -\frac{kmga}{2}\sin\frac{\theta}{2} - mga\cos\theta$

Substitute $\theta = \frac{\pi}{2}$ and $k = \sqrt{2}$

$\therefore \frac{d^2V}{d\theta^2} = -\frac{mga}{2} < 0$

$\therefore$ unstable so will not return to horizontal position.

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Edexcel AS and A Level Modular Mathematics

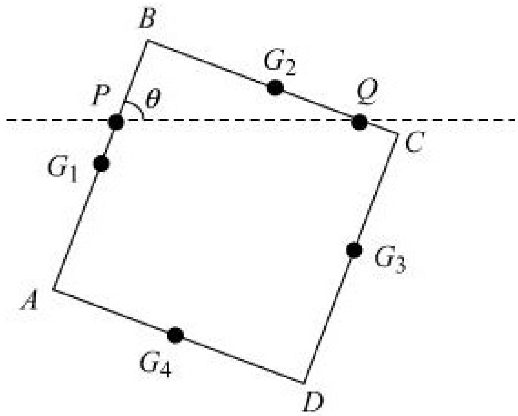
Stability
Exercise A, Question 10

Question:

Four equal uniform rods, each of length $2a$ and each of mass M are rigidly joined together to form a square frame. The frame hangs at rest in a vertical plane on two pegs P and Q which are at the same level as each other.
If $PQ = b$ and the pegs are each in contact with different rods, show that the potential energy V satisfies the equation $V = 2mg(b\sin 2\theta - 2a\sin\theta - 2a\cos\theta)$.
Find the three positions of equilibrium if $b = \sqrt{2}a$ and determine the stability of each of them.

Solution:

a



The horizontal through points P and Q is the zero level for potential energy.
(The mid-point of PQ will be vertically above the centre of the square.)
Label the square $ABCD$.
Let θ be the angle between AB and the horizontal.
Let G_1, G_2, G_3 and G_4 be the mid-points of the four rods as shown.
$BP = b\cos\theta$
$\therefore PG_1 = (a - b\cos\theta)$

$\therefore$ Potential Energy of rod $AB = -Mg(a - b\cos\theta)\sin\theta$
Similarly
$BQ = b\sin\theta$, and so $G_2Q = (b\sin\theta - a)$
$\therefore$ Potential Energy of rod $BC = Mg(b\sin\theta - a)\cos\theta$
Potential Energy of rod $CD = -Mg(2a - b\sin\theta)\cos\theta - Mga\sin\theta$ and
Potential Energy of rod $AD = -Mg(2a - b\cos\theta)\sin\theta - Mga\cos\theta$

$\therefore$ Total Potential Energy

$$\begin{aligned}V &= -Mga\sin\theta + Mgb\cos\theta\sin\theta \\ &\quad + Mgb\cos\theta\sin\theta - Mga\cos\theta \\ &\quad - Mga\sin\theta + Mgb\cos\theta\sin\theta - 2Mga\cos\theta \\ &\quad - 2Mga\sin\theta + Mgb\sin\theta\cos\theta - Mga\cos\theta\end{aligned}$$

i.e. $V = -4Mga\sin\theta + 4Mgb\sin\theta\cos\theta - 4Mga\cos\theta$

$$= 2Mg\left[b\sin 2\theta - 2a\sin\theta - 2a\cos\theta\right]$$

If $b = \sqrt{2}a$

$$V = 2\sqrt{2}Mga\left[\sin 2\theta - \sqrt{2}\sin\theta - \sqrt{2}\cos\theta\right]$$

$$\frac{dV}{d\theta} = 2\sqrt{2}Mga\left[2\cos 2\theta - \sqrt{2}\cos\theta + \sqrt{2}\sin\theta\right]$$

Put $\frac{dV}{d\theta} = 0$

Then $2\cos 2\theta - \sqrt{2}(\cos\theta - \sin\theta) = 0$

$\therefore 2(\cos^2\theta - \sin^2\theta) - \sqrt{2}(\cos\theta - \sin\theta) = 0$

$\therefore \sqrt{2}(\cos\theta - \sin\theta)\left[\sqrt{2}(\cos\theta + \sin\theta) - 1\right] = 0$

$$\therefore \cos\theta = \sin\theta \quad \text{or} \quad \cos\theta + \sin\theta = \frac{1}{\sqrt{2}}$$

$$\text{i.e. } \tan\theta = 1 \quad \text{or} \quad \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\text{i.e. } \theta = \frac{\pi}{4} \quad \text{or } \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{\pi}{3} + \frac{\pi}{4} \quad \text{or} \quad \theta = -\frac{\pi}{3} + \frac{\pi}{4}$$

$$\text{i.e. } \theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{7\pi}{12} \quad \text{or} \quad \theta = \frac{-\pi}{12}$$

$$\frac{d^2V}{d\theta^2} = 2\sqrt{2}Mga\left[-4\sin 2\theta + \sqrt{2}\sin\theta + \sqrt{2}\cos\theta\right]$$

when $\theta = \frac{\pi}{4}$ $\frac{d^2V}{d\theta^2} = 2\sqrt{2}Mga\,[-4+1+1] = -4\sqrt{2}Mga < 0$ $\therefore$ unstable.

when $\theta = \frac{7\pi}{12}$ $\frac{d^2V}{d\theta^2} = 2\sqrt{2}Mga\,[2+1] = 6\sqrt{2}Mga > 0$ $\therefore$ stable.

when $\theta = -\frac{\pi}{12}$ $\frac{d^2V}{d\theta^2} = 2\sqrt{2}Mga\left[+2 + \frac{1}{2} - \frac{\sqrt{3}}{2} + \frac{1}{2} + \frac{\sqrt{3}}{2}\right]$

$= 6\sqrt{2}Mga > 0$ $\therefore$ stable

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Edexcel AS and A Level Modular Mathematics

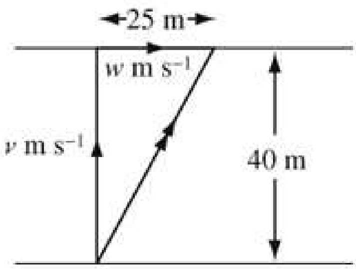
Review Exercise 1
Exercise A, Question 1

Question:

A river of width 40 m flows with uniform and constant speed between straight banks. A swimmer crosses as quickly as possible and takes 30 s to reach the other side. She is carried 25 m downstream.
Find

a the speed of the river,
b the speed of the swimmer relative to the water. **[E]**

Solution:



For the quickest crossing, the swimmer must swim on a course which aims directly across the river; that is the course which is always perpendicular to the river banks.

a Let the speed of the river be w m s^{-1}.

R($\rightarrow$) $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

$$w = \frac{25}{30} = \frac{5}{6}$$

The speed of the river is $\frac{5}{6}$ m s^{-1}.

As the course of the swimmer is perpendicular to the river bank, her only motion parallel to the bank is due to the flow of the river. She moves 25 m downstream in 30 s.

b Let the speed of the swimmer relative to the water be v m s^{-1}.

R($\uparrow$) $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

$$v = \frac{40}{30} = \frac{4}{3}$$

Across the stream, the swimmer moves 40 m in 30 s.

The speed of the swimmer relative to the water is $\frac{4}{3}$ m s^{-1}.

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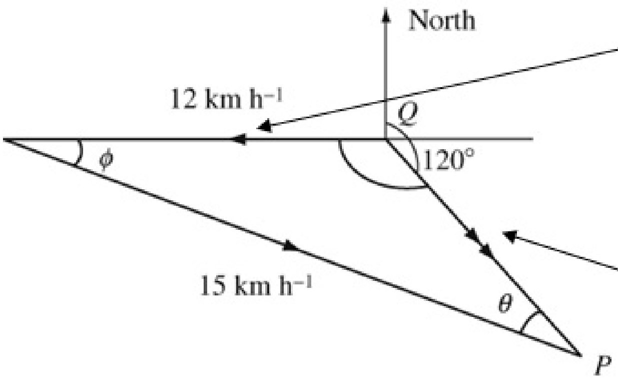
Review Exercise 1
Exercise A, Question 2

Question:

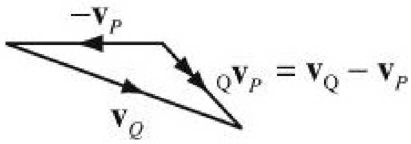
At noon, a boat P is on a bearing of $120°$ from boat Q. Boat P is moving due east at a constant speed of 12 km h^{-1}. Boat Q is moving in a straight line with a constant speed of 15 km h^{-1} on a course to intercept P. Find the direction of motion of Q, giving your answer as a bearing. **[E]**

Solution:



You fix P by introducing the velocity that is 'minus the velocity of P'. This vector represents a velocity equal in magnitude to the velocity of P but in the opposite direction.

This vector represents the velocity of Q relative to P; ${}_Q\mathbf{v}_P = \mathbf{v}_Q - \mathbf{v}_P$.
The diagram can be illustrated as



As Q wants to intercept P, ${}_Q\mathbf{v}_P$ is in direction QP.

Using the sine rule

$$\frac{\sin\theta}{12} = \frac{\sin 150°}{15}$$

$$\sin\theta = \frac{12\sin 150°}{15} = 0.4$$

$\theta = 24°$, to the nearest degree

$\phi = 180° - 150° - \theta$

$\quad = 6°$ (nearest degree)

The direction of motion of Q, as a bearing, is $090° + \phi = 096°$ (nearest degree)

Bearings are measured from north, clockwise, and are usually given to the nearest degree or nearest tenth of a degree.

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Review Exercise 1
Exercise A, Question 3

Question:

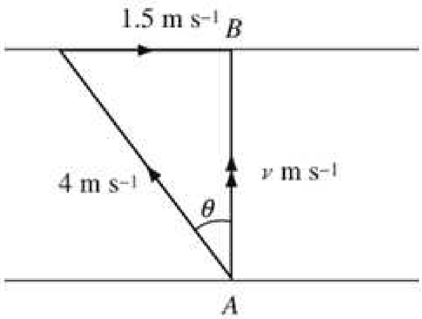
Points A and B are directly opposite each other on the parallel banks of a river. A motorboat, which travels at 4 m s^{-1} relative to the water, crosses from A to B. Given that the distance AB is 400 m and that the river is flowing at 1.5 m s^{-1} parallel to the banks, calculate

a the angle, to the nearest degree, between AB and the direction in which the boat is being steered,

b the speed, in m s^{-1} to 2 significant figures, of the motorboat relative to the bank,

c the time, to the nearest second, taken by the motorboat to cross the river. **[E]**

Solution:



As the motorboat goes directly from A to B, the resultant of the velocity of the motorboat and the velocity of the river must be in the direction AB.

a Let the angle between AB and the direction in which the boat is being steered be θ.

$$\tan\theta = \frac{1.5}{4} = 0.375$$

$$\theta = 21^\circ \text{(nearest degree)}$$

b Let v m s^{-1} be the speed of the motorboat relative to the bank.
Using a Pythagoras relation

$$v^2 = 4^2 - 1.5^2 = 13.75$$

$$v = \sqrt{13.75} \approx 3.71$$

The speed of the motorboat relative to the bank is 3.7 m s^{-1} (2 s.f.)

The triangle of velocities drawn in the diagram is used to answer both parts **a** and **b**. The mathematics involved is elementary trigonometry and Pythagoras' theorem.

c

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{400}{\sqrt{13.75}} \text{ s}$$

$$= 108 \text{ s (nearest second)}$$

The speed found in part **b** is the effective speed with which the motorboat progresses from A to B and the time is found using distance = speed × time.

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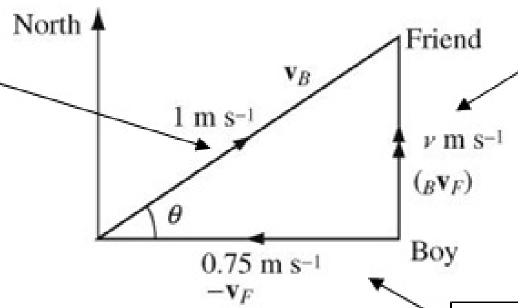
Review Exercise 1
Exercise A, Question 4

Question:

A boy enters a large horizontal field and sees a friend 100 m due north. The friend is walking in an easterly direction at a constant speed of 0.75 m s^{-1}. The boy can walk at a maximum speed of 1 m s^{-1}.
Find the shortest time for the boy to intercept his friend and the bearing on which he must travel to achieve this. **[E]**

Solution:



This is the direction the boy must take.

For interception, the velocity of the boy relative to the friend must be the direction of the line joining the initial position of the boy to the initial position of the friend.

You fix the position of the friend by introducing the velocity that is 'minus the velocity of the friend'. This vector represents a velocity equal in magnitude to the velocity of the friend but in the opposite direction.

Let the speed of the boy relative to the friend be $v \text{ m s}^{-1}$.

$$v^2 = 1^2 - 0.75^2 = 0.4375$$

$$\text{times} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{100}{\sqrt{0.4375}} \text{ s} \approx 151.2 \text{ s}$$

This can be thought of as the 'relative distance', 100 m, divided by the 'relative speed', $\sqrt{0.4375} \text{ m s}^{-1}$.

$$\cos\theta = \frac{0.75}{1} \Rightarrow \theta \approx 41.4°$$

The bearing is $090° - \theta \approx 048.6°$

Bearings are measured from north, clockwise, and are usually given to the nearest degree or nearest tenth of a degree.

The shortest time for the boy to intercept his friend is 151 s (nearest second), and the bearing on which he must travel is 049° (nearest degree).

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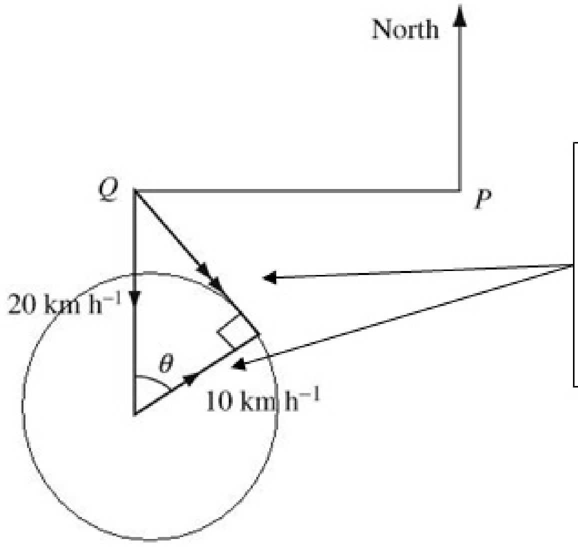
Review Exercise 1
Exercise A, Question 5

Question:

A cyclist P is cycling due north at a constant speed of 20 km h^{-1}. At 12 noon another cyclist Q is due west of P. The speed of Q is constant at 10 km h^{-1}.
Find the course which Q should set in order to pass as close to P as possible, giving your answer as a bearing. **[E]**

Solution:



For the closest approach, the direction of motion of Q, shown here by a vector with a single arrow of magnitude 10 km h^{-1}, must be perpendicular to the velocity of Q relative to P, shown here by the vector with a double arrow.

$$\cos\theta = \frac{10}{20} = \frac{1}{2}$$

$$\theta = 60°$$

The course Q should set is $060°$

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Review Exercise 1
Exercise A, Question 6

Question:

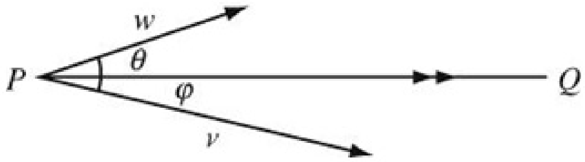
[*In this question* **i** *and* **j** *are horizontal unit vectors due east and due north respectively.*]
An aeroplane makes a journey from a point P to point Q which is due east of P. The wind velocity is $w(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})$, where w is a positive constant. The velocity of the aeroplane relative to the wind is $v(\cos\phi\mathbf{i}-\sin\phi\mathbf{j})$, where v is a constant and $v>w$.
Given that θ and ϕ are both acute angles,

a show that $v\sin\phi=w\sin\theta$,

b find, in terms of v, w and θ, the speed of the aeroplane relative to the ground. **[E]**

Solution:

a



Let ${}_a\mathbf{v}_w$ be the velocity of the aeroplane relative to the wind,

$\mathbf{v}_w$ be the velocity of the wind and

$\mathbf{v}_a$ be the velocity of the aeroplane relative to the ground.

If x is the speed of the aeroplane relative to the ground

$\mathbf{v}_a = x\mathbf{i}$

> As the aeroplane moves from P to Q, that is due east, the velocity of the aeroplane is in the direction of $\mathbf{i}$. The magnitude of the velocity is the speed x and so $\mathbf{v}_a = x\mathbf{i}$.

${}_a\mathbf{v}_w = \mathbf{v}_a - \mathbf{v}_w$

$v(\cos\phi\mathbf{i} - \sin\phi\mathbf{j}) = x\mathbf{i} - w(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$

Equating $\mathbf{j}$ components

$-v\sin\phi = -w\sin\theta$

$v\sin\phi = w\sin\theta$, as required

> By definition, the velocity of A relative to B is given by ${}_A\mathbf{v}_B = \mathbf{v}_A - \mathbf{v}_B$. The specification for M4 requires you to know this formula.

b Equating the $\mathbf{i}$ components

$v\cos\phi = x - w\cos\theta$

$x = v\cos\phi + w\cos\theta$

From part **a**

$\sin\phi = \frac{w}{v}\sin\theta$

$\cos^2\phi = 1 - \sin^2\phi = 1 - \frac{w^2}{v^2}\sin^2\theta$

> The question asks you to find the speed in terms of v, w and θ, so you must eliminate ϕ. You do this using the answer to part **a** and the identity $\sin^2\phi + \cos^2\phi = 1$.

Hence

$$x = v\left(1 - \frac{w^2}{v^2}\sin^2\theta\right)^{\frac{1}{2}} + w\cos\theta$$

$$= \left(v^2 - w^2\sin^2\theta\right)^{\frac{1}{2}} + w\cos\theta$$

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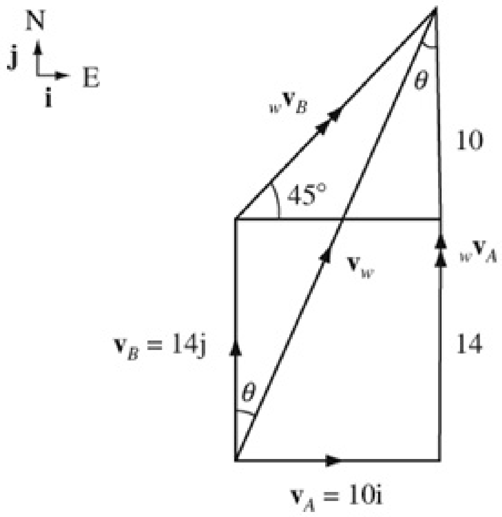
Review Exercise 1
Exercise A, Question 7

Question:

Boat A is sailing due east at a constant speed of 10 km h^{-1}. To an observer on A, the wind appears to be blowing from due south. A second boat B is sailing due north at a constant speed of 14 km h^{-1}. To an observer on B, the wind appears to be blowing from the south west. The velocity of the wind relative to the Earth is constant and is the same for both boats.
Find the velocity of the wind relative to the Earth, stating its magnitude and direction.
[E]

Solution:



If you draw a diagram combining the velocity vector triangles for the velocity of the wind relative to A and the velocity of the wind relative to B, then it is possible just to write down, from the diagram, that the velocity of the wind is $(10\mathbf{i} + 24\mathbf{j})\text{m s}^{-1}$. An alternative solution using vectors is given below.

Let $\mathbf{v}_w$ be the velocity of the wind relative to the ground,
$\mathbf{v}_A$ be the velocity of A and
$_w\mathbf{v}_A$ be the velocity of the wind relative to A.
Taking $\mathbf{i}$ and $\mathbf{j}$ as horizontal unit vectors due east and due north respectively

$_w\mathbf{v}_A = \mathbf{v}_w - \mathbf{v}_A = \lambda\mathbf{j}$, say
$\Rightarrow \mathbf{v}_w - 10\mathbf{i} = \lambda\mathbf{j}$
$\mathbf{v}_w = 10\mathbf{i} + \lambda\mathbf{j}$ ①

From A, the wind appears to blow from the south, so the velocity of the wind relative to A is a multiple of $\mathbf{j}$.

Let $\mathbf{v}_B$ be the velocity of B and
$_w\mathbf{v}_B$ be the velocity of the wind relative to B.

$_w\mathbf{v}_B = \mathbf{v}_B - \mathbf{v}_w = \mu\mathbf{i} + \mu\mathbf{j}$, say
$\Rightarrow \mathbf{v}_w - 14\mathbf{j} = \mu\mathbf{i} + \mu\mathbf{j}$
$\mathbf{v}_w = \mu\mathbf{i} + (\mu + 14)\mathbf{j}$ ②

From B, the wind appears to be blowing from the south west, so the velocity of the wind relative to B must be parallel to $\mathbf{i} + \mathbf{j}$.

Equating the $\mathbf{i}$ components of ① and ②
$\mu = 10$
Hence
$\mathbf{v}_w = 10\mathbf{i} + 24\mathbf{j}$

Substituting $\mu = 10$ into ②.

$|\mathbf{v}_w|^2 = 10^2 + 24^2 = 676 \Rightarrow v_w = \sqrt{676} = 26$

$\tan\theta = \frac{10}{24} \Rightarrow \theta \approx 22.6°$

The velocity of the wind relative to the Earth has magnitude 26 m s^{-1} and is on a bearing 023° (nearest degree).

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Review Exercise 1
Exercise A, Question 8

Question:

Ship A is steaming on a bearing of $060°$ at 30 km h^{-1} and at 9 a.m. it is 20 km due west of a second ship B. Ship B steams in a straight line.

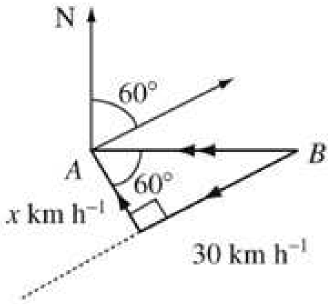
a Find the least speed of B if it is to intercept A.

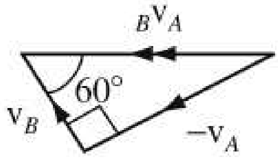
Given that the speed of B is 24 km h^{-1},

b find the earliest time at which it can intercept A. **[E]**

Solution:

a



> The side representing the velocity of B in the triangle of velocity will have the least possible magnitude when it is perpendicular to the side representing the negative of the velocity of A.
>
> 
>

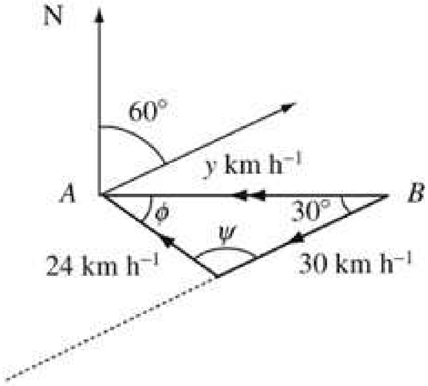
Let the least speed be x km h^{-1}

$$\frac{30}{x} = \tan 60°$$

$$x = \frac{30}{\tan 60°} = \frac{30}{\sqrt{3}} = 10\sqrt{3}$$

The least speed of B if it is to intercept A is $10\sqrt{3}$ km h^{-1}.

b



> The diagram for part **a** must be modified as the velocity of B is no longer perpendicular to path of A. In these circumstances, it is advisable to draw a separate diagram.

Using the sine rule

$$\frac{\sin\phi}{30} = \frac{\sin 30°}{24}$$

$$\sin\phi = \frac{30\sin 30°}{24} = \frac{5}{8}$$

Working to 2 decimal places

$$\phi = 38.68°$$

$$\psi = 180° - 30° - 38.68° = 111.32°$$

> There is a second solution where $\phi \approx 141.32°$ but this would give a smaller value of y and a later time of interception.

Let y km h^{-1} be the magnitude of the velocity of B relative to A.

Using the sine rule

$$\frac{y}{\sin 111.32°} = \frac{24}{\sin 30°}$$

$$y = 44.72$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{20}{44.72}\text{ h} = 0.45\text{ h}$$

$$= 0.45 \times 60\text{ min} = 27\text{ min}$$

> You can think of this as the 'relative distance', 20 km divided by the 'relative speed', 44.72 km h^{-1}.

The earliest time at which B can intercept A is 9.27 a.m. (nearest minute).

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Review Exercise 1
Exercise A, Question 9

Question:

A cyclist C is moving with a constant speed of 10 m s^{-1} due south. Cyclist D is moving with a constant speed of 16 m s^{-1} on a bearing of $240°$.

a Show that the magnitude of the velocity of C relative to D is 14 m s^{-1}.

At 2 p.m., D is 4 km due east of C.

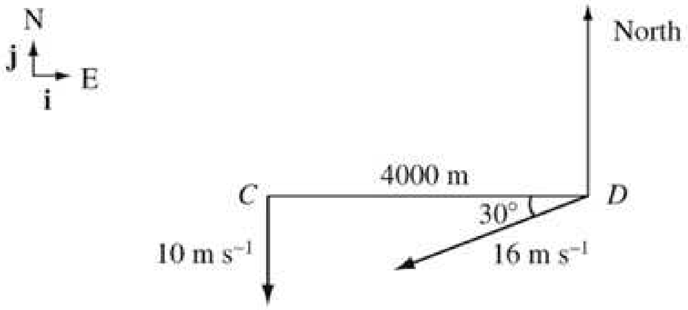
b Find

i the shortest distance between C and D during the subsequent motion,

ii the time, to the nearest minute, at which this shortest distance occurs. **[E]**

Solution:

a



Let **i** and **j** be horizontal unit vectors due east and due north respectively.
Let $\mathbf{v}_C$ m s^{-1} be the velocity of C and $\mathbf{v}_D$ m s^{-1} be the velocity of D.

$\mathbf{v}_C = -10\mathbf{j}$

$\mathbf{v}_D = -16\cos 30°\mathbf{i} - 16\sin 30°\mathbf{j}$

$\quad = -8\sqrt{3}\mathbf{i} - 8\mathbf{j}$

You resolve the velocity of D along the directions of **i** and **j**.

The velocity of C relative to D is given by

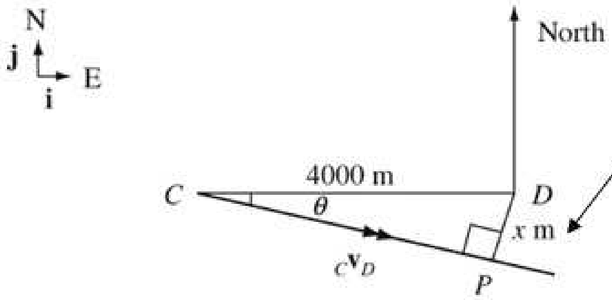
$${}_C\mathbf{v}_D = \mathbf{v}_C - \mathbf{v}_D$$
$$= -10\mathbf{j} - \left(-8\sqrt{3}\mathbf{i} - 8\mathbf{j}\right)$$
$$= 8\sqrt{3}\mathbf{i} - 2\mathbf{j}$$

$$|{}_C\mathbf{v}_D|^2 = \left(8\sqrt{3}\right)^2 + \left(-2\right)^2 = 192 + 4 = 196$$

$$|{}_C\mathbf{v}_D| = \sqrt{196} = 14$$

The magnitude of velocity of C relative to D is 14 m s^{-1}, as required.

b i



The shortest distance, is given by the perpendicular from D to the direction of the velocity of C relative to D.

Let the foot of the perpendicular from D to the direction of the velocity of C relative to D be P. Let $DP = x$ m and $CP = y$ m .

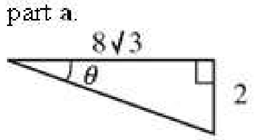
From part **a**

$${}_C\mathbf{v}_D = 8\sqrt{3}\mathbf{i} - 2\mathbf{j}$$

Hence

$$\tan\theta = \frac{2}{8\sqrt{3}} \Rightarrow \theta = 8.213° \text{ (3 d.p)}$$

You find the angle CP makes with CD using the vector form of the relative velocity you found in part **a**.



In ΔCPD

$$\frac{x}{4000} = \sin\theta$$

$x = 4000 \sin 8.213° = 571\text{ m}$ (nearest whole number)

The shortest distance between C and D during the subsequent motion is 571 m (nearest metre).

ii In ΔCPD

$$\frac{y}{4000} = \cos 8.213°$$

$$y = 4000 \cos 8.213° = 3959 \text{ (nearest whole number)}$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{3959}{14}\text{ s} = 283\text{ s, to the nearest second.}$$

$$= 5 \text{ minutes (nearest minute)}$$

The time, to the nearest minute, is 2.05 p.m.

$$\text{time} = \frac{\text{relative distance travelled}}{\text{relative speed}}$$

$$283\text{ s} = \frac{283}{60}\text{ minutes} \approx 5 \text{ minutes}$$

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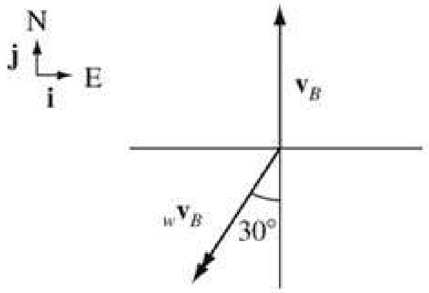
Review Exercise 1
Exercise A, Question 10

Question:

A boat is sailing north at a speed of $15\,\text{km h}^{-1}$. To an observer on the boat the wind appears to blow from a direction $030°$.
The boat turns round and sails due south at the same speed. The velocity of the wind relative to the Earth remains constant, but to an observer on the boat it now appears to blow from $120°$.
Find the velocity of the wind relative to the Earth. **[E]**

Solution:



Let $\mathbf{i}$ and $\mathbf{j}$ be horizontal unit vectors due east and due north respectively.
Let $\mathbf{v}_B\,\text{km h}^{-1}$ be the velocity of the boat,
$\mathbf{v}_W\,\text{km h}^{-1}$ be the velocity of the wind and
${}_W\mathbf{v}_B\,\text{km h}^{-1}$ be the velocity of the wind relative to the boat.
If the velocity of the wind relative to the boat has magnitude $x\text{ m s}^{-1}$ then

$$ {}_W\mathbf{v}_B = -x\sin 30°\mathbf{i} - x\cos 30°\mathbf{j} $$
$$ = -\frac{x}{2}\mathbf{i} - \frac{x\sqrt{3}}{2}\mathbf{j} $$

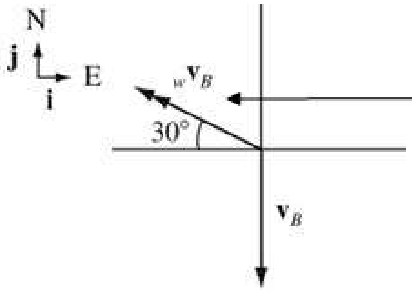
You resolve the relative velocity along the directions of $\mathbf{i}$ and $\mathbf{j}$.

Also

$$ {}_W\mathbf{v}_B = \mathbf{v}_W - \mathbf{v}_B $$
$$ -\frac{x}{2}\mathbf{i} - \frac{x\sqrt{3}}{2}\mathbf{j} = \mathbf{v}_W - 15\mathbf{j} $$
$$ \mathbf{v}_W = -\frac{x}{2}\mathbf{i} + \left(15 - \frac{x\sqrt{3}}{2}\right)\mathbf{j} \quad ① $$

As the boat is moving north, that is in the direction of $\mathbf{j}$, at $15\,\text{km h}^{-1}$, $\mathbf{v}_B = 15\mathbf{j}$.

Solve this equation for $\mathbf{v}_W$.



The boat now reverses its direction and the wind appears to come from a different direction.

Now let the velocity of the wind relative to the boat have magnitude y m s^{-1}.

$${}_W\mathbf{v}_B = -y\cos 30°\mathbf{i} + y\sin 30°\mathbf{j}$$

$$= -\frac{y\sqrt{3}}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$$

You again resolve the relative velocity along the directions of **i** and **j**.

Also

$${}_W\mathbf{v}_B = \mathbf{v}_W - \mathbf{v}_B$$

$$-\frac{y\sqrt{3}}{2}\mathbf{i} + \frac{y}{2}\mathbf{j} = \mathbf{v}_W - (-15\mathbf{j})$$

As the boat is moving south, that is in the direction of $-\mathbf{j}$, at 15 km h^{-1}, $\mathbf{v}_B = -15\mathbf{j}$.

$$\mathbf{v}_W = -\frac{y\sqrt{3}}{2}\mathbf{i} + \left(\frac{y}{2} - 15\right)\mathbf{j} \quad ②$$

You now have two equations for the velocity of the wind and equating the **i** and **j** components will give you a pair of simultaneous equations in x and y.

Equating the **i** components in equations ① and ②

$$-\frac{x}{2} = -\frac{y\sqrt{3}}{2} \Rightarrow x = y\sqrt{3}$$

Equating the **j** components in equations ① and ②

$$\frac{y}{2} - 15 = 15 - \frac{x\sqrt{3}}{2}$$

Substituting $x = y\sqrt{3}$

$$\frac{y}{2} - 15 = 15 - \frac{3y}{2}$$

$$2y = 30 \Rightarrow y = 15$$

Substituting $y = 15$ into ②

$$\mathbf{v}_W = -\frac{15\sqrt{3}}{2}\mathbf{i} - \frac{15}{2}\mathbf{j}$$

The velocity of the wind relative to the Earth is

$$\left(-\frac{15\sqrt{3}}{2}\mathbf{i} - \frac{15}{2}\mathbf{j}\right) \text{km h}^{-1}$$

This is an acceptable vector form for the velocity but the answer can be given in other forms. For example, as a speed of 15 km h^{-1} blowing from the direction 060°. There are also many alternative ways of solving this question.

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Review Exercise 1
Exercise A, Question 11

Question:

A pilot flying an aircraft at a constant speed of 2000 km h^{-1} detects an enemy aircraft 100 km away on a bearing of $045°$. The enemy aircraft is flying at a constant velocity of 1500 km h^{-1} due west.

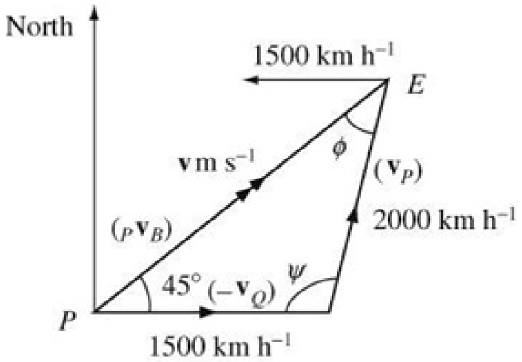
Find

a the course, as a bearing to the nearest degree, that the pilot should set in order to intercept the enemy aircraft,

b the time, to the nearest s, that the pilot will take to reach the enemy aircraft. **[E]**

Solution:

a



For interception, the velocity of the pilot (P) relative to the enemy (E) must be the direction of the line joining the initial position of P to the initial position of E. In this diagram, this relative velocity is shown with a double arrow.

Using the sine rule

$$\frac{\sin\phi}{1500}=\frac{\sin 45^\circ}{2000}$$

$$\sin\phi=\frac{3}{4\sqrt{2}}$$

$$\phi = 32.028^\circ \quad (3\text{ d.p.})$$

$$\psi = 180^\circ - 45^\circ - \phi = 102.972^\circ$$

As ϕ is opposite the smallest side in the vector triangle, it must be acute.

The bearing on which the pilot must fly is

$$\psi - 90^\circ = 013^\circ \quad \text{(nearest degree)}$$

Bearings are measured from north, clockwise. This question requires you to give your answer to the nearest degree.

b Let the magnitude of the velocity of the pilot relative to the enemy be v m s^{-1}

Using the cosine rule

$$v^2 = 1500^2 + 2000^2 - 2\times1500\times2000\cos\psi$$

$$= 7\,596\,849\ldots$$

$$v = 2756.238\ldots$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{100}{v}\text{ m} \approx 0.03628\text{ h}$$

$$= 131\text{ s} \quad \text{(nearest second)}$$

$0.03628\text{ h} = 0.03628\times3600\text{ s} \approx 130.6\text{ s}$

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Review Exercise 1
Exercise A, Question 12

Question:

At noon, two boats A and B are 6 km apart with A due east of B. Boat B is moving due north at a constant speed of 13 km h^{-1}.

Boat A is moving with constant speed 12 km h^{-1} and sets a course so as to pass as close as possible to boat B. Find

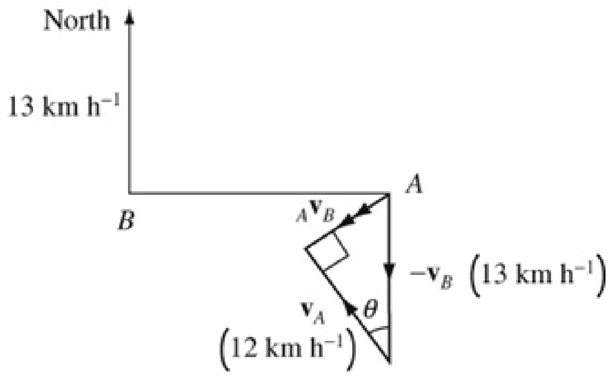
a the direction of motion of A, giving your answer as a bearing,

b the time when the boats are closest,

c the shortest distance between the boats. **[E]**

Solution:

a



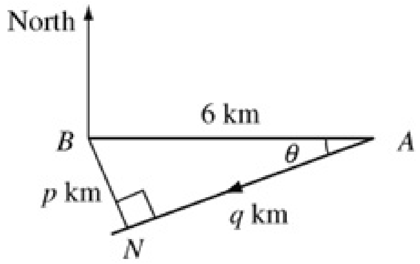
For the closest approach, the direction of motion of A, shown here by a vector with a single arrow of magnitude 12 km h^{-1}, must be perpendicular to the velocity of A relative to B, shown here by the vector with a double arrow.

$\cos\theta = \dfrac{12}{13} \Rightarrow \theta \approx 22.62°$

The bearing on which A moves is $360° - \theta = 337°$ (nearest degree)

b Let the magnitude of the velocity of A relative to B be x m s^{-1}

$x^2 = 13^2 - 12^2 = 25 \Rightarrow x = 5$



Let the foot of the perpendicular from B to the direction of travel of A relative to B be N and let $BN = p$ km and $AN = q$ km.

In ΔBNA

$\dfrac{q}{6} = \cos\theta$

$q = 6\cos\theta = 6\cos 22.62° = 5.538$ (3 d.p.)

The θ in part **b** is the θ you found in part **a**. The two angles are equal.

$\text{time} = \dfrac{\text{distance}}{\text{speed}}$

$= \dfrac{5.538}{5}\text{ h} = 1.108\text{ h}$

$0.108\text{ h} = 0.108 \times 60 \text{ minutes} \approx 6 \text{ minutes}$

$= 1\text{h } 6\text{min}$

The time when the boats are closest is 1306, (nearest minute)

c In ΔBNA

$\dfrac{p}{6} = \sin\theta$

$p = 6\sin\theta = 6\sin 22.62° = 2.30\ldots$

The shortest distance is 2.3 km (nearest 0.1 km)

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Review Exercise 1
Exercise A, Question 13

Question:

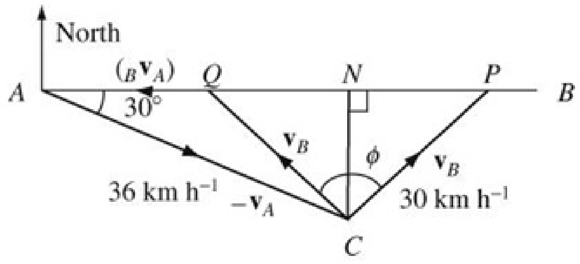
A ship A has maximum speed 30 km h^{-1}. At time $t = 0$, A is 70 km due west of B which is moving at a constant speed of 36 km h^{-1} on a bearing of $300°$. Ship A moves on a straight course at constant speed and intercepts B. The course of A makes an angle θ with due north.

a Show that $-\arctan\frac{4}{3} \le \theta \le \arctan\frac{4}{3}$.

b Find the least time for A to intercept B. **[E]**

Solution:

a



In this diagram, the minimum speed for interception is represented by CN and the maximum speed of A, 30 km h^{-1}, by CP and CQ. If $\angle NCP = \phi$, then θ can vary from $-\phi$ to ϕ.

In ΔANC

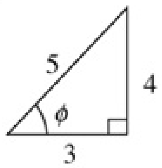
$$\frac{CN}{36} = \sin 30°$$

$$CN = 36\sin 30° = 18$$

In ΔCNP

$$\cos\phi = \frac{CN}{CP} = \frac{18}{30} = \frac{3}{5}$$

$$\tan\phi = \frac{4}{3} \Rightarrow \phi = \arctan\frac{4}{3}$$



From a 3, 4, 5 triangle, you can see that if $\cos\phi = \frac{3}{5} \Rightarrow \tan\phi = \frac{4}{3}$.

As

$$-\phi \le \theta \le \phi$$

$$-\arctan\frac{4}{3} \le \theta \le \arctan\frac{4}{3}, \quad \text{as required.}$$

b

$$AP = AN + NP$$

$$= 36\cos 30° + 30\sin\phi$$

$$= 36 \times \frac{\sqrt{3}}{2} + 30 \times \frac{4}{5} = 18\sqrt{3} + 24$$

The greatest possible velocity of B relative to A is represented on the diagram by AP. The greatest velocity of B relative to A will give you the least time.

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{70}{18\sqrt{3}+24}\text{ h} = 1.27\text{ h} \quad (3\text{ s.f.})$$

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Review Exercise 1
Exercise A, Question 14

Question:

At 12 noon, ship A is 20 km from ship B, on a bearing of $300°$. Ship A is moving at a constant speed of 15 km h^{-1} on a bearing of $070°$. Ship B moves in a straight line with constant speed V km h^{-1} and intercepts A.

a Find, giving your answer to 3 significant figures, the minimum possible value for V.

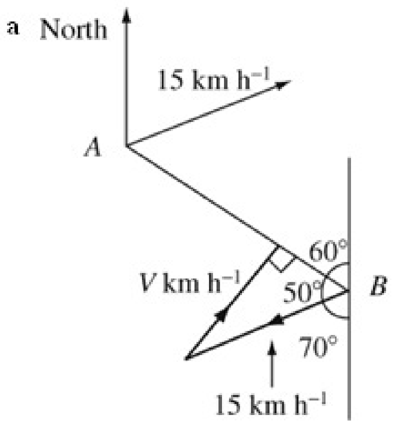
It is now given that $V = 13$.

b Explain why there are two possible times at which ship A can intercept ship B.

c Find, giving your answer to the nearest minute, the earlier time at which ship B can intercept ship A. **[E]**

Solution:

a



You fix A by introducing a vector equal and opposite to the velocity of A to the system.

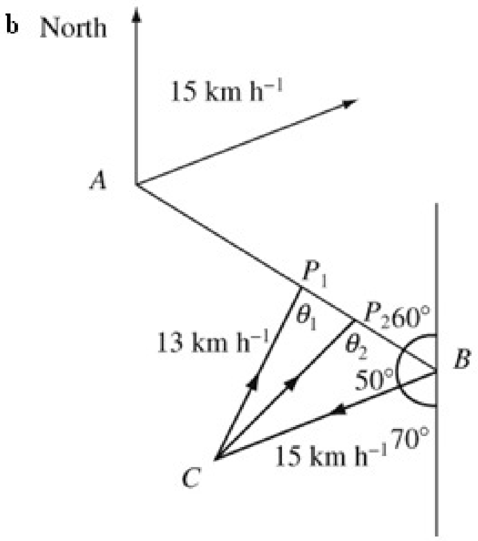
The smallest value of V is given by a vector perpendicular to the direction of the line joining A to B.

The smallest value of V is given by

$$\frac{V}{15} = \sin 50°$$

$$V = 15 \sin 50° = 11.5 \quad (3 \text{ s.f.})$$

b



If $V = 13$, there are two possible courses for interception represented by the vectors $\overrightarrow{CP_1}$ and $\overrightarrow{CP_2}$ on the diagram above. When you calculate the value of θ using the sine rule

$$\frac{\sin\theta}{15} = \frac{\sin 50°}{13},$$

the sine rule is ambiguous.

As $\sin\theta = \sin(180° - \theta)$, there are two possible values of the angle θ which satisfy the rule, related by the relation $\theta_1 + \theta_2 = 180°$.

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Review Exercise 1
Exercise A, Question 15

Question:

At time $t = 0$ particles P and Q start simultaneously from points which have position vectors $(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$m and $(-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$m respectively, relative to a fixed origin O.

The velocities of P and Q are $(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ m s^{-1} and $(2\mathbf{i} + \mathbf{k})$m s^{-1} respectively.

a Show that P and Q collide and find the position vector of the point at which they collide.

A third particle R moves in such a way that its velocity relative to P is parallel to the vector $(-5\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ and its velocity relative to Q is parallel to the vector $(-2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

Given that all three particles collide simultaneously, find

b **i** the velocity of R,

ii the position vector of R at time $t = 0$. **[E]**

Solution:

a The path of P is given by

$$\mathbf{p} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + t\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

You may use either the **i**, **j**, **k** notation vectors or, as is used here, column vectors. In three dimensions, column vectors can often be written more quickly.

The path of Q is given by

$$\mathbf{q} = \begin{pmatrix}-1\\2\\-1\end{pmatrix} + t\begin{pmatrix}2\\0\\1\end{pmatrix}$$

In this question, you need to use the equation of a line using vectors in three dimensions and you need to know how to show that two lines intersect. These are topics in C4. The prerequisites for M4 require the knowledge of books C1 to C4 as well as M1 to M3.

Equating the **i** components

$1 + t = -1 + 2t \Rightarrow t = 2$

When $t = 2$

$$\mathbf{p} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + 2\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$$

and

$$\mathbf{q} = \begin{pmatrix}-1\\2\\-1\end{pmatrix} + 2\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix} = \mathbf{p}$$

Hence, when $t = 2$, P and Q collide at the point with position vector $(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$m .

b i Let ${}_R\mathbf{v}_P = \lambda\begin{pmatrix}-5\\4\\-1\end{pmatrix}$

As the velocity of R relative to P is in the direction of $(-5\mathbf{i} + 4\mathbf{j} - \mathbf{k})$, it must be a multiple of $(-5\mathbf{i} + 4\mathbf{j} - \mathbf{k})$.

$${}_R\mathbf{v}_P = \mathbf{v}_R - \mathbf{v}_P$$

$$\lambda\begin{pmatrix}-5\\4\\-1\end{pmatrix} = \mathbf{v}_R - \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$\mathbf{v}_R = \begin{pmatrix}1-5\lambda\\2+4\lambda\\-1-\lambda\end{pmatrix} \quad ①$$

Let ${}_R\mathbf{v}_Q = \mu\begin{pmatrix}-2\\2\\-1\end{pmatrix}$

$${}_R\mathbf{v}_Q = \mathbf{v}_R - \mathbf{v}_Q$$

$$\mu\begin{pmatrix}-2\\2\\-1\end{pmatrix} = \mathbf{v}_R - \begin{pmatrix}2\\0\\1\end{pmatrix}$$

$$\mathbf{v}_R = \begin{pmatrix}2-2\mu\\2\mu\\1-\mu\end{pmatrix} \quad ②$$

You now have two expressions for $\mathbf{v}_R$. Equating any two of the components will give you a pair of simultaneous equations in λ and μ.

Equating the $\mathbf{i}$ components of ① and ②

$1-5\lambda = 2-2\mu$

$-5\lambda+2\mu = 1 \quad ③$

Equating the $\mathbf{j}$ components of ① and ②

$2+4\lambda = 2\mu$

$-4\lambda+2\mu = 2$

Subtracting ④ − ③

$\lambda = 1$

Substituting $\lambda = 1$ into ①

$$\mathbf{v}_R = \begin{pmatrix}1-5\\2+4\\-1-1\end{pmatrix} = \begin{pmatrix}-4\\6\\-2\end{pmatrix}$$

The velocity of R is $(-4\mathbf{i}+6\mathbf{j}-2\mathbf{k})\ \text{m s}^{-1}$.

ii The path of R is given by

$$\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} + t\begin{pmatrix}-4\\6\\-2\end{pmatrix}$$

The path of a particle moving with constant velocity can be written as $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where t is the time, $\mathbf{r}_0$ the position vector when $t = 0$, and $\mathbf{v}$ the velocity.

where $(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$ m is the position vector of R at time $t = 0$.

From part **a**, when $t = 2, \mathbf{r} = \begin{pmatrix}3\\2\\1\end{pmatrix}$

When $t = 2$, all three particles are at the point with position vector $(3\mathbf{i}+2\mathbf{j}+\mathbf{k})$ m.

$$\begin{pmatrix}x\\y\\z\end{pmatrix} + 2\begin{pmatrix}-4\\6\\-2\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3+8\\2-12\\1+4\end{pmatrix} = \begin{pmatrix}11\\-10\\5\end{pmatrix}$$

The position vector of R at time $t = 0$ is $(11\mathbf{i}-10\mathbf{j}+5\mathbf{k})$ m.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 16

Question:

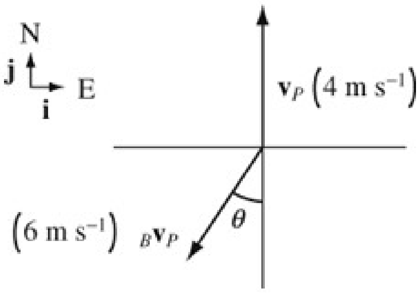
A rugby player is running due north with speed 4 m s^{-1}. He throws the ball horizontally and the ball has an initial velocity relative to the player of 6 m s^{-1} in the direction $\theta°$ west of south, i.e. on a bearing of $(180+\theta)°$, where $\tan\theta° = \frac{4}{3}$.

a Find the magnitude and direction of the initial velocity of the ball relative to a stationary spectator.

b Find also the bearing on which the ball appears to move initially to the referee who is running with speed $2\sqrt{2} \text{ m s}^{-1}$ in a north-westerly direction. **[E]**

Solution:

a



Let **i** and **j** be horizontal unit vectors due east and due north respectively.

$$\mathbf{v}_P = 4\mathbf{j}$$

$$_B\mathbf{v}_P = -6\sin\theta\mathbf{i} - 6\cos\theta\mathbf{j}$$

You resolve the velocity of the player, $\mathbf{v}_P$, and the velocity of the ball relative to the player, $_B\mathbf{v}_P$, parallel to the unit vectors **i** and **j**.

$$\tan\theta = \frac{4}{3} \Rightarrow \sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}$$

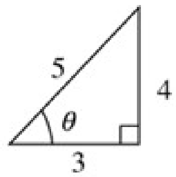
$$_B\mathbf{v}_P = \mathbf{v}_B - \mathbf{v}_P$$

$$-6\times\frac{4}{5}\mathbf{i} - 6\times\frac{3}{5}\mathbf{j} = \mathbf{v}_B - 4\mathbf{j}$$

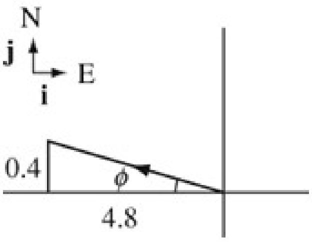
$$\mathbf{v}_B = -4.8\mathbf{i} + (4-3.6)\mathbf{j} = -4.8\mathbf{i} + 0.4\mathbf{j}$$

$$|\mathbf{v}_B|^2 = (-4.8)^2 + (0.4)^2 = 23.2$$

$$|\mathbf{v}_B| = 4.82 \quad (3\text{ s.f.})$$



You can see from a sketch that if $\tan\theta = \frac{4}{3}$, then $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.



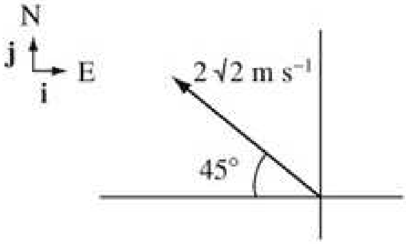
$$\tan\phi = \frac{0.4}{4.8} = 0.083$$

$$\phi = 5^\circ \text{(nearest degree)}$$

The velocity of the ball has magnitude 4.82 m s^{-1} (3 s.f.) and is on a bearing of 275° (nearest degree).

No accuracy is specified in this question and any reasonable accuracy would be accepted. In this context, 2 or 3 significant figures is sensible.

b



The velocity of the referee is given by

$\mathbf{v}_R = -2\sqrt{2}\cos 45^\circ\mathbf{i} + 2\sqrt{2}\sin 45^\circ\mathbf{j}$

$= -2\mathbf{i} + 2\mathbf{j}$

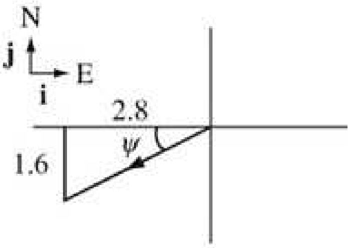
Using $\cos 45^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}}$.

The velocity of the ball relative to the referee is given by

${}_B\mathbf{v}_R = \mathbf{v}_B - \mathbf{v}_R$

$= -4.8\mathbf{i} + 0.4\mathbf{j} - (-2\mathbf{i} + 2\mathbf{j})$

$= -2.8\mathbf{i} - 1.6\mathbf{j}$



There is an interesting interpretation of this question. The ball is actually being passed forward but appears to be backwards to both the player and the referee.

$\tan\psi = \frac{1.6}{2.8} \Rightarrow \psi = 29.7^\circ \text{ (1 d.p.)}$

The bearing on which the ball will appear to move to the referee is 240° (nearest degree).

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Review Exercise 1
Exercise A, Question 17

Question:

Two ships A and B are travelling with constant speeds $2u$ m s^{-1} and u m s^{-1} respectively, A on a bearing θ and B on a bearing $90° + \theta$. It is also assumed that a third ship C has a constant, but unknown, velocity which is taken to be a speed v m s^{-1} on a bearing ϕ. To an observer on ship B the velocity of C appears to be due north.

a Show that $\dfrac{u}{\sin\phi} = \dfrac{v}{\cos\theta}$.

To an observer on ship A the velocity of C appears to be on a bearing of $135°$.

b Show that $2u(\cos\theta + \sin\theta) = v(\cos\phi + \sin\phi)$.

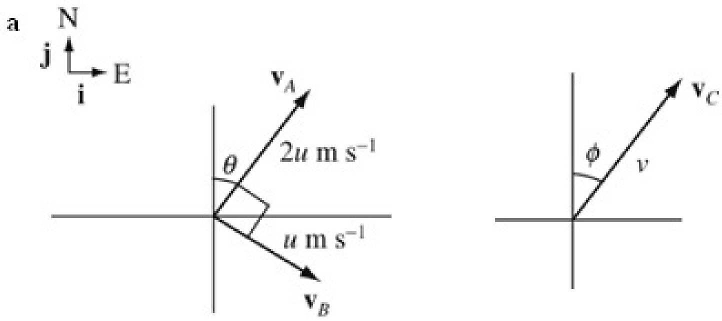
c Hence, find $\tan\phi$ in terms of $\tan\theta$.

Given that $\theta = 30°$ and $u = 10$,

d find the true velocity of C, giving your answer to 3 significant figures. **[E]**

Solution:

a



Let $\mathbf{i}$ and $\mathbf{j}$ be horizontal unit vectors due east and due north respectively.

$\mathbf{v}_A = 2u\sin\theta\mathbf{i} + 2u\cos\theta\mathbf{j}$

$\mathbf{v}_B = u\cos\theta\mathbf{i} - u\sin\theta\mathbf{j}$

$\mathbf{v}_C = v\sin\phi\mathbf{i} + v\cos\phi\mathbf{j}$

You resolve the velocities of A, B and C in the directions of $\mathbf{i}$ and $\mathbf{j}$.

${}_C\mathbf{v}_B = \mathbf{v}_C - \mathbf{v}_B = \lambda\mathbf{j}$, say

$v\sin\phi\mathbf{i} + v\cos\phi\mathbf{j} - (u\cos\theta\mathbf{i} - u\sin\theta\mathbf{j}) = \lambda\mathbf{j}$

As the velocity of C relative to B, ${}_C\mathbf{v}_B$, is due north then it must have the form $\lambda\mathbf{j}$, where λ is a constant.

Equating $\mathbf{i}$ components

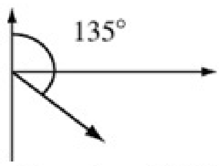
$v\sin\phi - u\cos\theta = 0$

$v\sin\phi = u\cos\theta$

$\frac{u}{\sin\phi} = \frac{v}{\cos\theta}$, as required

b ${}_C\mathbf{v}_A = \mathbf{v}_C - \mathbf{v}_A = \mu(\mathbf{i} - \mathbf{j})$, say

$v\sin\phi\mathbf{i} + v\cos\phi\mathbf{j} - (2u\sin\theta\mathbf{i} + 2u\cos\theta\mathbf{j}) = \mu(\mathbf{i} - \mathbf{j})$



A bearing of 135° is in the direction $\mathbf{i} - \mathbf{j}$.

Equating $\mathbf{i}$ components

$v\sin\phi - 2u\sin\theta = \mu$ ①

Equating $\mathbf{j}$ components

$v\cos\phi - 2u\cos\theta = -\mu$ ②

Eliminating μ between ① and ②

$v\sin\phi - 2u\sin\theta = -v\cos\phi + 2u\cos\theta$

$2u(\cos\theta + \sin\theta) = v(\cos\phi + \sin\phi)$, as required.

c From the answer to part **a**

$$u = \frac{v\sin\phi}{\cos\theta}$$

Hence

$$2\frac{v\sin\phi}{\cos\theta}(\cos\theta+\sin\theta) = v(\cos\phi+\sin\phi)$$

$$2\left(\frac{\cos\theta+\sin\theta}{\cos\theta}\right) = \frac{\cos\phi+\sin\phi}{\sin\phi}$$

$$2+2\tan\theta = \cot\phi+1$$

$$\cot\phi = 1+2\tan\theta$$

$$\tan\phi = \frac{1}{1+2\tan\theta}$$

The first step in part **c** is to eliminate u between the answers to part **a** and part **b**. When you do this v also 'cancels' and you obtain a trigonometric relation between θ and ϕ.

d $\tan\phi = \dfrac{1}{1+2\tan 30^\circ} = 0.4641\ldots$

$\phi = 24.9^\circ$ (3 s.f.)

$$v = \frac{u\cos\theta}{\sin\phi} = \frac{10\cos 30^\circ}{\sin 24.896\ldots^\circ} = 20.6 \quad (3\text{ s.f.})$$

The true velocity of C has magnitude 20.6 m s^{-1} (3 s.f.) and is on the bearing 024.9° (3 s.f.).

An answer in vector form is also acceptable. This would be $(8.66\mathbf{i}+18.7\mathbf{j})\text{m s}^{-1}$.

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Review Exercise 1
Exercise A, Question 18

Question:

[*In this question* **i** *and* **j** *are horizontal unit vectors due east and due north respectively*.]
The airport B is due north of airport A. On a particular day the velocity of the wind is $(70\mathbf{i} + 25\mathbf{j})\text{km h}^{-1}$. Relative to the air, an aircraft flies with constant speed 250 km h^{-1}.
When the aircraft flies directly from A to B, find

a its speed relative to the ground,

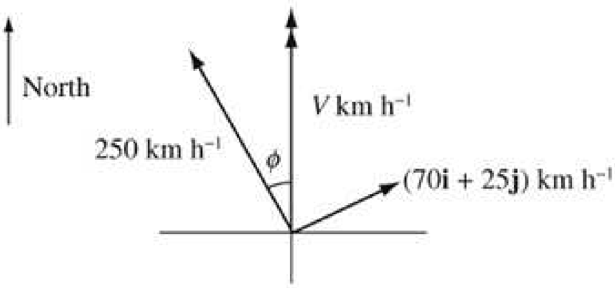
b its direction, as a bearing to the nearest degree, in which it must head.

After flying from A to B, the aircraft returns directly to A.

c Calculate the ratio of the time taken on the outward journey to the time taken on the return flight. **[E]**

Solution:

a



Let the aircraft fly on a bearing of $(360^\circ - \phi)$ and its speed relative to the ground be V km h^{-1}.

$R(\rightarrow)$ $\quad 70 - 250\sin\phi = 0$

$$\sin\phi = \frac{7}{25}$$

The velocity relative to the ground is the resultant of the velocity relative to the air and the velocity of the wind. As this resultant is north, the sum of the component velocities in the easterly direction must be 0.

$R(\uparrow)$ $\quad V = 250\cos\phi + 25$

$$= 250 \times \frac{24}{25} + 25 = 265$$

$\cos^2\phi = 1 - \sin^2\phi = 1 - \frac{7^2}{25^2} = \frac{576}{25^2} = \left(\frac{24}{25}\right)^2$

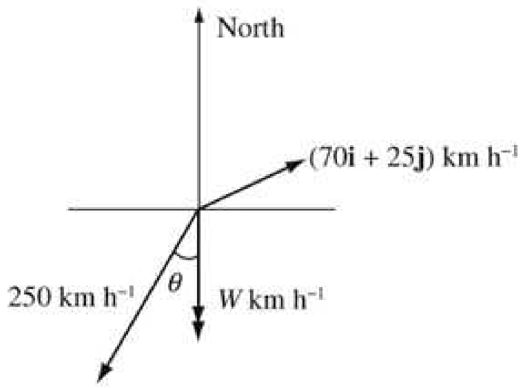
As ϕ is acute, $\cos\phi = \frac{24}{25}$.

The speed of the aircraft relative to the ground is 265 km h^{-1}.

b $\sin\phi = \frac{7}{25} \Rightarrow \phi = 16^\circ$ (nearest degree)

The direction of the aircraft is on the bearing $(360^\circ - \phi) = 344^\circ$ (nearest degree).

c



On the return journey, let the aircraft fly on a bearing of $(180^\circ + \theta)$ and its velocity relative to the ground be W km h^{-1}.

$R(\rightarrow)$ $\quad 70 - 250\sin\theta = 0$

$$\sin\theta = \frac{7}{25}$$

$R(\downarrow)$ $\quad W = 250\cos\theta - 25$

$$= 250 \times \frac{24}{25} - 25 = 215$$

θ has the same value as ϕ in part a

Let T_1 be the time for the outward journey and T_2 the time for the return journey.

$$\frac{T_1}{T_2} = \frac{W}{V} = \frac{215}{265} = \frac{43}{53}$$

As the distances are the same, the time of a journey is inversely proportional to the true speed of the aircraft, $T \propto \frac{1}{V}$. The actual distance travelled is not relevant in part c.

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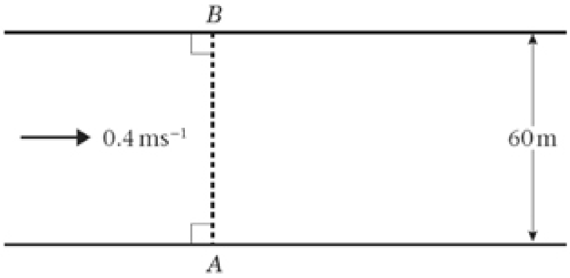
Review Exercise 1
Exercise A, Question 19

Question:

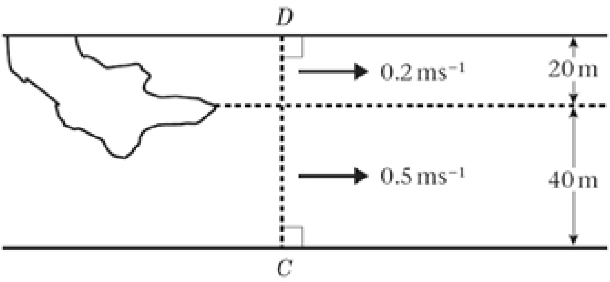


Mary swims in still water at 0.85 m s^{-1}. She swims across a straight river which is 60 m wide and flowing at 0.4 m s^{-1}. She sets off from a point A on the near bank and lands at a point B, which is directly opposite A on the far bank, as shown in the figure above.

Find

a the angle between the near bank and the direction in which Mary swims,

b the time she takes to cross the river.



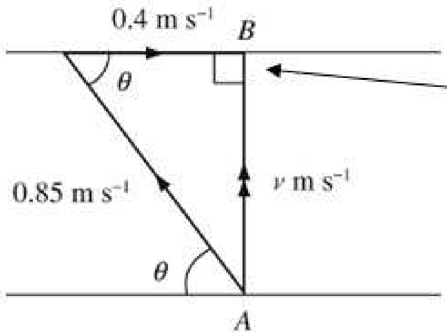
A little further downstream a large tree has fallen from the far bank into the river. The river is modelled as flowing at 0.5 m s^{-1} for a width of 40 m from the near bank, and 0.2 m s^{-1} beyond that. Nassim swims at 0.85 m s^{-1} in still water. He swims across the river from a point C on the near bank. The point D on the far bank is directly opposite C as shown above. Nassim swims at the same angle to the near bank as Mary.

c Find the maximum distance, downstream from CD, of Nassim during the crossing.

d Show that he will land at the point D. **[E]**

Solution:

a



As Mary crosses directly from A to B, the resultant of the velocity of her swimming and the velocity of the water must be perpendicular to the bank. You then use trigonometry and Pythagoras' theorem to solve parts **a** and **b**.

Let the angle between the near bank and the direction in which Mary swims be θ.

$\cos\theta = \dfrac{0.4}{0.85} = \dfrac{8}{17}$

$\theta = 61.9°$ (nearest 0.1°)

b Let Mary's speed relative to the bank be v m s^{-1}.

$v^2 = 0.85^2 - 0.4^2 = 0.5625$

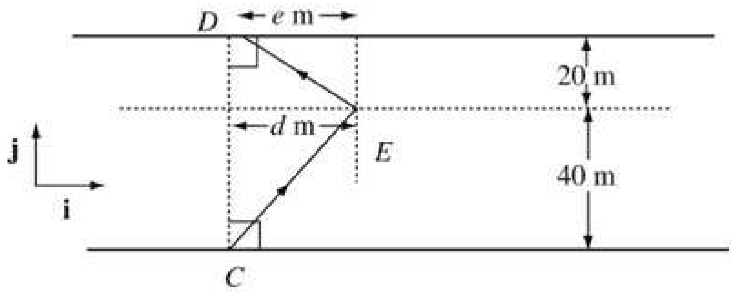
$v = \sqrt{0.5625} = 0.75$

time $= \dfrac{\text{distance}}{\text{speed}}$

$= \dfrac{60}{0.75}$ s $= 80$ s

Mary takes 80 s to cross the river.

c



Let E be the point where Nassim is furthest downstream.

Let **i** and **j** be unit vectors parallel and perpendicular to the banks in the directions shown in the diagram.

It is possible to solve parts **c** and **d** using vector triangles. However, to get full marks you must show that Nassim lands exactly at D and this is easier to show using a vector or component method.

As Nassim moves from C to E

$${}_N\mathbf{v}_W = -0.4\mathbf{i} + 0.75\mathbf{j}$$

The velocity of Nassim relative to the water, ${}_N\mathbf{v}_W$ m s^{-1}, is the same as Mary's in parts **a** and **b**, that is $(-0.4\mathbf{i} + v\mathbf{j})$ m s^{-1} and in part **b** you showed that $v = 0.75$.

$${}_N\mathbf{v}_W = \mathbf{v}_N - \mathbf{v}_W$$
$$-0.4\mathbf{i} + 0.75\mathbf{j} = \mathbf{v}_N - 0.5\mathbf{i}$$
$$\mathbf{v}_N = 0.1\mathbf{i} + 0.75\mathbf{j}$$

From C to E, the velocity of the water, $\mathbf{v}_W$ m s^{-1} has magnitude 0.5 m s^{-1}.

Considering Nassim's motion parallel to **j** as he moves from C to E

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$
$$= \frac{40}{0.75}\text{ s} = \frac{160}{3}\text{ s}$$

Parallel to **j**, Nassim moves a distance of 40 m with speed 0.75 m s^{-1}.

Let the distance moved downstream be d m.
Considering Nassim's motion parallel to **i** as he moves from C to E

$$\text{distance} = \text{speed} \times \text{time}$$
$$d = 0.1 \times \frac{160}{3} = \frac{16}{3}$$

The resultant velocity $(0.1\mathbf{i} + 0.75\mathbf{j})$ m s^{-1} shows that Nassim is pushed downstream at a rate of 0.1 m s^{-1}.

The maximum distance downstream of Nassim during the crossing is $\frac{16}{3}$ m.

d From E to the far bank the velocity of the water has magnitude 0.2 m s^{-1} and equation * becomes

$$-0.4\mathbf{i} + 0.75\mathbf{j} = \mathbf{v}_N - 0.2\mathbf{i} \quad *$$
$$\mathbf{v}_N = -0.2\mathbf{i} + 0.75\mathbf{j}$$

You repeat the method you used in part **c** to find the distance that Nassim moves upstream as he swims from E to the far bank.

Considering Nassim's motion parallel to **j** as he moves from E to the far bank

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$
$$= \frac{20}{0.75} \text{ s} = \frac{80}{3} \text{ s}$$

Parallel to **j**, Nassim moves a distance of 20 m with speed 0.75 m s^{-1}. In this direction, his speed has not changed.

Let the distance moved upstream be e m.
Considering Nassim's motion parallel to **i** as he moves from C to E

$$\text{distance} = \text{speed} \times \text{time}$$
$$e = 0.2 \times \frac{80}{3} = \frac{16}{3}$$

As $e = d$, Nassim lands at the point D.

The resultant velocity $(-0.2\mathbf{i} + 0.75\mathbf{j}) \text{ m s}^{-1}$ shows that Nassim moves upstream at a rate of 0.2 m s^{-1}.

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 20

Question:

A girl wishes to swim across a river from a fixed point O on the bank, to a point B on the opposite bank. The position vector of B relative to O is $20\mathbf{j}$m. In a simple model the water is assumed to be flowing with uniform velocity $u\mathbf{i}$ m s^{-1} and the girl intends to swim in such a way that she moves along the line OB.

a Given that $u = 0.6$ and that the speed of the girl relative to the water is 1 m s^{-1}, show that the time taken to swim across the river is 25 s.

A geographer points out that the flow of the river will be faster nearer the middle than closer to the banks and the model for the flow of the river is refined. When the girl is at a point R on the river, with position vector $(x\mathbf{i}+y\mathbf{j})$m , the velocity of the river at that point is $v\mathbf{i}$ m s^{-1}, where

$v = \frac{y}{25}(20-y), \quad 0 \le y \le 20.$

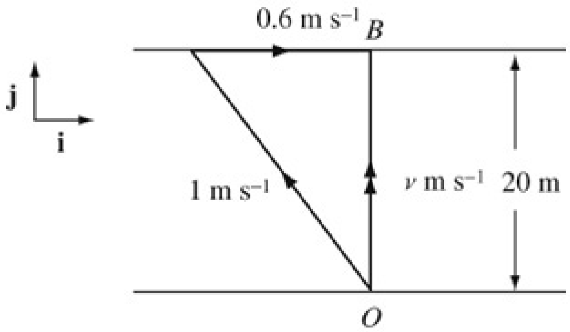
The girl swims with velocity $(-p\mathbf{j}+q\mathbf{j})$m s^{-1} relative to the water, where p and q are positive constants. The girl starts to swim from O at time $t = 0$ and the time taken to cross from O to B is now 50 s.

b Find the value of q and hence show that, at time t seconds, $y = 0.4t$.

c By considering the motion of the girl in the $\mathbf{i}$ direction, find the value of p. **[E]**

Solution:

a



Let the speed of the swimmer relative to the bank be v m s^{-1}.

$v^2 = 1^2 - 0.6^2 = 0.64$

$v = \sqrt{0.64} = 0.8$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{20}{0.8}\text{ s} = 25\text{ s}, \quad \text{as required}$$

b Let $\mathbf{v}_W$ be the velocity of the water, $\mathbf{v}_G$ m s^{-1} = $(\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$ m s^{-1} be the velocity of the girl, and ${}_G\mathbf{v}_W$ m s^{-1} be the velocity of the girl relative to the water.

You interpret the conditions of the question as

$\mathbf{v}_W = \frac{y}{25}(20 - y)\mathbf{i}$ and

${}_G\mathbf{v}_W = -p\mathbf{i} + q\mathbf{j}$.

$${}_G\mathbf{v}_W = \mathbf{v}_G - \mathbf{v}_W$$

$$-p\mathbf{i} + q\mathbf{j} = \mathbf{v}_G - \frac{y}{25}(20 - y)\mathbf{i}$$

$$\mathbf{v}_G = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} = \left(\frac{y}{25}(20 - y) - p\right)\mathbf{i} + q\mathbf{j} \quad *$$

Equation * shows that, in the $\mathbf{j}$ direction, the girl is moving with the constant speed q m s^{-1}.

Considering the motion in the $\mathbf{j}$ direction

distance = speed × time

$20 = q \times 50 \Rightarrow q = 0.4$

At time t seconds

distance = speed × time

$y = qt = 0.4t, \quad \text{as required}$

c Taking the **i** components of equation *

$$\dot{x} = \frac{dx}{dt} = \frac{y}{25}(20 - y) - p$$

$$= \frac{4y}{5} - \frac{y^2}{25} - p$$

$$= \frac{1.6t}{5} - \frac{0.16t^2}{25} - p$$

Using the result of part **a** by substituting $y = 0.4t$.

Integrating

$$x = \frac{1.6t^2}{10} - \frac{0.16t^3}{75} - pt + A$$

A is a constant of integration.

When $t = 0, x = 0 \Rightarrow A = 0$

$$x = \frac{1.6t^2}{10} - \frac{0.16t^3}{75} - pt$$

When $t = 50$, $x = 0$

When the girl reaches B after 50 s, there has been no displacement upstream or downstream.

$$0 = \frac{1.6 \times 50^2}{10} - \frac{0.16 \times 50^3}{75} - 50p$$

$$50p = 400 - \frac{800}{3}$$

$$p = 8 - \frac{16}{3} = \frac{8}{3}$$

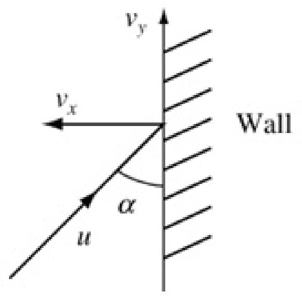
Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 21

Question:

A smooth uniform sphere S of mass m is moving on a smooth horizontal plane when it collides with a fixed smooth vertical wall. Immediately before the collision, the speed of S is U and its direction of motion makes an angle α with the wall. The coefficient of restitution between S and the wall is e.
Find the kinetic energy of S immediately after the collision. **[E]**

Solution:



Let the components of the velocity perpendicular and parallel to the wall immediately after the collision be v_x and v_y respectively.

Parallel to the wall

$v_y = u\cos\alpha$

The impulse is perpendicular to the wall and so the component of the velocity parallel to the wall is unchanged.

Perpendicular to the wall

Newton's law of restitution

$v_x = eu\sin\alpha$

Perpendicular to the wall, Newton's law of restitution gives that, for the velocity, the component after collision $= e \times$ the component before collision
The component of the velocity perpendicular to the wall before collision is $u\sin\alpha$.

The kinetic energy of S after the collision is given by

$$\frac{1}{2}m\left(v_x^2 + v_y^2\right)$$
$$= \frac{1}{2}m\left(e^2u^2\sin^2\alpha + u^2\cos^2\alpha\right)$$
$$= \frac{1}{2}mu^2\left(e^2\sin^2\alpha + \cos^2\alpha\right)$$

If v is the velocity after collision, the kinetic energy of S after the collision is $\frac{1}{2}mv^2$ and $v^2 = v_x^2 + v_y^2$.

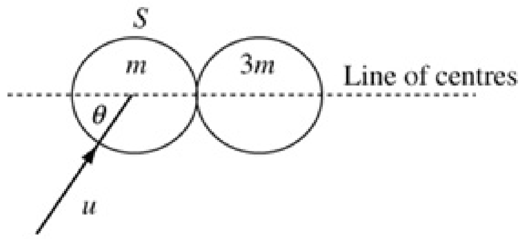
Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 22

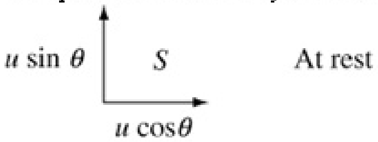
Question:

A smooth sphere S, of mass m, is moving with speed u on a horizontal plane when it collides with another smooth sphere, of mass $3m$ and having the same radius as S, which is at rest on the horizontal plane. The direction of motion of S before impact makes an angle $\theta, 0 < \theta < \frac{\pi}{2}$, with the line of centres of the two spheres. The coefficient of restitution between the spheres is e. After impact the spheres are moving in directions which are perpendicular to each other.
Find the value of e. **[E]**

Solution:



Components of velocity before impact



Let the components of the velocity after impact be

Before the impact, the second sphere is at rest and the impulse on this sphere acts along the line of centres. So the second sphere must move along the line of centres. The question gives you that, after the impact, the spheres are moving in perpendicular directions, so S is moving perpendicular to the line of centres.

Parallel to the line of centres

Conservation of linear momentum

$mu\cos\theta = 3mx$

$$x = \frac{1}{3}u\cos\theta \quad ①$$

Newton's law of restitution

The velocity of S has no component along the line of centres and so the velocity of separation in this direction is just the velocity of the second sphere, x.

velocity of separation $= e \times$ velocity of approach

$$x = eu\cos\theta \quad ②$$

From ① and ②

$\frac{1}{3}u\cos\theta = eu\cos\theta$

Hence

$e = \frac{1}{3}$

You could find y. As the component of the velocity of S perpendicular to the line of centres is unchanged, $y = u\sin\theta$. However, in this question, this is not required.

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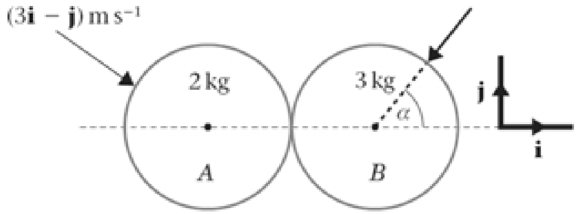
Review Exercise 1
Exercise A, Question 23

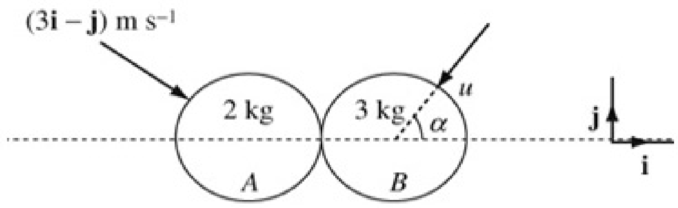
Question:



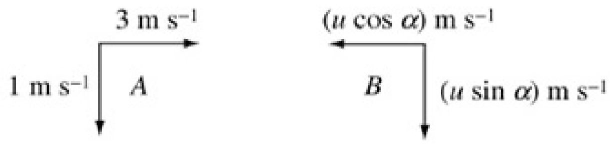
Two smooth uniform spheres A and B, of equal radius, are moving on a smooth horizontal plane. Sphere A has mass 2 kg and sphere B has mass 3 kg. The spheres collide and at the instant of collision the line joining their centres is parallel to **i**. Before the collision A has velocity $(3\mathbf{i} - \mathbf{j})\text{m s}^{-1}$ and after the collision it has velocity $(-2\mathbf{i} - \mathbf{j})\text{m s}^{-1}$. Before the collision the velocity of B makes an angle α with the line of centres, as shown in the figure, where $\tan \alpha = 2$. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find, in terms of **i** and **j**, the velocity of B before the collision. **[E]**

Solution:

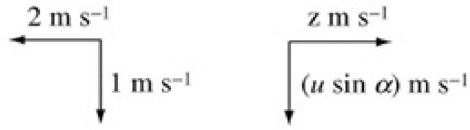
Let the speed of B before the collision be $u \text{ m s}^{-1}$



Components of velocity before collision



Let the components of velocity after collision be



Parallel to $\mathbf{j}$, the components of the velocity are unchanged and you can just write them down.

Parallel to $\mathbf{i}$

Conservation of linear momentum

$$2\times 3 - 3\times u\cos\alpha = 2\times(-2) + 3z$$

$$6 - 3u\cos\alpha = -4 + 3z$$

$$3u\cos\alpha + 3z = 10 \quad ①$$

Equations ① and ② are a pair of simultaneous equations in $u\cos\alpha$ and z. The question asks you to find the velocity of B before the collision. You do not need to know z, so eliminate it.

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$2 + z = \frac{1}{2}(3 + u\cos\alpha)$$

$$u\cos\alpha - 2z = 1 \quad ②$$

① $\times 2$

$$6u\cos\alpha + 6z = 20 \quad ③$$

② $\times 3$

$$3u\cos\alpha - 6z = 3 \quad ④$$

③ + ④

$$9u\cos\alpha = 23$$

$$u\cos\alpha = \frac{23}{9}$$

$$\tan\alpha = \frac{u\sin\alpha}{u\cos\alpha} = 2$$

The question gives you that $\tan\alpha = 2$ and, as you have found $u\cos\alpha$, you can use this result to find $u\sin\alpha$.

$$u\sin\alpha = 2u\cos\alpha = 2\times\frac{23}{9} = \frac{46}{9}$$

The velocity of B before the collision is

$$(-u\cos\alpha\mathbf{i} - u\sin\alpha\mathbf{j})\text{ m s}^{-1} = \left(-\frac{23}{9}\mathbf{i} - \frac{46}{9}\mathbf{j}\right)\text{ m s}^{-1}$$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 24

Question:

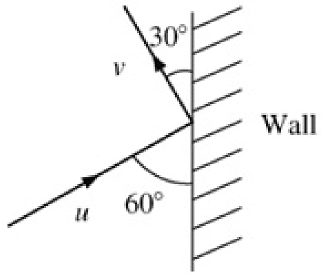
A small ball is moving on a horizontal plane when it strikes a smooth vertical wall. The coefficient of restitution between the ball and the wall is e. Immediately before the impact the direction of motion of the ball makes an angle of $60°$ with the wall. Immediately after the impact the direction of motion of the ball makes an angle of $30°$ with the wall.

a Find the fraction of the kinetic energy of the ball which is lost in the impact.

b Find the value of e. **[E]**

Solution:

a



Let the speed of the ball before impact be u m s^{-1} and the speed of the ball after impact be v m s^{-1}.

Parallel to the wall

$u\cos 60° = v\cos 30°$

$$\frac{1}{2}u = \frac{\sqrt{3}}{2}v \Rightarrow v = \frac{u}{\sqrt{3}}$$

As the impulse of the wall on the ball is perpendicular to the wall, parallel to the wall the component of the velocity of the ball is unchanged.

The kinetic energy lost is

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{1}{2}m\left(\frac{u}{\sqrt{3}}\right)^2$$

$$= \frac{1}{2}mu^2 - \frac{1}{6}mu^2 = \frac{1}{3}mu^2$$

Substituting $v = \frac{u}{\sqrt{3}}$.

The fraction of the kinetic energy lost is

$$\frac{\frac{1}{3}mu^2}{\frac{1}{2}mu^2} = \frac{2}{3}$$

You find the $\frac{\text{loss in kinetic energy}}{\text{original kinetic energy}}$.

b *Perpendicular to the wall*

Newton's law of restitution

$v\sin 30° = eu\cos 60°$

$$\frac{1}{2}v = \frac{\sqrt{3}}{2}eu$$

Perpendicular to the wall, Newton's law of restitution gives that, for the velocity, component after collision $= e \times$ component before collision.

As $v = \frac{u}{\sqrt{3}}$

$$\frac{1}{2} \times \frac{u}{\sqrt{3}} = \frac{\sqrt{3}}{2}eu \Rightarrow e = \frac{1}{\sqrt{3}\times\sqrt{3}} = \frac{1}{3}$$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 25

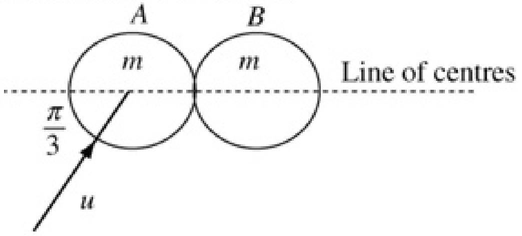
Question:

A smooth sphere A moving with speed u collides with an identical sphere B which is at rest. The directions of motion of A before and after impact makes angles $\frac{\pi}{3}$ and β respectively with the line of centres at the moment of impact. The coefficient of restitution between the spheres is 0.8.
Show that $\tan B = 10\sqrt{3}$. **[E]**

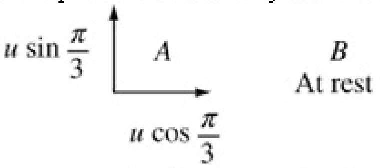
Solution:

Let the mass of the spheres be m, the speed of A after the collision be v and the speed of B after the collision be w.

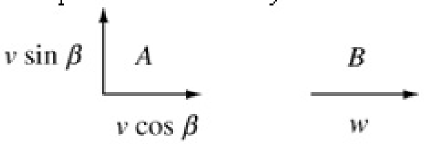
In questions about identical spheres, the mass of the spheres will usually cancel out of any equations but it is sensible to introduce a variable for the mass, m, so that you can write down the equation for conservation of linear momentum.



Components of velocity before the collision



Components of velocity after the collision



Before the collision, B is at rest and the impulse on B acts along the line of centres. Hence, after the collision, B must move along the line of centres.

Perpendicular to the line of centres

$$v\sin\beta = u\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}u \quad ①$$

Perpendicular to the line of centres the component of the velocity is unchanged.

Parallel to line of centres

Conservation of linear momentum

$$mu\cos\frac{\pi}{3} = mv\cos\beta + mw$$

$$w = \frac{1}{2}u - v\cos\beta \quad ②$$

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$w - v\cos\beta = 0.8u\cos\frac{\pi}{3}$$

$$w = 0.4u + v\cos\beta \ldots \quad ③$$

The value of w is not required, so you eliminate it between equations ② and ③.

Eliminating w from ② and ③

$$\frac{1}{2}u - v\cos\beta = 0.4u + v\cos\beta$$

$$v\cos\beta = 0.05u \quad ④$$

Dividing ① by ④

You eliminate u and v between equations ① and ④ by dividing the equations.

$$\frac{\not{v}\sin\beta}{\not{v}\cos\beta} = \frac{\frac{\sqrt{3}}{2}\not{u}}{0.05\not{u}}$$

$$\tan\beta = 10\sqrt{3}, \quad \text{as required}$$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 26

Question:

A smooth uniform sphere P of mass m is falling vertically and strikes a fixed smooth inclined plane with speed u. The plane is inclined at an angle $\theta, \theta < 45°$, to the horizontal.
The coefficient of restitution between P and the inclined plane is e. Immediately after P strikes the plane, P moves horizontally.

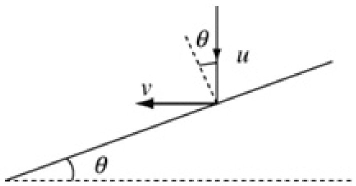
a Show that $e = \tan^2\theta$.

b Show that the magnitude of the impulse exerted by P on the plane is $mu\sec\theta$.

[E]

Solution:

a



Let the speed of P immediately after the impact be v.

Parallel to the plane
$u\sin\theta = v\cos\theta$ ①

Parallel to the plane, the component of the velocity is unchanged by the impact.

Perpendicular to the plane

Newton's law of restitution
$v\sin\theta = eu\cos\theta$
$eu\cos\theta = v\sin\theta$ ②
Dividing ② by ①

$$\frac{eu\cos\theta}{u\sin\theta} = \frac{v\sin\theta}{v\cos\theta}$$

$$e = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta, \quad \text{as required}$$

b Resolving perpendicular to the plane, the impulse is given by

$$I = mv\sin\theta - m(-u\cos\theta)$$
$$= m(v\sin\theta + u\cos\theta)$$
$$= m\left(\frac{u\sin\theta}{\cos\theta}\sin\theta + u\cos\theta\right)$$
$$= mu\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta}\right) = mu\left(\frac{1}{\cos\theta}\right)$$
$$= mu\sec\theta, \quad \text{as required}$$

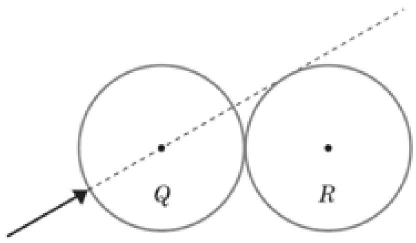
The impulse is perpendicular to the plane and its magnitude is the difference in the linear momenta resolved perpendicular to the plane.

From equation ①, in part a, $v = \dfrac{u\sin\theta}{\cos\theta}$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

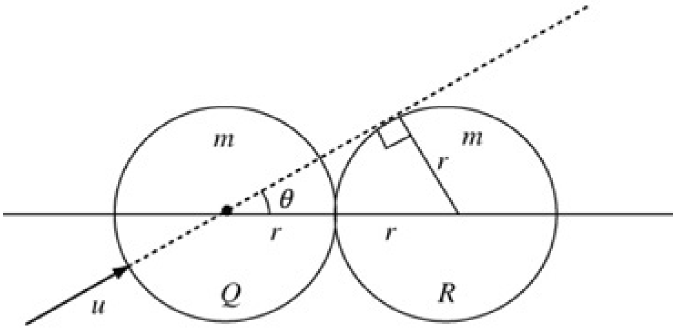
Review Exercise 1
Exercise A, Question 27

Question:



A smooth uniform sphere P is at rest on a smooth horizontal plane, when it is struck by an identical sphere Q moving on the plane. Immediately before the impact, the line of motion of the centre of Q is tangential to the sphere P, as shown the figure. The direction of motion of Q is turned through $30°$ by the impact. Find the coefficient of restitution between the spheres. **[E]**

Solution:



Let the mass of each of the spheres be m and the radius of each of the spheres be r.
Let the angle the direction of motion of Q makes with the line of centres before the impact be θ.

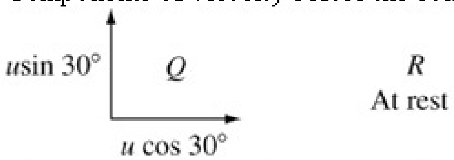
$\sin\theta = \frac{r}{2r} = \frac{1}{2} \Rightarrow \theta = 30^\circ$

Hence, the angle the direction of motion of Q makes with the line of centres, after the impact, is 60°.

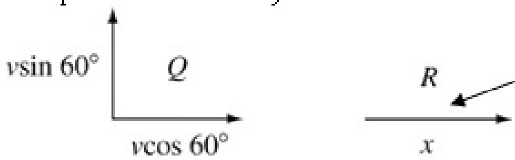
Q is turned through 30° so, after the collision, it makes an angle of $30^\circ + 30^\circ = 60^\circ$ with the line of centres.

Let the speed of Q immediately after the impact be v and the speed of R immediately after the impact be x.

Components of velocity before the collision



Components of velocity after the collision



Initially, R is at rest and the impulse of Q on R acts along the line of centres. So, after the impact, R moves along the line of centres.

Perpendicular to the line of centres
For Q

$u\sin 30^\circ = v\sin 60^\circ$

$\frac{1}{2}u = \frac{\sqrt{3}}{2}v \Rightarrow u = v\sqrt{3}.$ ①

As the impulse is along the line of centres, the component of the velocity of Q perpendicular to the line of centres is unchanged.

Parallel to the line of centres
Conservation of linear momentum

$mu\cos 30^\circ = mv\cos 60^\circ + mx$

$x = \frac{\sqrt{3}}{2}u - \frac{1}{2}v$ ②

Newton's law of restitution
velocity of separation $= e \times$velocity of approach

$x - v\cos 60^\circ = eu\cos 30^\circ$

$$x = \frac{\sqrt{3}}{2}eu + \frac{1}{2}v \qquad ③$$

Eliminating x from ② and ③

$$\frac{\sqrt{3}}{2}eu + \frac{1}{2}v = \frac{\sqrt{3}}{2}u - \frac{1}{2}v$$

$$v = \frac{\sqrt{3}}{2}u(1-e)$$

You now use equation ① to eliminate u and v.

Using ①

$$v = \frac{\sqrt{3}}{2}v\sqrt{3}(1-e)$$

$$1 = \frac{3}{2}(1-e) \Rightarrow \frac{2}{3} = 1-e$$

Dividing both sides by v and using $\sqrt{3}\times\sqrt{3} = 3$.

$$e = \frac{1}{3}$$

The coefficient of restitution between the spheres is $\frac{1}{3}$.

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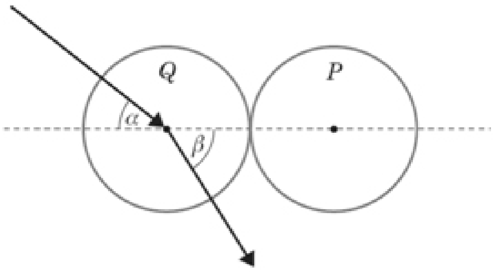
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 28

Question:

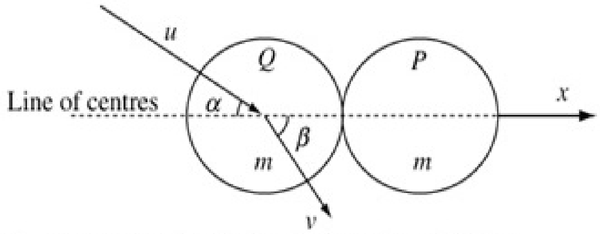


A smooth sphere P lies at rest on a smooth horizontal plane. A second identical sphere Q, moving on the plane, collides with the sphere P. Immediately before the collision the direction of motion of Q makes an angle α with the line joining the centres of the spheres. Immediately after the collision the direction of motion of Q makes an angle β with the line joining the centres of spheres, as shown in the figure. The coefficient of restitution between the spheres is e.
Show that $(1-e)\tan\beta = 2\tan\alpha$. **[E]**

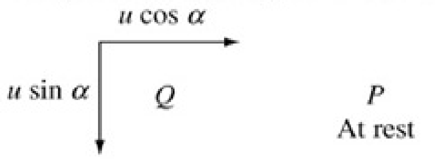
Solution:

Let the mass of each of the spheres be m.
Let the speed of Q immediately before the collision be u and its speed immediately after the collision be v.
Let the speed of P immediately after the collision be x.

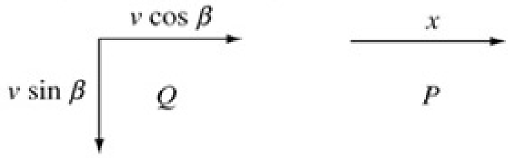
You need to introduce a number of variables to solve this question and you need to make clear to an examiner what the variables stand for. You can do this with a clearly labelled diagram.



Components of velocity before the collision



Components of velocity after the collision



Perpendicular to the line of centres
For Q
$u\sin\alpha = v\sin\beta$ ①

As the impulse is along the line of centres, the component of the velocity of Q perpendicular to the line of centres is unchanged.

Along line of centres

Conservation of linear momentum
$\not{m}u\cos\alpha = \not{m}v\sin\beta + \not{m}x$
$x = u\cos\alpha - v\cos\beta$ ②

Use these two equations to eliminate x, the speed of P.

Newton's law of restitution
velocity of separation $= e \times$velocity of approach
$x - v\cos\beta = eu\cos\alpha$
$x = eu\cos\alpha + v\cos\beta$ ③

Eliminating x between ② and ③
$u\cos\alpha - v\cos\beta = eu\cos\alpha + v\cos\beta$
$(1-e)u\cos\alpha = 2v\cos\beta$ ④

Dividing ① by ④
$$\frac{u\sin\alpha}{(1-e)u\cos\alpha} = \frac{v\sin\beta}{2v\cos\beta}$$
$$\frac{\tan\alpha}{1-e} = \frac{\tan\beta}{2}$$
$(1-e)\tan\beta = 2\tan\alpha$, as required

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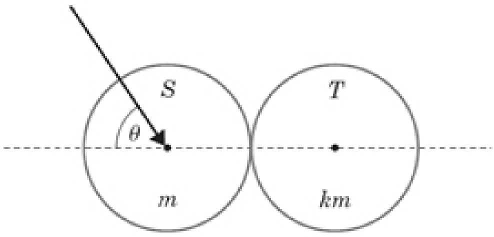
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 29

Question:

A smooth uniform sphere S of mass m is moving on a smooth horizontal table. The sphere S collides with another smooth uniform sphere T, of the same radius as S but of mass $km, k > 1$, which is at rest on the table. The coefficient of restitution between the spheres is e. Immediately before the spheres collide the direction of motion of S makes an angle θ with the line joining their centres, as shown in the figure.



Immediately after the collision the directions of motion of S and T are perpendicular.

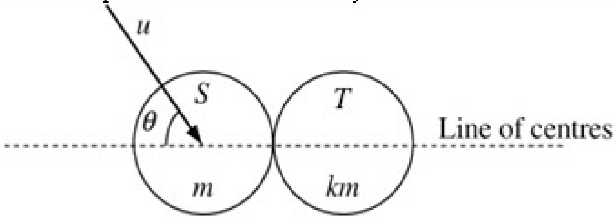
a Show that $e = \frac{1}{k}$.

Given that $k = 2$ and that the kinetic energy lost in the collision is one quarter of the initial kinetic energy,

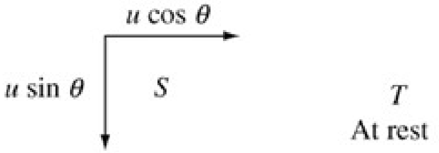
b find the value of θ. **[E]**

Solution:

a Let the speed of S immediately before the collision be u and its speed immediately after the collision be v.
Let the speed of T immediately after the collision be x.



Components of velocity before the collision



Components of velocity after the collision

Before the collision, T is at rest and the impulse on T acts along the line of centres. So, after the collision, T moves along the line of centres. After the collision, the spheres are moving in perpendicular directions, so S is moving perpendicular to the line of centres.

Parallel to the line of centres

Conservation of linear momentum

$mu\cos\theta = kmx$

$kx = u\cos\theta$ ①

Newton's law of restitution

velocity of separation $= e \times$velocity of approach

$x = eu\cos\theta$ ②

Parallel to the line of centres, the velocity of separation of the spheres is just x.

Dividing ② by ①

$\frac{x}{kx} = \frac{eu\cos\theta}{u\cos\theta}$

$e = \frac{1}{k}$, as required

b If $k = 2$, then $e = \frac{1}{2}$.

Using the printed answer to part **a**.

From equation ② in part **a**

$x = \frac{1}{2}u\cos\theta$

Perpendicular to the line of centres

For S

$v = u\sin\theta$

The initial kinetic energy is $\frac{1}{2}mu^2$

The final kinetic energy is

The mass of T is km.

$$\frac{1}{2}mv^2 + \frac{1}{2}kmx^2 = \frac{1}{2}m(u\sin\theta)^2 + \frac{1}{2}km\left(\frac{1}{2}u\cos\theta\right)^2$$

$$= \frac{1}{2}mu^2\sin^2\theta + \frac{1}{4}mu^2\cos^2\theta$$

As $k = 2$.

The total loss in kinetic energy is

$$\frac{1}{2}mu^2 - \left(\frac{1}{2}mu^2\sin^2\theta + \frac{1}{4}mu^2\cos^2\theta\right)$$

$$= \frac{1}{2}mu^2(1-\sin^2\theta) - \frac{1}{4}mu^2\cos^2\theta$$

$$= \frac{1}{2}mu^2\cos^2\theta - \frac{1}{4}mu^2\cos^2\theta = \frac{1}{4}mu^2\cos^2\theta$$

Hence

$$\frac{1}{4}mu^2\cos^2\theta = \frac{1}{4}\times\frac{1}{2}mu^2$$

$$\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = 45°$$

The loss of energy is one quarter of the initial kinetic energy; that is one quarter of $\frac{1}{2}mu^2$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 30

Question:

A smooth uniform sphere A has mass $2m$ kg and another smooth uniform sphere B, with the same radius as A, has mass m kg. The spheres are moving on a smooth horizontal plane when they collide. At the instant of collision the line joining the centres of the spheres is parallel to $\mathbf{j}$. Immediately **after** the collision, the velocity of A is $(3\mathbf{i}-\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(2\mathbf{i}+\mathbf{j})\text{m s}^{-1}$. The coefficient of restitution between the spheres is $\frac{1}{2}$.

a Find the velocities of the two spheres immediately before the collision.

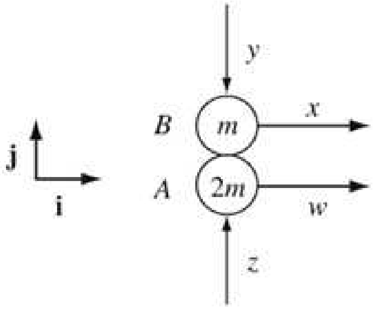
b Find the magnitude of the impulse in the collision.

c Find, to the nearest degree, the angle through which the direction of motion of A is deflected by the collision. **[E]**

Solution:

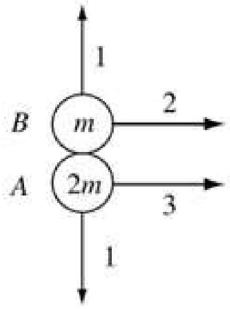
a Let the velocity of B before the collision be $(x\mathbf{i} - y\mathbf{j})\text{m s}^{-1}$ and the velocity of A before the collision be $(w\mathbf{i} + z\mathbf{j})\text{m s}^{-1}$.

Before the collision



The components of the velocities are in m s^{-1}.

After the collision



Parallel to **i**

$x = 2, w = 3$

As the impulse is in the direction of **j**, the components of the velocities of both A and B in the direction of **i** are unchanged.

Parallel to **j**

Conservation of linear momentum

$-my + 2mz = m \times 1 - 2m \times 1$

$-y + 2z = -1$ ①

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$1 - (-1) = \frac{1}{2}(y + z)$$

$$y + z = 4 \quad ②$$

① + ②

$3z = 3 \Rightarrow z = 1$

Substituting $z = 1$ into ②

$y + 1 = 4 \Rightarrow y = 3$

The velocity of A is $(3\mathbf{i} + \mathbf{j})\text{m s}^{-1}$.

The velocity of B is $(2\mathbf{i} - 3\mathbf{j})\text{m s}^{-1}$.

b Considering the change in momentum of A in the direction of $\mathbf{j}$.

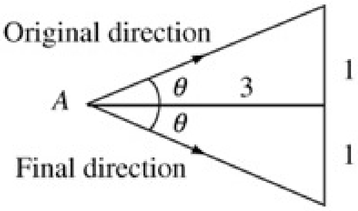
As the impulse is in the direction of $\mathbf{j}$, you can consider the change of momentum of either A or B in the direction of $\mathbf{j}$.

$$I = 2mv - 2mu$$
$$= 2m \times 1 - 2m \times (-1) = 4m$$

The mass of A is $2m$.

The magnitude of the impulse is $4m$ N s.

c



A is deflected from the direction of $(3\mathbf{i} + \mathbf{j})$ to the direction of $(3\mathbf{i} - \mathbf{j})$.

The angle of deflection is given by

$$2\theta = 2\arctan\frac{1}{3} = 37^\circ \quad \text{(nearest degree)}$$

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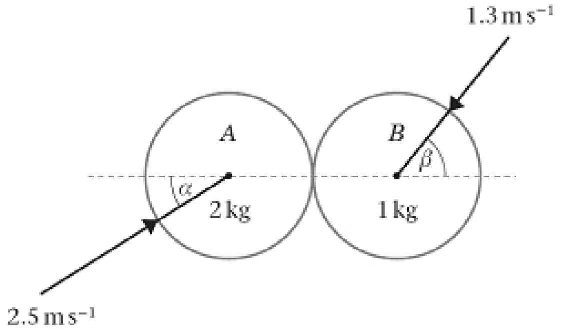
Review Exercise 1
Exercise A, Question 31

Question:



Two smooth uniform spheres A and B of equal radius have masses 2 kg and 1 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is 2.5 m s^{-1} and the speed of B is 1.3 m s^{-1}. When they collide the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B, where $\tan \alpha = \frac{4}{3}$ and $\tan \beta = \frac{12}{5}$, as shown in the figure.

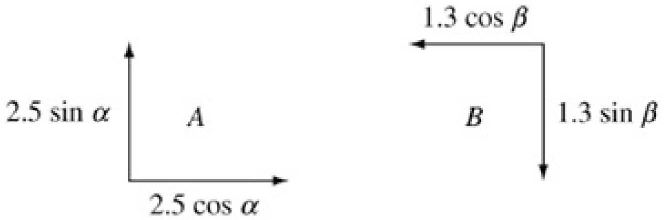
a Find the components of the velocities of A and B perpendicular and parallel to the line of centres immediately before the collision.

The coefficient of restitution between A and B is $\frac{1}{2}$.

b Find, to one decimal place, the speed of each sphere after the collision. **[E]**

Solution:

a Components of the velocity before the collision.
All velocities are in m s^{-1}

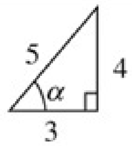


The component of the velocity of A perpendicular to the line of centres immediately before the collision is

$2.5\sin\alpha \text{ m s}^{-1} = 2.5\times\frac{4}{5}\text{ m s}^{-1} = 2\text{ m s}^{-1}$

The component of the velocity of A parallel to the line of centres immediately before the collision is

$2.5\cos\alpha \text{ m s}^{-1} = 2.5\times\frac{3}{5}\text{ m s}^{-1} = 1.5\text{ m s}^{-1}$



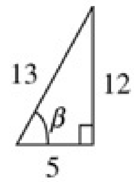
This sketch illustrates that, as $3^2+4^2=5^2$, if $\tan\beta=\frac{4}{3}$, then $\sin\beta=\frac{4}{5}$ and $\cos\beta=\frac{3}{5}$.

The component of the velocity of B perpendicular to the line of centres immediately before the collision is

$1.3\sin\beta \text{ m s}^{-1} = 1.3\times\frac{12}{13}\text{ m s}^{-1} = 1.2\text{ m s}^{-1}$

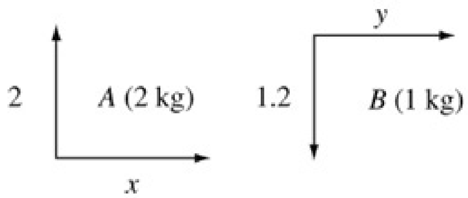
The component of the velocity of B parallel to the line of centres immediately before the collision is

$1.3\cos\beta \text{ m s}^{-1} = 1.3\times\frac{5}{13}\text{ m s}^{-1} = 0.5\text{ m s}^{-1}$



This sketch illustrates that, as $5^2+12^2=13^2$, if $\tan\beta=\frac{12}{5}$, then $\sin\beta=\frac{12}{13}$ and $\cos\beta=\frac{5}{13}$.

b Let the components of the velocity after the collision be, with all velocities in m s^{-1},



The components perpendicular to the line of centres are unchanged.

Parallel to the line of centres

Conservation of linear momentum

$2x + y = 2\times 1.5 - 1\times 0.5$

$2x + y = 2.5$ ①

Newton's law of restitution

velocity of separation $= e\times$velocity of approach

$y - x = \frac{1}{2}(1.5 + 0.5)$

$y - x = 1$ ②

A approaches the point of the collision with speed 1.5 m s^{-1} along the line of centres. B approaches the point of the collision with speed 0.5 m s^{-1} along the line of centres. So the two spheres are approaching one another at the speed of $(1.5 + 0.5) \text{ m s}^{-1} = 2 \text{ m s}^{-1}$

① − ②

$3x = 1.5 \Rightarrow x = 0.5$

From ②
$y = 1 + x = 1.5$

The speed of A is
$\sqrt{(0.5^2 + 2^2)} \text{ m s}^{-1} = \sqrt{4.25} \text{ m s}^{-1} = 2.1 \text{ m s}^{-1}$ (1 d.p.)

The speed of B is
$\sqrt{(1.2^2 + 1.5^2)} \text{ m s}^{-1} = \sqrt{3.69} \text{ m s}^{-1} = 1.9 \text{ m s}^{-1}$ (1 d.p.)

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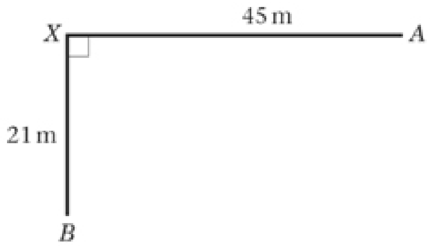
Review Exercise 1
Exercise A, Question 32

Question:



The figure represents the scene of a road accident. A car of mass 600 kg collided at the point X with a stationary van of mass 800 kg. After the collision the van came to rest at the point A having travelled a horizontal distance of 45 m, and the car came to rest at the point B having travelled a horizontal distance of 21 m. The angle AXB is $90°$. The accident investigators are trying to establish the speed of the car before the collision and they model both vehicle as small spheres.

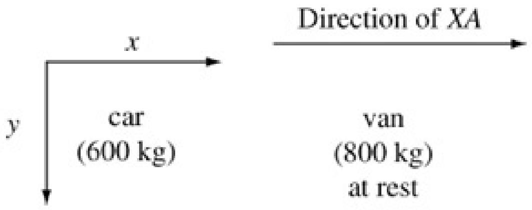
a Find the coefficient of restitution between the car and the van.

The investigators assume that after the collision, and until the vehicles came to rest, the van was subject to a constant horizontal force of 500 N acting along AX and the car to a constant horizontal force of 300 N along BX.

b Find the speed of the car immediately before the collision. **[E]**

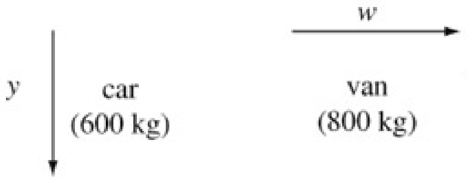
Solution:

a Let the components of the velocity of the car before the collision, with all components in m s^{-1}, be



As the van is at rest, after the collision it must travel along the line of centres of the car and the van. In the diagram in the question, *XA* must be the line of centres, so you consider the components of velocity perpendicular and parallel to *XA*.

Let the components of the velocity of the car and van after the collision, with all components in m s^{-1}, be



After the collision, the van is moving along *XA* and the car is moving perpendicular to *XA*.

Parallel to XA

Conservation of linear momentum

$$600x = 800w \Rightarrow w = \frac{3}{4}x \quad *$$

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$w = ex$$

Hence

$$\frac{3}{4}x = ex$$

$$e = \frac{3}{4}$$

b For the van

$$\mathbf{F} = m\mathbf{a}$$

$$-500 = 800a \Rightarrow a = -0.625$$

$$v^2 = u^2 + 2as$$

$$0^2 = w^2 - 2 \times 0.625 \times 45$$

$$w^2 = 56.25 \Rightarrow w = 7.5$$

For the car

$$\mathbf{F} = m\mathbf{a}$$

$$-300 = 600a \Rightarrow a = -0.5$$

$$v^2 = u^2 + 2as$$

$$0^2 = y^2 - 2 \times 0.5 \times 21$$

$$y^2 = 21 \Rightarrow y = \sqrt{21}$$

To find *w* (and hence *x*) and *y*, you need to use both Newton's second law and the kinematic equation for constant acceleration, $v^2 = u^2 + 2as$.

From equation * in part **a**

$w = \frac{3}{4}x$

$7.5 = \frac{3}{4}x \Rightarrow x = \frac{7.5}{\frac{3}{4}} = 10$

Let the speed of the car immediately before the collision be U m s^{-1}

$U^2 = x^2 + y^2 = 10^2 + 21 = 121$

$U = \sqrt{121} = 11$

The speed of the car immediately before the collision is 11 m s^{-1}.

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Review Exercise 1
Exercise A, Question 33

Question:

A smooth sphere T is at rest on a smooth horizontal table. An identical sphere S moving on the table with speed U collides with T. The directions of motion of S before and after impact make angles of $30°$ and $\beta°$ $(0 < \beta < 90)$ respectively with L, the line of centres at the moment of impact. The coefficient of restitution between S and T is e.

a Show that V, the speed of T immediately after impact, is given by $V = \frac{U\sqrt{3}}{4}(1+e)$.

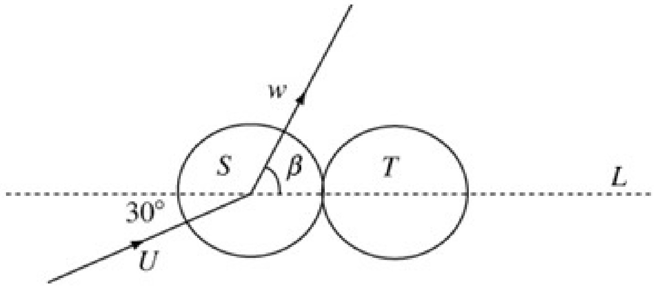
b Find the components of the velocity of S, parallel and perpendicular to L, immediately after impact.

Given that $e = \frac{2}{3}$,

c find, to 1 decimal place, the value of β. **[E]**

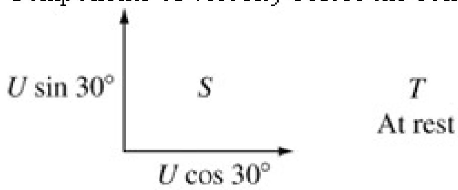
Solution:

a

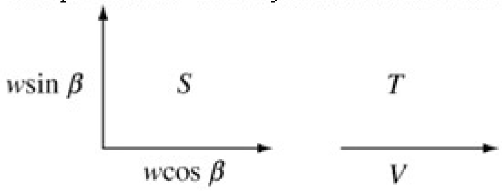


Let the speed of S after the collision be w and the mass of both S and T be m.

Components of velocity before the collision



Components of velocity after the collision



As the impact is along L, sphere T must move along L.

Parallel to the L

Conservation of linear momentum

$mU\cos 30° = mw\cos\beta + mV$

$w\cos\beta + V = \frac{\sqrt{3}}{2}U$ ① ← Using $\cos 30° = \frac{\sqrt{3}}{2}$.

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$V - w\cos\beta = eU\cos 30°$$

$$V - w\cos\beta = \frac{\sqrt{3}}{2}eU \quad ②$$

① + ②

$2V = \frac{\sqrt{3}}{2}U + \frac{\sqrt{3}}{2}eU = \frac{U\sqrt{3}}{2}(1+e)$

$V = \frac{U\sqrt{3}}{4}(1+e)$, as required

b Subtracting ② from ①

$$2w\cos\beta = \frac{\sqrt{3}}{2}U - \frac{\sqrt{3}}{2}eU = \frac{\sqrt{3}}{2}(1-e)U$$

$$w\cos\beta = \frac{\sqrt{3}}{4}(1-e)U \quad ③$$

The components of the velocity of S, parallel and perpendicular to L, immediately after impact, are $w\cos\beta$ and $w\sin\beta$ respectively. You find $w\cos\beta$ using the equations ① and ② in part **a**.

Perpendicular to L
For S

$$w\sin\beta = U\sin 30° = \frac{1}{2}U \quad ④$$

The component of the velocity of S perpendicular to the impulse is unchanged.

The components of the velocity of S, parallel and perpendicular to L, immediately after impact are $\frac{\sqrt{3}}{4}(1-e)U$ and $\frac{1}{2}U$, respectively.

c If $e = \frac{2}{3}$, from ③

$$w\cos\beta = \frac{\sqrt{3}}{4}\left(1-\frac{2}{3}\right)U = \frac{\sqrt{3}}{12}U$$

Also $w\sin\beta = \frac{1}{2}U$

Dividing

$$\frac{w\sin\beta}{w\cos\beta} = \frac{\frac{1}{2}U}{\frac{\sqrt{3}}{12}U}$$

$$\tan\beta = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\beta = 73.9° \quad (1\text{ d.p.})$$

Use your calculator to complete the question. As no mode is specified, $\beta = 1.3$ radians is also acceptable.

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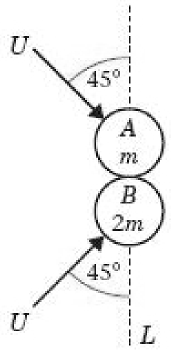
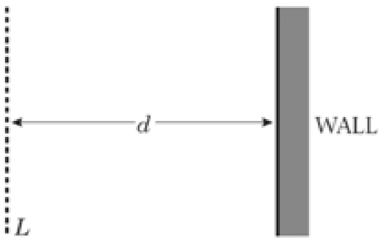
Review Exercise 1
Exercise A, Question 34

Question:

Two small spheres A and B, of equal size and of mass m and $2m$ respectively, are moving initially with the same speed U on a smooth horizontal floor. The spheres collide when their centres are on a line L. Before the collision the spheres are moving towards each other, with their directions of motion perpendicular to each other and each inclined at an angle $45°$ to the line L, as shown in the figure below. The coefficient of restitution between the spheres is $\frac{1}{2}$.



a Find the magnitude of the impulse which acts on A in the collision.

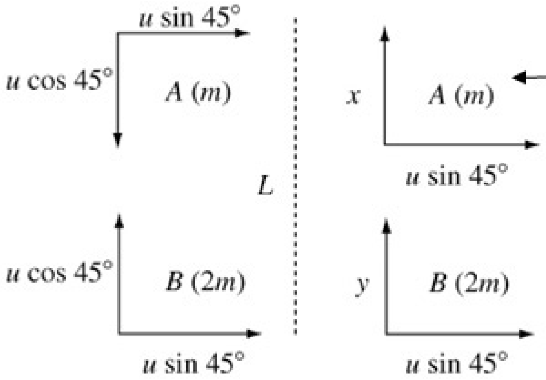


The line L is parallel to and a distance d from a smooth vertical wall, as shown in the second figure.

b Find, in terms of d, the distance between the points at which the spheres first strike the wall. **[E]**

Solution:

a Components before collision Components after collision



The components perpendicular to L are unchanged by the collision. In this case, all of the components perpendicular to L are $u \sin 45° = \frac{u}{\sqrt{2}}$.

Parallel to L

Conservation of linear momentum $(\uparrow)$

$$2mu\cos 45° - mu\cos 45° = mx + 2my$$

$$x + 2y = \frac{u}{\sqrt{2}} \quad ①$$

Newton's law of restitution

velocity of separation $= e \times$velocity of approach

$$x - y = \frac{1}{2}\left(u\cos 45° + u\cos 45°\right)$$

$$x - y = \frac{u}{\sqrt{2}} \quad ②$$

Use $\cos 45° = \frac{1}{\sqrt{2}}$.

① − ②

$3y = 0 \Rightarrow y = 0$

Hence $x = \frac{u}{\sqrt{2}}$

As $y = 0$, after the collision B is travelling perpendicular to L. You will need this to solve part **b**.

The impulse on A is given by

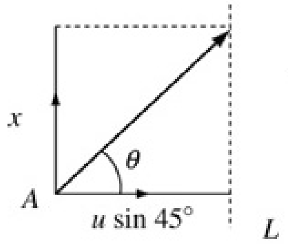
$(\uparrow) \quad I =$ final momentum of $A -$ initial momentum of A

$$= \frac{mu}{\sqrt{2}} + \frac{mu}{\sqrt{2}} = \frac{2mu}{\sqrt{2}} = \sqrt{2}mu$$

$m\left(-u\sin 45°\right)$

The magnitude of the impulse which acts on A in the collision is $\sqrt{2}mu$.

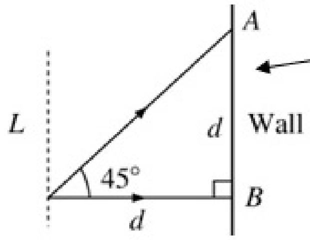
b



This diagram shows the components of the velocity of A parallel and perpendicular to L. As both x and $u\sin 45^\circ$ are $\frac{u}{\sqrt{2}}$, A moves at an angle of 45° to the direction of motion of B.

The direction of motion of A is given by

$$\tan\theta = \frac{x}{u\sin 45^\circ} = \frac{\frac{u}{\sqrt{2}}}{\frac{u}{\sqrt{2}}} = 1 \Rightarrow \theta = 45^\circ$$



This diagram shows the paths of A and B from the point of collision to the points where they hit the wall. The triangle is isosceles and the required distance is d.

The distance between the points at which the spheres first strike the wall is d.

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Review Exercise 1
Exercise A, Question 35

Question:

Two smooth uniform spheres A and B have equal radii. Sphere A has mass m and sphere B has mass km. The spheres are at rest on a smooth horizontal table. Sphere A is then projected along the table with speed u and collides with B. Immediately before the collision, the direction of motion of A makes an angle of $60°$ with the line joining the centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$.

a Show that the speed of B immediately after the collision is $\frac{3u}{4(k+1)}$.

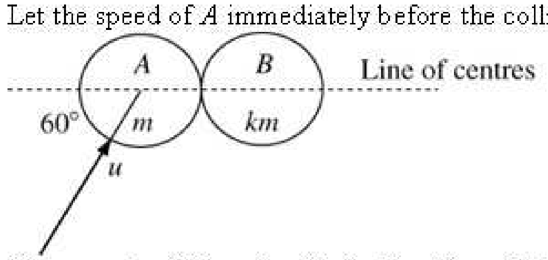
Immediately after the collision the direction of motion of A makes an angle $\arctan(2\sqrt{3})$ with the direction of motion of B.

b Show that $k = \frac{1}{2}$.

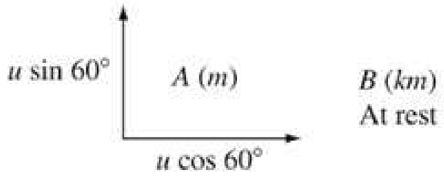
c Find the loss of kinetic energy due to the collision. **[E]**

Solution:

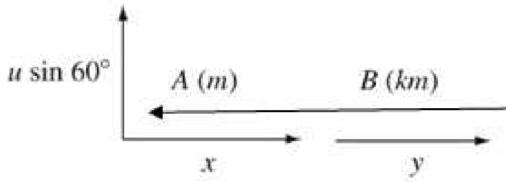
a Let the speed of A immediately before the collision be u.



Components of the velocities before the collision



Let the components of the velocities after collision be



As the impulse is along the line of centres, the component of the velocity of A perpendicular to the line of centres is unchanged.

Parallel to the line of centres

Conservation of linear momentum

$mu\cos 60° = mx + kmy$

$x + ky = \frac{1}{2}u$ ①

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$y - x = \frac{1}{2}u\cos 60° = \frac{1}{4}u \quad ②$$

$$ky + y = \frac{3u}{4}$$

$$y = \frac{3u}{4(k+1)}\text{, as required}$$

As B moves along the line of centres, the component, y, of the velocity of B along the line of centres is the velocity of B. So to solve part **a**, you must find y from this pair of simultaneous equations.

b From ②

$$x = y - \frac{u}{4} = \frac{3u}{4(k+1)} - \frac{u}{4} = \frac{3u - u(k+1)}{4(k+1)}$$

$$= \frac{(2-k)u}{4(k+1)}$$

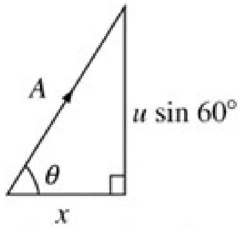
The direction of motion of A is given by

$$\tan\theta = \frac{u\sin 60^\circ}{x} = \frac{\frac{\sqrt{3}u}{2}}{\frac{(2-k)u}{4(k+1)}}$$

$$= \frac{4(k+1)\sqrt{3}}{2(2-k)} = 2\sqrt{3}$$

$$k+1 = 2-k$$

$$2k = 1 \Rightarrow k = \frac{1}{2}, \text{ as required}$$



The question gives you that the direction of motion of A makes an angle $\arctan(2\sqrt{3})$ with the line of centres, so $\tan\theta = 2\sqrt{3}$. This gives an equation that you can solve for k.

c If $k = \frac{1}{2}$,

$$y = \frac{3u}{4\left(\frac{1}{2}+1\right)} = \frac{1}{2}u$$

$$x = \frac{\left(2-\frac{1}{2}\right)u}{4\left(\frac{1}{2}+1\right)} = \frac{1}{4}u$$

The kinetic energy of the system after the collision is

$$\frac{1}{2}m\left(x^2 + (u\sin 60^\circ)^2\right) + \frac{1}{2}kmy^2$$

$$= \frac{1}{2}m\left(\frac{u^2}{16} + \frac{3u^2}{4}\right) + \frac{1}{4}m \times \frac{1}{4}u^2$$

$$= \frac{1}{2}mu^2\left(\frac{1}{16} + \frac{3}{4} + \frac{1}{8}\right) = \frac{15}{32}mu^2$$

After the collision the velocity of A has components x and $u\sin 60^\circ$. So the kinetic energy of A after the collision is $\frac{1}{2}m\left(x^2 + (u\sin 60^\circ)^2\right)$.

The loss in kinetic energy is

$$\frac{1}{2}mu^2 - \frac{15}{32}mu^2 = \frac{1}{32}mu^2$$

Before the collision only A is moving and it has speed u, so the initial kinetic energy of the system is $\frac{1}{2}mu^2$.

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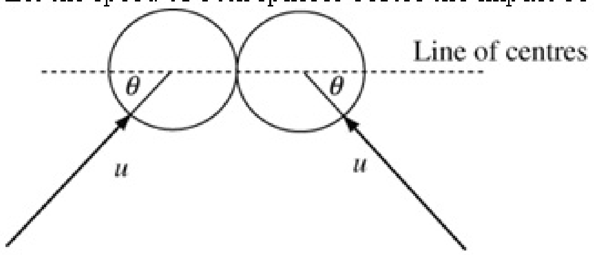
Review Exercise 1
Exercise A, Question 36

Question:

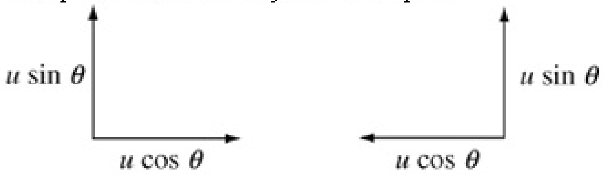
Two equal smooth spheres approach each other from opposite directions with equal speeds. The coefficient of restitution between the spheres is e. At the moment of impact, their common normal is inclined at an angle θ to the original direction of motion. After impact, each sphere moves at right angles to its original direction of motion.
Show that $\tan\theta = \sqrt{e}$. **[E]**

Solution:

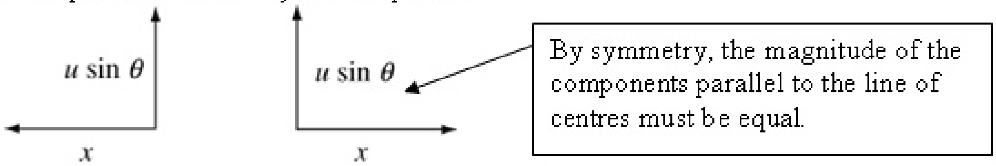
Let the speed of both spheres before the impact be u and the mass of each sphere m.



Components of velocity before impact



Let the components of velocity after impact be



Parallel to the line of centres

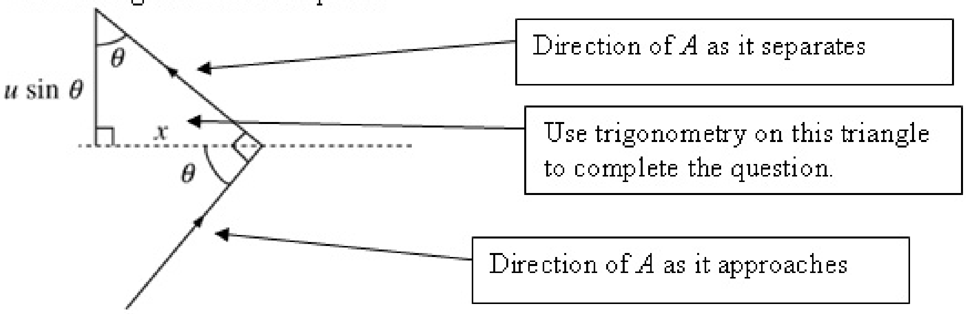
Newton's law of restitution

velocity of separation $= e \times$velocity of approach

$$2x = e2u\cos\theta$$

$$x = eu\cos\theta$$

Considering the left hand sphere



$$\tan\theta = \frac{x}{u\sin\theta} = \frac{eu\cos\theta}{u\sin\theta}$$

$$e = \tan\theta \times \frac{\sin\theta}{\cos\theta} = \tan^2\theta$$

$\tan\theta = \sqrt{e}$, as required

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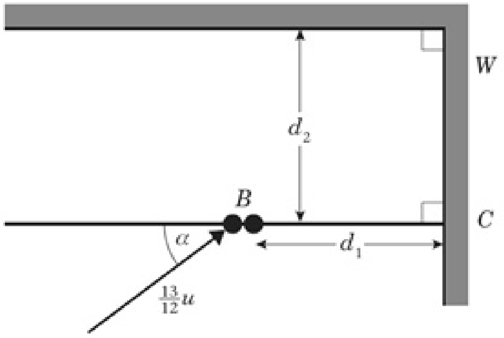
Review Exercise 1
Exercise A, Question 37

Question:



A small ball Q of mass $2m$ is at rest at the point B on a smooth horizontal plane. A second small ball P of mass m is moving on the plane with speed $\frac{13}{12}u$ and collides with Q. Both the balls are smooth, uniform and of the same radius. The point C is on a smooth vertical wall W which is at a distance d_1 from B, and BC is perpendicular to W. A second smooth vertical wall is perpendicular to W and at a distance d_2 from B. Immediately before the collision occurs, the direction of motion of P makes an angle α with BC, as shown in the figure, where $\tan\alpha = \frac{5}{12}$.

The line of centres of P and Q is parallel to BC. After the collision Q moves towards C with speed $\frac{3}{5}u$.

a Show that, after the collision, the velocity components of P parallel and perpendicular to CB are $\frac{1}{5}u$ and $\frac{5}{12}u$ respectively.

b Find the coefficient of restitution between P and Q.

c Show that when Q reaches C, P is at a distance $\frac{4}{3}d_1$ from W.

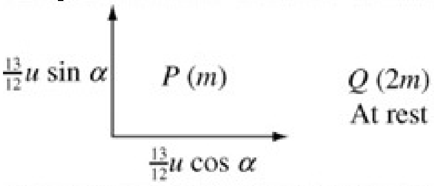
For each collision between a ball and a wall the coefficient of restitution is $\frac{1}{2}$. Given that the balls collide with each other again,

d show that the time between the two collisions of the balls is $\frac{15d_1}{u}$,

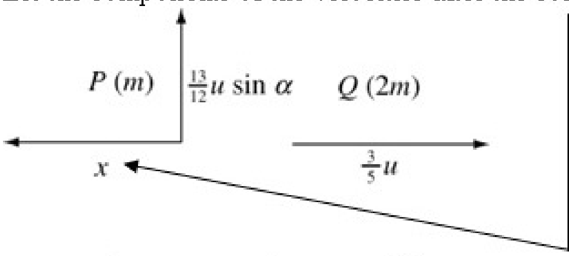
e find the ratio $d_1 : d_2$. **[E]**

Solution:

a Components of the velocities before the collision

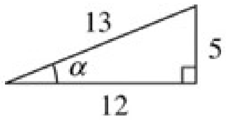


Let the components of the velocities after the collision be



The direction of the component of the velocity of P along the line of centres, here called x, is not obvious. If you put it in the opposite direction to that shown here, you would get a negative value of x and your solution would still be valid.

$$\tan\alpha = \frac{5}{12} \Rightarrow \sin\alpha = \frac{5}{13}, \cos\alpha = \frac{12}{13}$$



This sketch illustrates that, as $5^2 + 12^2 = 13^2$, if $\tan\alpha = \frac{5}{12}$, then $\sin\alpha = \frac{5}{13}$ and $\cos\alpha = \frac{12}{13}$.

Perpendicular to the line of centres CB

In this direction, the component of the velocity of P is unchanged and is

$\frac{13}{12}u\sin\alpha = \frac{13}{12}u \times \frac{5}{13} = \frac{5}{12}u$, as required

Perpendicular to the line of centres CB

Conservation of linear momentum

$$m \times \frac{13}{12}u\cos\alpha = -mx + 2m \times \frac{3}{5}u$$

$x = \frac{6}{5}u - \frac{13}{12}u \times \frac{12}{13} = \frac{6}{5}u - u = \frac{1}{5}u$, as required

b Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$x + \frac{3}{5}u = e\frac{13}{12}u\cos\alpha$$

$$\frac{13}{12}u\cos\alpha = \frac{13}{12}u \times \frac{12}{13} = u$$

$$\frac{1}{5}u + \frac{3}{5}u = eu$$

$$e = \frac{4}{5}$$

c Let the time after the collision for Q to reach C be t_1.

distance = speed × time

$$d_1 = \frac{3}{5}ut_1 \Rightarrow t_1 = \frac{5d_1}{3u}$$

Perpendicular to W, in time t_1, P travels a distance s given by

distance = speed × time

$$s = \frac{1}{5}u \times t_1 = \frac{1}{5}u \times \frac{5d_1}{3u} = \frac{1}{3}d_1$$

Perpendicular to W, the component of the velocity of P after the collision is $\frac{1}{5}u$. To find the distance of P from W, you need consider only this component.

The distance of P from W is

$d_1 + s = d_1 + \frac{1}{3}d_1 = \frac{4}{3}d_1$, as required

d Before hitting W, Q has speed $\frac{3}{5}u$

After hitting W, Q has speed $e\frac{3}{5}u = \frac{1}{2} \times \frac{3}{5}u = \frac{3}{10}u$

In the direction CB, the velocity of Q relative to P is

$$\frac{3}{10}u - \frac{1}{5}u = \frac{1}{10}u$$

The time, t_2, for Q to travel from C to the point of the second collision is given by

$$t_2 = \frac{\frac{4}{3}d_1}{\frac{1}{10}u} = \frac{40d_1}{3u}$$

In the direction CB, the time is given by the distance of Q relative to $P\left(\frac{4}{3}d_1\right)$ divided by the velocity of Q relative to $P\left(\frac{1}{10}u\right)$.

The time between the two collisions is

$t_1 + t_2 = \frac{5d_1}{3u} + \frac{40d_1}{3u} = \frac{45d_1}{3u} = \frac{15d_1}{u}$, as required

e Before hitting the perpendicular wall, P has a component velocity $\frac{5}{12}u$ perpendicular to CB.

After hitting the wall, this component becomes

$$e\frac{5}{12}u = \frac{1}{2} \times \frac{5}{12}u = \frac{5}{24}u$$

If t_3 is the time for P to move from B to the wall and t_4 is the time for P to move from the wall back to CB, then

As Q moves along CB, the second collision must occur on CB. So you need to find the time it takes for P to move to the wall and return to CB.

$$t_3 = \frac{d_2}{\frac{5}{12}u} = \frac{12d_2}{5u}$$

and

$$t_4 = \frac{d_2}{\frac{5}{24}u} = \frac{24d_2}{5u}$$

$$t_3 + t_4 = \frac{15d_1}{u}$$

Q is moving along CB. So, for the second collision, P must travel from CB to the wall, which is perpendicular to W, and back to the line CB, in time $\frac{15d_1}{u}$.

$$\frac{12d_2}{5u} + \frac{24d_2}{5u} = \frac{15d_1}{u}$$

$$36d_2 = 75d_1$$

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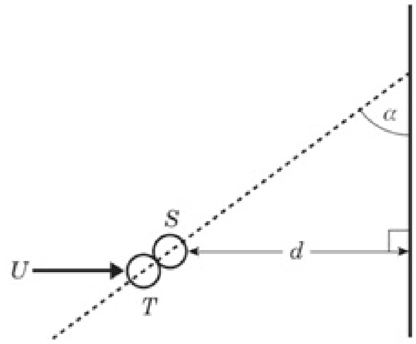
Review Exercise 1
Exercise A, Question 38

Question:



A small smooth uniform sphere S is at rest on a smooth horizontal floor at a distance d from a straight vertical wall. An identical sphere T is projected along the floor with speed U towards S and in a direction which is perpendicular to the wall. At the instant when T strikes S the line joining their centres makes an angle α with the wall, as shown in the figure.

Each sphere is modelled as having negligible diameter in comparison with d. The coefficient of restitution between the spheres is e.

a Show that the components of the velocity of T after the impact, parallel and perpendicular to the line of centres, are $\frac{1}{2}U(1-e)\sin\alpha$ and $U\cos\alpha$ respectively.

b Show that the components of the velocity of T after the impact, parallel and perpendicular to the wall are $\frac{1}{2}U(1+e)\cos\alpha\sin\alpha$ and $\frac{1}{2}U[2-(1+e)\sin^2\alpha]$ respectively.

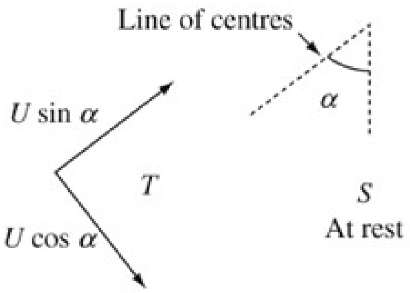
The spheres S and T strike the wall at the points A and B respectively.

Given that $e=\frac{2}{3}$ and $\tan\alpha=\frac{3}{4}$,

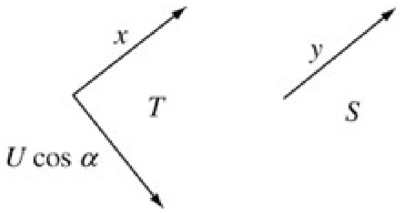
c find, in terms of d, the distance AB. **[E]**

Solution:

a The components of velocity before the collision perpendicular and parallel to the line of centres are



Let the components of velocities after the collision be



Let the mass of each sphere be m

Perpendicular to the line of centres
The component of the velocity is unchanged, so the component of the velocity of T after the impact perpendicular to the line of centres is $U \cos \alpha$, as required.

Parallel to the line of centres
Conservation of linear momentum

$mU \sin \alpha = mx + my$

$x + y = u \sin \alpha$ ①

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

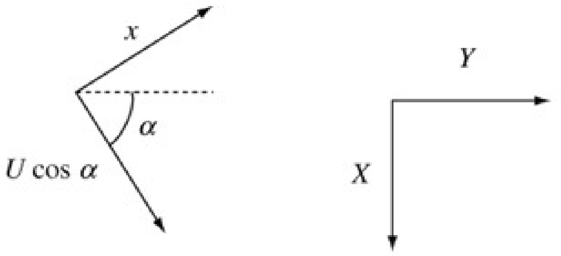
$y - x = eU \sin \alpha$ ②

① − ②

$2x = U \sin \alpha - eU \sin \alpha = U(1-e)\sin \alpha$

$x = \frac{1}{2}U(1-e)\sin \alpha$, as required

b Let the components of the velocity of T after the impact, parallel and perpendicular to the wall be X and Y respectively



$$R(\downarrow) X = U\cos\alpha\sin\alpha - x\cos\alpha$$
$$= U\cos\alpha\sin\alpha - \frac{1}{2}U(1-e)\sin\alpha\cos\alpha$$
$$= U\cos\alpha\sin\alpha\left(1 - \frac{1}{2} + \frac{1}{2}e\right) = U\cos\alpha\sin\alpha\left(\frac{1}{2} + \frac{1}{2}e\right)$$
$$= \frac{1}{2}U(1+e)\cos\alpha\sin\alpha, \text{ as required}$$

$$R(\rightarrow) Y = U\cos\alpha\cos\alpha + x\sin\alpha$$
$$= U\cos^2\alpha + \frac{1}{2}U(1-e)\sin\alpha\sin\alpha$$
$$= U(1-\sin^2\alpha) + \frac{1}{2}U(1-e)\sin^2\alpha$$
$$= \frac{1}{2}U(2 - 2\sin^2\alpha + \sin^2\alpha - e\sin^2\alpha)$$
$$= \frac{1}{2}U(2 - \sin^2\alpha - e\sin^2\alpha)$$
$$= \frac{1}{2}U\left[2-(1+e)\sin^2\alpha\right], \text{ as required}$$

c $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

> You can just write these down but, if you can't remember these relations, you can find the sine and cosine by sketching a 3, 4, 5 triangle.

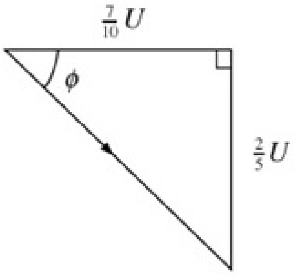
With $e = \frac{2}{3}$, the components in part **b** become

$$X = \frac{1}{2}U\left(1+\frac{2}{3}\right)\times\frac{3}{5}\times\frac{4}{5} = \frac{2}{5}U$$

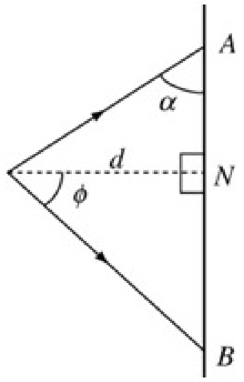
$$Y = \frac{1}{2}U\left[2-\left(1+\frac{2}{3}\right)\times\frac{9}{25}\right] = \frac{1}{2}U\left(2-\frac{3}{5}\right) = \frac{7}{10}U$$

The direction of motion of S is along the line of centres
The direction of motion of T is given by

To find the points where S and T strike the wall, you need to know the direction of motion after the collision of both S and T.



$$\tan\phi = \frac{\frac{2}{5}U}{\frac{7}{10}U} = \frac{4}{7}$$



N is the foot of the perpendicular from the point of collision to the wall.

$$\frac{d}{AN} = \tan\alpha = \frac{3}{4} \Rightarrow AN = \frac{4}{3}d$$

$$\frac{NB}{d} = \tan\phi = \frac{4}{7} \Rightarrow NB = \frac{4}{7}d$$

$$AB = AN + NB = \frac{4}{3}d + \frac{4}{7}d = \frac{40}{21}d$$

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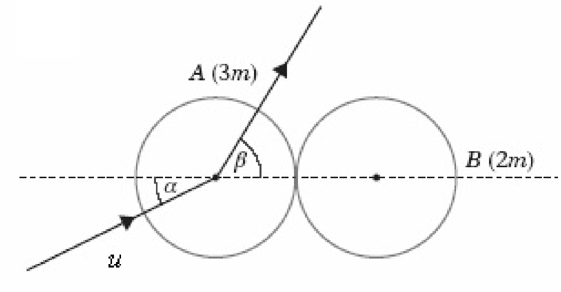
Review Exercise 1
Exercise A, Question 39

Question:



A uniform small smooth sphere of mass $3m$ moving with speed u on a smooth horizontal table collides with a stationary small sphere B of the same size as A and of mass $2m$. The direction of motion of A before impact makes an angle α with the line of centres of A and B, and the direction of motion of A after the impact makes an angle β with the same line, as shown in the figure. The coefficient of restitution between the spheres is $\frac{2}{3}$.

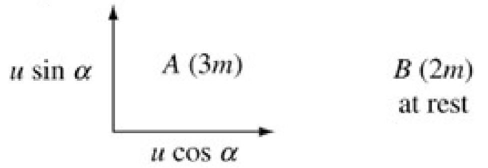
a Show that $\tan\beta = 3\tan\alpha$.

b Express $\tan(\beta - \alpha)$ in terms of t, where $t = \tan\alpha$.

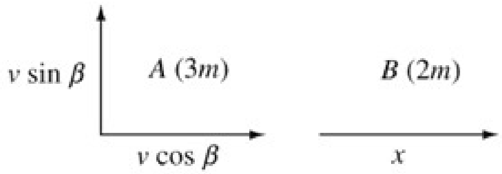
c Hence find, as α varies, the maximum angle of deflection of A caused by the impact. **[E]**

Solution:

a Let the speed of A before the collision be u,
the speed of A after the collision be v and the speed of B after the collision be x
Components before the collision



Components after the collision



Perpendicular to the line of centres

$v\sin\beta = u\sin\alpha$ ①

Parallel to the line of centres

Conservation of linear momentum

$$3mu\cos\alpha = 3mv\cos\beta + 2mx$$

$2x + 3v\cos\beta = 3u\cos\alpha$ ②

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$x - v\cos\beta = \frac{2}{3}u\cos\alpha \quad ③$$

> The relation you are asked to prove contains only angles, so you must eliminate the three velocities, u, v and x, from these 3 equations.

② $-2\times$③

$$5v\cos\beta = 3u\cos\alpha - \frac{4}{3}u\cos\alpha = \frac{5}{3}u\cos\alpha$$

$$v\cos\beta = \frac{1}{3}u\cos\alpha \quad ④$$

Divide ① by ④

$$\frac{v\sin\beta}{v\cos\beta} = \frac{u\sin\alpha}{\frac{1}{3}u\cos\alpha}$$

$\tan\beta = 3\tan\alpha$, as required

b $\tan(\beta-\alpha)=\dfrac{\tan\beta-\tan\alpha}{1+\tan\alpha\tan\beta}$

$=\dfrac{3\tan\alpha-\tan\alpha}{1+\tan\alpha\times3\tan\alpha}=\dfrac{2\tan\alpha}{1+3\tan^2\alpha}$

$=\dfrac{2t}{1+3t^2}$

c Let $f(t)=\dfrac{2t}{1+3t^2}$

The angle of deflection is the change in the angle due to the impact and that is $(\beta-\alpha)$. The maximum of $(\beta-\alpha)$ will correspond to the maximum of f(t), which can be found using calculus.

$f'(t)=\dfrac{2(1+3t^2)-2t(6t)}{(1+3t^2)^2}=\dfrac{2-6t^2}{(1+3t^2)^2}$

Using the quotient rule for differentiating.

For a maximum value $f'(t)=0$

$2-6t^2=0$

$t=\dfrac{1}{\sqrt{3}}$

For a collision, $0\le\alpha<90°$, so the negative solution can be ignored.

At $t=\dfrac{1}{\sqrt{3}}$

$f(t)=\dfrac{2\times\frac{1}{\sqrt{3}}}{1+3\left(\frac{1}{\sqrt{3}}\right)^2}=\dfrac{2\times\frac{1}{\sqrt{3}}}{1+1}=\dfrac{1}{\sqrt{3}}$

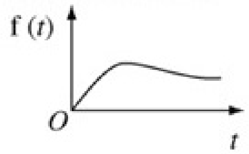
Hence

$\tan(\beta-\alpha)=\dfrac{1}{\sqrt{3}}$

$\beta-\alpha=30°$

The maximum angle of deflection of A caused by the impact is $30°$.

The least deflection is clearly when $\alpha=0$ (there is no deflection then) and the deflection initially increases as α increases so, as the function is continuous, the following stationary value must be a maximum.



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Review Exercise 1
Exercise A, Question 40

Question:

Two identical spheres A and B lie at rest on a smooth horizontal table. Sphere B is projected along the table towards sphere A with velocity $u\mathbf{i}+v\mathbf{j}$, where $\mathbf{i}$ is the unit vector along the line of centres at the time of impact and $\mathbf{j}$ is a unit vector perpendicular to $\mathbf{i}$ and in the plane of the table. Given that the coefficient of restitution between the spheres is e,

a find the velocities of the spheres after impact.

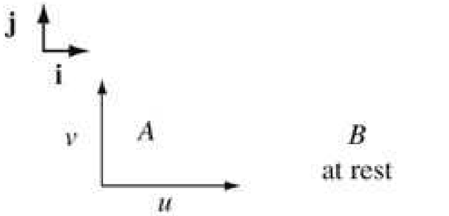
Given further that the velocity of B before impact makes an angle θ with the direction of $\mathbf{i}$ and that the velocity of B after impact makes an angle ϕ with the direction of $\mathbf{i}$,

b show that $\tan(\phi-\theta)=\dfrac{\tan\theta(1+e)}{1-e+2\tan^2\theta}$.

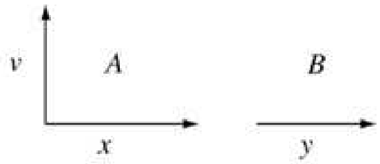
c Hence show that, as θ varies, the maximum value of the angle of deviation, $\phi-\theta$, occurs when $\tan^2\theta=\dfrac{1-e}{2}$. **[E]**

Solution:

Let the mass of both spheres be m
Components of velocity before impact



Let the components of the velocities after impact be



Parallel to **i**
Conservation of linear momentum

$mu = mx + my$

$x + y = u$ ①

Newton's law of restitution
velocity of separation $= e \times$ velocity of approach

$y - x = eu$ ②

① + ②

$2y = u + eu \Rightarrow y = \dfrac{(1+e)u}{2}$

① − ②

$2x = u - eu \Rightarrow x = \dfrac{(1-e)u}{2}$

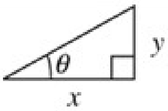
After the impact, the velocity of A is

$x\mathbf{i} + v\mathbf{j} = \dfrac{(1-e)u}{2}\mathbf{i} + v\mathbf{j}$

The component in the **j** direction is unchanged by the impulse as it is perpendicular to the impulse.

and the velocity of B is

$y\mathbf{i} = \dfrac{(1+e)u}{2}\mathbf{i}$



If $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, then $\tan\theta = \dfrac{y}{x}$.

b $\tan\theta = \dfrac{v}{u}$

$\tan\phi = \dfrac{v}{x} = \dfrac{2v}{(1-e)u}$

$= \dfrac{2}{1-e} \times \dfrac{v}{u} = \dfrac{2}{1-e}\tan\theta$

As $\dfrac{v}{u} = \tan\theta$

$\tan(\phi - \theta) = \dfrac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$

The formula for the difference of two tangents is in the C3 specification.

$= \dfrac{\dfrac{2}{1-e}\tan\theta - \tan\theta}{1 + \dfrac{2}{1-e}\tan\theta \times \tan\theta}$

Multiply the numerator and the denominator by $(1-e)$.

$= \dfrac{2\tan\theta - (1-e)\tan\theta}{1 - e + 2\tan^2\theta}$

$= \dfrac{\tan\theta(1+e)}{1 - e + 2\tan^2\theta}$, as required

c Let $\tan\theta = t$ and $\mathrm{f}(t) = \dfrac{(1+e)t}{1-e+2t^2}$

The maximum of $(\phi - \theta)$ will correspond to the maximum of f(t), which can be found using calculus.

$\mathrm{f}'(t) = \dfrac{(1-e+2t^2)(1+e) - (1+e)t(4t)}{(1-e+2t^2)^2}$

For $\mathrm{f}'(t) = 0$

$(1-e+2t^2)\cancel{(1+e)} - \cancel{(1+e)}4t^2 = 0$

$1 - e + 2t^2 - 4t^2 = 0$

$2t^2 = 1 - e$

$t^2 = \dfrac{1-e}{2}$

$\tan^2\theta = \dfrac{1-e}{2}$, as required

If you are very familiar with the pure mathematics involved, you may recognise that $\mathrm{f}'(t) = 0$ has a solution at $t = \infty$, which corresponds to $\theta = \dfrac{\pi}{2}$.
Examination of the situation shows this is not actually a collision (the spheres 'graze' each other) so this case can be ignored.

The least deflection is clearly when $\theta = 0$ (there is no deflection then) and the deflection initially increases as θ increases so, as the function is continuous, the following stationary value must be a maximum.

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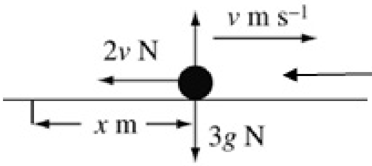
Review Exercise 1
Exercise A, Question 41

Question:

A particle P of mass 3 kg moves in a straight line on a smooth horizontal plane. When the speed of P is v m s^{-1}, the resultant force acting on P is a resistance to motion of magnitude $2v$ N.
Find the distance moved by P while slowing down from 5 m s^{-1} to 2 m s^{-1}. **[E]**

Solution:



The displacement, x m, of P must be measured from a fixed point. Here you can choose to measure the displacement from the point where the velocity of P is 5 m s^{-1}.

R($\rightarrow$) $\mathbf{F} = ma$

$$-2v = 3a = 3v\frac{dv}{dx}$$

$$\frac{dv}{dx} = -\frac{2}{3}$$

When resistance is a function of velocity and the question asks about a relation between distance and velocity, the formula $a = v\frac{dv}{dx}$ is normally used.

The resistance is in the direction of x decreasing and so has a negative sign in this equation.

Integrating with respect to x

$$v = -\frac{2}{3}x + C$$

When $x = 0, v = 5$

$5 = 0 + C \Rightarrow C = 5$

Hence

$$v = 5 - \frac{2}{3}x$$

As you are measuring the displacement from the point where the velocity of P is 5 m s^{-1}, you use $x = 0$ when $v = 5$ to evaluate the constant of integration.

When $v = 2$

$$2 = 5 - \frac{2}{3}x$$

$$\frac{2}{3}x = 5 - 2 = 3 \Rightarrow x = \frac{3}{2} \times 3 = 4.5$$

The distance moved by P while slowing down from 5 m s^{-1} to 2 m s^{-1} is 4.5 m.

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Review Exercise 1
Exercise A, Question 42

Question:

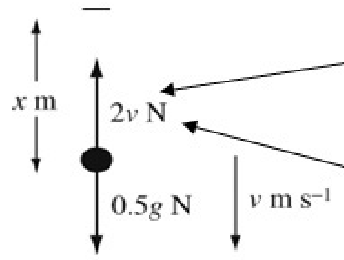
A particle P of mass 0.5 kg is released from rest at time $t = 0$ and falls vertically through a liquid. The motion P is resisted by a force of magnitude $2v$ N, where v m s^{-1} is the speed of P at time t seconds.

a Show that $5\frac{dv}{dt} = 49 - 20v$.

b Find the speed of P when $t = 1$. **[E]**

Solution:

a



The displacement, x m, must be measured from a fixed point. Here you can choose the point from which P is released.

As P is falling, the resistance opposes motion and acts upwards.

R(↓) $\mathbf{F} = ma$

$0.5g - 2v = 0.5a$

The weight acts in the direction of x increasing, so, in this equation, the term $0.5g$ is positive. The resistance acts in the direction of x decreasing, so, in this equation, the term $2v$ is negative.

Using $g = 9.8$ and $a = \frac{dv}{dt}$

$4.9 - 2v = 0.5\frac{dv}{dt}$

$5\frac{dv}{dt} = 49 - 20v$, as required

You multiply this equation by 10 and rearrange the terms to obtain the differential equation printed in the question.

b $\int \frac{5}{49-20v}\,dv = \int 1\,dt$

The answer to part **a** is a separable differential equation and the first step in solving it is to separate the variables.

$-\frac{1}{4}\int \frac{-20}{49-20v}\,dv = \int 1\,dt$

$-\frac{1}{4}\ln(49-20v) = t + A$

$\ln(49-20v) = B - 4t$, where $B = -4A$

$\int \frac{f'(v)}{f(v)}\,dv = \ln f(v) + A$ and $\frac{d}{dv}(49-20v) = -20$. Using $5 = -\frac{1}{4} \times -20$, you adjust the constants so that the integral can just be written down.

When $t = 0, v = 0$

$\ln 49 = B$

Hence

$\ln(49-20v) = \ln 49 - 4t$

$\ln 49 - \ln(49-20v) = \ln\left(\frac{49}{49-20v}\right) = 4t$

You use the log rule $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$ to simplify the expression.

$\frac{49}{49-20v} = e^{4t}$

$49 - 20v = 49e^{-4t}$

$20v = 49 - 49e^{-4t} = 49(1-e^{-4t})$

$v = \frac{49}{20}(1-e^{-4t})$

Taking exponentials of both sides to the equation and using the rule $e^{\ln f(x)} = f(x)$.

When $t = 1$

$v = \frac{49}{20}(1-e^{-4}) \approx 2.4$

The speed of P when $t = 1$ is

$\frac{49}{20}(1-e^{-4})$ m s^{-1} = 2.4 m s^{-1} (2 s.f.)

As a numerical value of g has been used, the final answer should be given to 2 or 3 significant figures.

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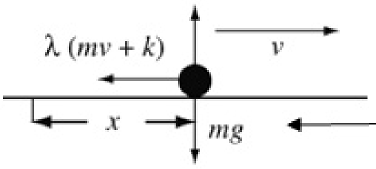
Review Exercise 1
Exercise A, Question 43

Question:

A particle of mass m moves in a straight line on a horizontal table against a resistance of magnitude $\lambda(mv+k)$, where λ and k are constants. Given that the particle starts with speed u at time $t=0$, show that the speed v of the particle at time t is

$v=\frac{k}{m}(e^{-\lambda t}-1)+ue^{-\lambda t}$. **[E]**

Solution:



The displacement x must be measured from a fixed point. Here you measure x from the point where the particle starts. Later, you will use $v=u$ when $t=0$ to evaluate the constant of integration.

R(→) $\mathbf{F}=m\mathbf{a}$

$$-\lambda(mv+k)=ma=m\frac{dv}{dt}$$

The resistance is in the direction of x decreasing and so has a negative sign in this equation.

Separating the variables

$$\int\frac{m}{mv+k}\,dv=-\int\lambda\,dt$$

$$\ln(mv+k)=-\lambda t+A$$

$$mv+k=e^{-\lambda t+A}=e^{A}e^{-\lambda t}$$

$$=B\,e^{-\lambda t}$$

e^{A}, where A is an arbitrary constant, is another arbitrary constant, B.

When $t=0, v=u$

$$mu+k=B$$

Hence

$$mv+k=(mu+k)e^{-\lambda t}$$

You make v the subject of this formula to complete the question.

$$mv=ke^{-\lambda t}-k+mu\,e^{-\lambda t}=k\left(e^{-\lambda t}-1\right)+mu\,e^{-\lambda t}$$

$$v=\frac{k}{m}\left(e^{-\lambda t}-1\right)+ue^{-\lambda t}$$, as required

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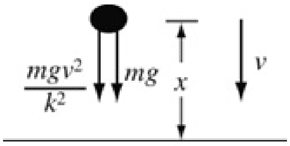
Review Exercise 1
Exercise A, Question 44

Question:

A particle P, of mass m, is projected upwards from horizontal ground with speed U. The motion takes place in a medium in which the resistance is of magnitude $\frac{mgv^2}{k^2}$, where v is the speed of P and k is a positive constant.
Show that P reaches its maximum height above ground after a time T given by
$T = \frac{k}{g}\arctan\left(\frac{U}{k}\right)$. **[E]**

Solution:



$$\mathbf{F} = m\mathbf{a}$$

$$-mg - \frac{mgv^2}{k^2} = ma = m\frac{dv}{dt}$$

$$-\cancel{m}g - \frac{\cancel{m}gv^2}{k^2} = \cancel{m}\frac{dv}{dt}$$

$$-g\left(\frac{k^2+v^2}{k^2}\right) = \frac{dv}{dt}$$

The displacement x is measured from the point of projection. In this question, both the weight of the particle and the resistance act in the direction of x decreasing and so the both terms representing these forces are negative in the equation of motion.

Separating the variables

$$\int \frac{1}{k^2+v^2}\,dv = -\int \frac{g}{k^2}\,dt$$

$$\frac{1}{k}\arctan\left(\frac{v}{k}\right) = -\frac{g}{k^2}t + A$$

The prerequisites given in the specification for M4 require you to know that $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$.

When $t = 0, v = U$

$$\frac{1}{k}\arctan\left(\frac{U}{k}\right) = A$$

Hence

$$\frac{1}{k}\arctan\left(\frac{v}{k}\right) = -\frac{g}{k^2}t + \frac{1}{k}\arctan\left(\frac{U}{k}\right)$$

At the maximum height $v = 0$ and $t = T$

When P reaches its maximum height above the ground its velocity is 0.

$$0 = -\frac{g}{k^2}T + \frac{1}{k}\arctan\left(\frac{U}{k}\right)$$

$$T = \frac{k}{g}\arctan\left(\frac{U}{k}\right), \text{ as required}$$

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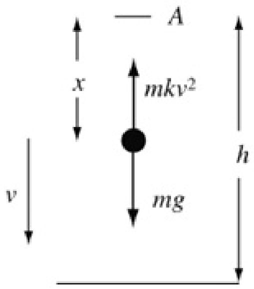
Review Exercise 1
Exercise A, Question 45

Question:

At time $t = 0$ a particle of mass m falls from rest at the point A which is at a height h above a horizontal plane. The particle is subject to a resistance of magnitude mkv^2, where v is the speed of the particle at time t and k is a positive constant. The particle strikes the plane with speed V.
Show that $kV^2 = g(1 - e^{-2kh})$. **[E]**

Solution:



$R(\downarrow)$ $\quad \mathbf{F} = m\mathbf{a}$

$mg - mkv^2 = ma = mv\frac{dv}{dx}$

$\not{m}g - \not{m}kv^2 = \not{m}v\frac{dv}{dx}$

The displacement x is measured from A. In this question, the weight of the particle is in the direction of x increasing. In the equation of motion, the term representing the weight, mg, is positive.
The resistance acts in the direction of x decreasing and, in the equation of motion, the term representing the resistance is negative.

Separating the variables

$\int \frac{v}{g - kv^2} dv = \int 1 \, dx$

If you multiply both sides of this equation by $-2k$, on the left hand side, the numerator of the fraction, $-2kv$, is the differential of the denominator, $g - kv^2$, and you can integrate using the formula $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$.

Multiply throughout by $-2k$

$\int \frac{-2kv}{g - kv^2} dv = \int -2k \, dx$

$\ln(g - kv^2) = -2kx + A$

$g - kv^2 = e^{-2kx + A} = e^A e^{-2kx}$

$= B e^{-2kx}$

e^A, where A is an arbitrary constant, is another arbitrary constant, B.

At $x = 0, v = 0$

$g = B e^0 = B$

Hence

$g - kv^2 = g e^{-2kx}$

$kv^2 = g(1 - e^{-2kx})$

At $x = h, v = V$

$kV^2 = g(1 - e^{-2kh})$, as required

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Review Exercise 1
Exercise A, Question 46

Question:

A particle of mass m is projected vertically upwards with speed U. It is subject to air resistance of magnitude mkv^2, where v is its speed and k is a positive constant.

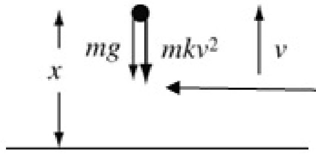
a Show that the greatest height of the particle above its point of projection is

$$\frac{1}{2k}\ln\left(1+\frac{kU^2}{g}\right).$$

b Find an expression for the total work done against air resistance during the upward motion. **[E]**

Solution:

a



As the particle is moving upwards, the resistance, mkv^2, opposes the motion and acts downwards.

R($\uparrow$) $\mathbf{F} = ma$

$$-mg - mkv^2 = ma = mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$-\not{m}g - \not{m}kv^2 = \not{m}v\frac{\mathrm{d}v}{\mathrm{d}x}$$

When resistance is a function of velocity and the question asks about a relation between distance and velocity, you usually use the formula $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$.

Separating the variables

$$\int \frac{v}{g + kv^2}\,\mathrm{d}v = -\int 1\,\mathrm{d}x$$

If you multiply both sides of this equation by $2k$, on the left hand side, the numerator of the fraction, $2kv$, is the differential of the denominator, $g + kv^2$, and you can integrate using the formula $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln f(x)$.

Multiply throughout by $2k$

$$\int \frac{2kv}{g + kv^2}\,\mathrm{d}v = -\int 2k\,\mathrm{d}x$$

$$\ln\left(g + kv^2\right) = -2kx + A$$

At $x = 0, v = U$

$$\ln\left(g + kU^2\right) = A$$

Hence

$$\ln\left(g + kv^2\right) = -2kx + \ln\left(g + kU^2\right)$$

The particle reaches its maximum height when $v = 0$.

$$\ln g = -2kx + \ln\left(g + kU^2\right)$$

$$2kx = \ln\left(g + kU^2\right) - \ln g = \ln\left(\frac{g + kU^2}{g}\right)$$

Using the law of logarithms $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.

$$x = \frac{1}{2k}\ln\left(1 + \frac{kU^2}{g}\right)\text{, as required}$$

b Work done = loss in energy

$$= \frac{1}{2}mU^2 - mgx$$

$$= \frac{1}{2}mU^2 - mg \times \frac{1}{2k}\ln\left(1 + \frac{kU^2}{g}\right)$$

$$= \frac{1}{2}m\left(U^2 - \frac{g}{k}\ln\left(1 + \frac{kU^2}{g}\right)\right)$$

The particle starts with kinetic energy $\frac{1}{2}mU^2$ and, at its maximum height, has, relative to the ground, potential energy $mg \times$ height above the ground. The difference between these energies is the work done by the resistance.

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Review Exercise 1
Exercise A, Question 47

Question:

A lorry of mass M is moving along a straight horizontal road. The engine produces a constant driving force of magnitude F. The total resistance to motion is modelled as having magnitude kv^2, where k is a constant, and v is the speed of the lorry.

Given that the lorry moves with constant speed V,

a show that $V = \sqrt{\left(\frac{F}{k}\right)}$.

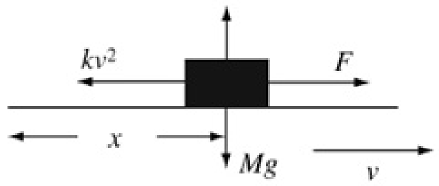
Given instead that the lorry starts from rest,

b show that the distance travelled by the lorry in attaining a speed $\frac{1}{2}V$ is $\frac{M}{2k}\ln\left(\frac{4}{3}\right)$.

[E]

Solution:

a



R(→) $\mathbf{F} = m\mathbf{a}$

$F - kV^2 = 0$

When the lorry is moving at a constant speed V, the acceleration of the lorry is 0.

$V^2 = \frac{F}{k}$

$V = \sqrt{\left(\frac{F}{k}\right)}$, as required

The velocity of the lorry is taken in the direction of x increasing and you can ignore the possibility of a negative square root.

b R(→) $\mathbf{F} = m\mathbf{a}$

$F - kv^2 = Ma = Mv\frac{dv}{dx}$

Separating the variables

$\int \frac{v}{F - kv^2} dv = \int \frac{1}{M} dx$

If you multiply both sides of this equation by $-2k$, on the left hand side, the numerator of the fraction, $-2kv$, is the differential of the denominator $F - kv^2$, and you can integrate using the formula $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$.

Multiply throughout by $-2k$

$\int \frac{-2kv}{F - kv^2} dv = -\int \frac{2k}{M} dx$

$\ln\left(F - kv^2\right) = -\frac{2k}{M}x + A$

$\frac{2k}{M}x = A - \ln\left(F - kv^2\right)$

At $x = 0, v = 0$

$0 = A - \ln F \Rightarrow A = \ln F$

Hence

$\frac{2k}{M}x = \ln F - \ln\left(F - kv^2\right) = \ln\left(\frac{F}{F - kv^2}\right)$

Using the law of logarithms $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.

$x = \frac{M}{2k}\ln\left(\frac{F}{F - kv^2}\right)$

When $v = \frac{1}{2}V$, $v^2 = \frac{1}{4}V^2 = \frac{F}{4k}$

You use the expression for V from part **a**.

$x = \frac{M}{2k}\ln\left(\frac{F}{F - k \times \frac{F}{4k}}\right) = \frac{M}{2k}\ln\left(\frac{F}{\frac{3}{4}F}\right)$

$= \frac{M}{2k}\ln\left(\frac{4}{3}\right)$, as required

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Review Exercise 1
Exercise A, Question 48

Question:

A train of mass m is moving along a straight horizontal railway line. At time t, the train is moving with speed v and the resistance to motion has magnitude kv, where k is a constant. The engine of the train is working at a constant rate P.

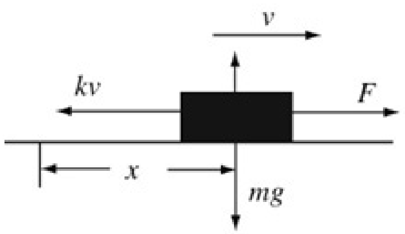
a Show that, when $v > 0$, $mv\frac{\mathrm{d}v}{\mathrm{d}t} + kv^2 = P$.

When $t = 0$, the speed of the train is $\frac{1}{3}\sqrt{\left(\frac{P}{k}\right)}$.

b Find, in terms of m and k, the time taken for the train to double its initial speed. **[E]**

Solution:

a



power = force × velocity

$P = Fv \Rightarrow F = \frac{P}{v}$

As the velocity increases, the tractive force F decreases and, hence, the acceleration will decrease.

R(→) $\mathbf{F} = m\mathbf{a}$

$F - kv = ma$

$\frac{P}{v} - kv = m\frac{\mathrm{d}v}{\mathrm{d}t}$

Multiply this equation throughout by v and rearrange the result to obtain the printed answer.

$P - kv^2 = mv\frac{\mathrm{d}v}{\mathrm{d}t}$

$mv\frac{\mathrm{d}v}{\mathrm{d}t} + kv^2 = P$, as required

b $mv\frac{\mathrm{d}v}{\mathrm{d}t} = P - kv^2$

Separating the variables

$\int \frac{v}{P - kv^2}\,\mathrm{d}v = \int \frac{1}{m}\,\mathrm{d}t$

If you multiply both sides of this equation by $-2k$, on the left hand side, the numerator of the fraction, $-2kv$, is the differential of the denominator, $P - kv^2$, and you can integrate using the formula $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln \mathrm{f}(x)$.

Multiply throughout by $-2k$

$\int \frac{-2kv}{P - kv^2}\,\mathrm{d}v = -\int \frac{2k}{m}\,\mathrm{d}t$

$\ln\left(P - kv^2\right) = -\frac{2k}{m}t + A$

When $t = 0$, $v = \frac{1}{3}\sqrt{\left(\frac{P}{k}\right)} \Rightarrow v^2 = \frac{P}{9k}$

$$\ln\left(P-\frac{P}{9}\right) = A \Rightarrow A = \ln\left(\frac{8P}{9}\right)$$

$$\ln\left(P-kv^2\right) = -\frac{2k}{m}t + \ln\left(\frac{8P}{9}\right)$$

You are asked to find an expression for the time, so it is sensible to rearrange this formula, making t the subject of the formula. This will save you time later.

$$t = \frac{m}{2k}\left[\ln\left(\frac{8P}{9}\right) - \ln\left(P-kv^2\right)\right]$$

When $v = \frac{2}{3}\sqrt{\left(\frac{P}{k}\right)} \Rightarrow v^2 = \frac{4P}{9k}$

The initial speed is $\frac{1}{3}\sqrt{\left(\frac{P}{k}\right)}$, so double the initial speed is $\frac{2}{3}\sqrt{\left(\frac{P}{k}\right)}$.

$$t = \frac{m}{2k}\left[\ln\left(\frac{8P}{9}\right) - \ln\left(P-\frac{4P}{9}\right)\right]$$

$$= \frac{m}{2k}\left[\ln\left(\frac{8P}{9}\right) - \ln\left(\frac{5P}{9}\right)\right]$$

$$= \frac{m}{2k}\ln\left(\frac{\frac{8P}{9}}{\frac{5P}{9}}\right) = \frac{m}{2k}\ln\left(\frac{8}{5}\right)$$

Using the law of logarithms $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 49

Question:

The engine of a car of mass 800 kg works at a constant rate of 32 kW. The car travels along a straight horizontal road and the resistance to motion of the car is proportional to the speed of the car. At time t seconds, $t \geq 0$, the car has a speed v m s^{-1} and when $t = 0$, its speed is 10 m s^{-1}.

a Show that $800v\frac{dv}{dt} = 32\,000 - kv^2$, where k is a positive constant.

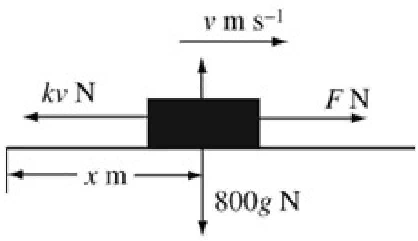
Given that the limiting speed of the car is 40 m s^{-1}, find

b the value of k,

c v^2 in terms of t. **[E]**

Solution:

a



power = force × velocity

$32\,000 = Fv \Rightarrow F = \frac{32\,000}{v}$

To use the relation power = force × velocity, you must convert 32 kW to 32 000 W.

Let the resistance be kv, where k is a constant

R(→) $\mathbf{F} = m\mathbf{a}$

$F - kv = 800a$

$\frac{32\,000}{v} - kv = 800\frac{dv}{dt}$

$32\,000 - kv^2 = 800v\frac{dv}{dt}$

$800v\frac{dv}{dt} = 32\,000 - kv^2$, as required

As the resistance is proportional to the velocity, you can let the resistance = kv, where k is the constant of proportionality.

You multiply this equation throughout by v.

b When $\frac{dv}{dt} = 0, v = 40$

$0 = 32\,000 - k \times 1600$

$k = \frac{32\,000}{1600} = 20$

At the limiting, or terminal, speed the speed of the car does not increase, so the acceleration is zero. This enables you to evaluate the constant of proportionality.

c $800v\frac{dv}{dt} = 32\,000 - 20v^2$

Separating the variables

$$\int \frac{800v}{32\,000 - 20v^2}\,dv = \int 1\,dt$$

$$-20\int \frac{-40v}{32\,000 - 20v^2}\,dv = \int 1\,dt$$

$-20\ln(32\,000 - 20v^2) = t + A$

$\ln(32\,000 - 20v^2) = -\frac{t}{20} + B$, where $B = -\frac{1}{20}A$

$32\,000 - 20v^2 = e^{-\frac{t}{20}}e^B = Ce^{-\frac{t}{20}}$, where $C = e^B$

$20v^2 = 32\,000 - Ce^{-\frac{t}{20}}$

$\int \frac{f'(v)}{f(v)}\,dv = \ln f(v) + A$ and $\frac{d}{dv}(32\,000 - 20v^2) = -40v$. Using $800 = -20 \times -40$, you adjust the constants so that the integral can just be written down.

When $t = 0, v = 10$

$2000 = 32\,000 - C \Rightarrow C = 30\,000$

$20v^2 = 32\,000 - 30\,000e^{-\frac{t}{20}}$

$v^2 = 1600 - 1500e^{-\frac{t}{20}}$

You use the initial conditions to find the arbitrary constant. You can choose at which stage you do this. In this solution, you could have found the constant earlier and evaluated A or B instead of C.

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Review Exercise 1
Exercise A, Question 50

Question:

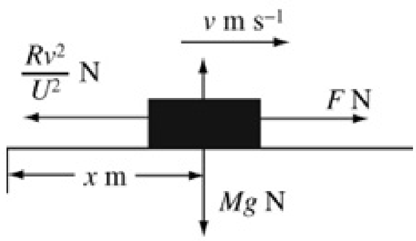
A car of mass M kg is driven by an engine working at a constant power RU watts, where R and U are positive constants. When the speed of the car is v m s^{-1}, the resistance to motion is $\frac{Rv^2}{U^2}$ newtons.

a Show that the acceleration of the car, a m s^{-2}, when its speed is v m s^{-1}, is given by $R(U^3 - v^3) = MU^2va$.

b Hence show that the distance, in m, travelled by the car as it increases its speed from u_1 m s^{-1} to u_2 m s^{-1}$(u_1 < u_2 < U)$ is $\frac{MU^2}{3R}\ln\left(\frac{U^3 - u_1^3}{U^3 - u_2^3}\right)$. **[E]**

Solution:

a



power $= \text{force} \times \text{velocity}$

$RU = Fv \Rightarrow F = \dfrac{RU}{v}$

R(→) $\mathbf{F} = ma$

$F - \dfrac{Rv^2}{U^2} = Ma$

$\dfrac{RU}{v} - \dfrac{Rv^2}{U^2} = Ma$

Multiply throughout by $U^2 v$

$RU^3 - Rv^3 = MU^2 va$

$R\left(U^3 - v^3\right) = MU^2 va$, as required

The question asks for the distance travelled as the speed increases. Time does not come into the question directly so you must use the relation $a = v\dfrac{dv}{dx}$.

b $R\left(U^3 - v^3\right) = MU^2 v\left(v\dfrac{dv}{dx}\right) = MU^2 v^2 \dfrac{dv}{dx}$

Separating the variables

$\displaystyle\int \frac{v^2}{U^3 - v^3}\,dv = \int \frac{R}{MU^2}\,dx$

Multiply throughout by -3

$\displaystyle\int \frac{-3v^2}{U^3 - v^3}\,dv = -3\int \frac{R}{MU^2}\,dx$

$\dfrac{d}{dv}\left(U^3 - v^3\right) = -3v^2$, so multiplying both sides of the equation by -3 gives, on the left hand side, a fraction in which the numerator is the differential of the denominator which gives a log integral.

Integrating both sides using the limits $v = u_1$ and $v = u_2$, and the limits $x = 0$ and $x = s$

$$\left[\ln(U^3 - v^3)\right]_{u_1}^{u_2} = -\frac{3R}{MU^2}\left[x\right]_0^s$$

$$\ln(U^3 - u_2^3) - \ln(U^3 - u_1^3) = -\frac{3R}{MU^2}s$$

$$s = \frac{MU^2}{3R}\left[\ln(U^3 - u_1^3) - \ln(U^3 - u_2^3)\right]$$

$$= \frac{MU^2}{3R}\ln\left(\frac{U^3 - u_1^3}{U^3 - u_2^3}\right), \text{ as required}$$

Using limits is sometimes a convenient way of avoiding evaluating the constant of integration. On the left hand side, v increases from u_1 to u_2. On the right hand side, you take x as increasing from 0 to s. s will then represent the distance travelled by the car as the speed increases from u_1 to u_2.

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Review Exercise 1
Exercise A, Question 51

Question:

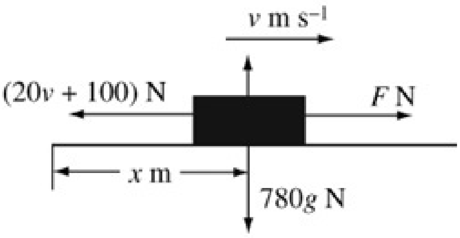
A car of mass 780 kg is moving along a straight horizontal road with the engine of the car working at 21 kW. The total resistance to the motion of the car is $(20v+100)$N, where v m s^{-1} is the speed of the car at time t seconds.

a Show that $39v\frac{dv}{dt}=(30-v)(35+v)$.

b Find an expression for the time taken for the car to accelerate from 15 m s^{-1} to V m s^{-1}. **[E]**

Solution:

a



$$\text{power} = \text{force} \times \text{velocity}$$

$$21\,000 = Fv \Rightarrow F = \frac{21\,000}{v}$$

The relation power = force × velocity, which is part of the M2 specification, lets you express the tractive force in terms of the velocity. To use this relation you must convert 21 kW to 21 000 W.

$$R(\rightarrow) \quad \mathbf{F} = m\mathbf{a}$$

$$F - (20v + 100) = 780\frac{dv}{dt}$$

$$\frac{21\,000}{v} - 20v - 100 = 780\frac{dv}{dt}$$

Divide throughout by 20

$$\frac{1050}{v} - v - 5 = 39\frac{dv}{dt}$$

Multiply throughout by v

$$1050 - v^2 - 5v = 39v\frac{dv}{dt}$$

$$39v\frac{dv}{dt} = 1050 - 5v - v^2$$

$$39v\frac{dv}{dt} = (30 - v)(35 + v)\text{, as required}$$

b Separating the variables

$$\int \frac{39v}{(30-v)(35+v)}\,dv = \int 1\,dt$$

Let $\frac{39v}{(30-v)(35+v)} = \frac{A}{30-v} + \frac{B}{35+v}$

$$39v = A(35+v) + B(30-v)$$

To integrate $\frac{39v}{(30-v)(35+v)}$ you must break the expression up into partial fractions. You may use any method, including the cover-up rule, to find the partial fractions.

Let $v \rightarrow 30$

$$39 \times 30 = A \times 65$$

$$A = \frac{39 \times 30}{65} = 18$$

Let $v \rightarrow -35$

$$39 \times -35 = B \times 65$$

$$B = \frac{39 \times -35}{65} = -21$$

Hence

$$\int\left(\frac{18}{30-v}-\frac{21}{35+v}\right)\mathrm{d}v=\int 1\,\mathrm{d}t$$

$-18\ln(30-v)-21\ln(35+v)\ =t+A$

When $t=0, v=15$

You can take $v=15$ as the starting value of the velocity.

$-18\ln 15-21\ln 50=A$

Hence

$-18\ln(30-v)-21\ln(35+v)=t-18\ln 15-21\ln 50$

$t\ =18\ln 15-18\ln(30-v)+21\ln 50-21\ln(35+v)$

$\ =18\ln\left(\frac{15}{30-v}\right)+21\ln\left(\frac{50}{35+v}\right)$

When $v=V$

$t=18\ln\left(\frac{15}{30-V}\right)+21\ln\left(\frac{50}{35+V}\right)$

The question specifies no form of the answer and there are many possible alternative forms.

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Review Exercise 1
Exercise A, Question 52

Question:

A railway truck of mass 3000 kg moves along a straight, horizontal railway line. When its speed is v m s^{-1}, it experiences a total resistance to motion of $(2000 + 5v^2)$N. A cable is attached to the truck, and the tension in the cable exerts a constant tractive force of 6500 N on the truck.

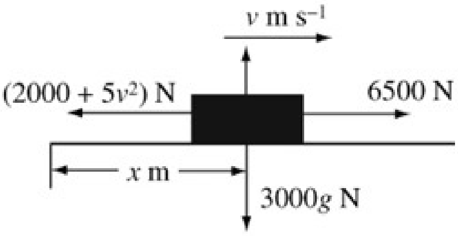
a Find the time taken for the truck to accelerate from rest to a speed of 20 m s^{-1}, giving your answer in seconds to 3 significant figures.

When the speed of the truck is 20 m s^{-1}, the cable breaks.

b Find the time taken after the cable breaks for the truck to come to rest, giving your answer in seconds to 3 significant figures. **[E]**

Solution:

a



R(→) $\mathbf{F} = ma$

$6500 - (2000 + 5v^2) = 3000a$

$4500 - 5v^2 = 3000\frac{dv}{dt}$

The tension in the cable is in the direction of x increasing and the resistance to motion acts in the direction of x decreasing.

÷5

$900 - v^2 = 600\frac{dv}{dt}$

Separating the variables

$\int 1 dt = 600\int \frac{1}{900 - v^2} dv$

To integrate $\frac{1}{900-v^2} = \frac{1}{(30-v)(30+v)}$ you must break the expression up into partial fractions. It is a common error to write $\int \frac{1}{900-v^2} dv = \ln(900-v^2)$.

Let $\frac{1}{900-v^2} = \frac{1}{(30-v)(30+v)} = \frac{A}{30-v} + \frac{B}{30+v}$

$\times(30-v)(30+v)$

$1 = A(30+v) + B(30-v)$

Let $v \to 30$

$1 = 60A \Rightarrow A = \frac{1}{60}$

Let $v \to -30$

$1 = 60B \Rightarrow B = \frac{1}{60}$

Hence

$\int 1\, dt = 600\int\left(\frac{1}{60(30+v)} + \frac{1}{60(30-v)}\right)dv$

$t = 10(\ln(30+v) - \ln(30-v)) + A$

$= 10\ln\left(\frac{30+v}{30-v}\right) + A$

$\int \frac{1}{30+v} dv = \ln(30+v) + A$ and $\int \frac{1}{30-v} dv = -\ln(30-v) + B$. However, after integrating both, you need only add one constant of integration.

When $t = 0, v = 0$

$0 = 10\ln\left(\frac{30}{30}\right) + A \Rightarrow A = 0$

Hence

$t = 10\ln\left(\frac{30+v}{30-v}\right)$

When $v = 20$

$$t = 10\ln\left(\frac{30+20}{30-20}\right) = 10\ln 5 \approx 16.1$$

The time taken for the truck to accelerate from rest to a speed of 20 m s^{-1} is 16.1 s (3 s.f.).

There is an exact answer here, 10 ln 5, but the conditions of the question require you to give your answers to 3 significant figures.

b After the rope breaks

R(→) **F** = *m***a**

$$-2000 - 5v^2 = 3000\frac{dv}{dt}$$

After the cable breaks, the only force acting horizontally on the truck is the total resistance acting in the direction of x decreasing.

÷5

$$-400 - v^2 = 600\frac{dv}{dt}$$

Separating the variables

$$\int \frac{1}{20^2 + v^2}\,dv = -\int \frac{1}{600}\,dt$$

$$\frac{1}{20}\arctan\left(\frac{v}{20}\right) = -\frac{1}{600}t + B$$

The prerequisites given in the specification for M4 require you to know that $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$.

When $t = 0, v = 20$

$$\frac{1}{20}\arctan 1 = B \Rightarrow B = \frac{1}{20}\times\frac{\pi}{4} = \frac{\pi}{80}$$

Using $\arctan 1 = \frac{\pi}{4}$.

Hence

$$\frac{1}{20}\arctan\left(\frac{v}{20}\right) = -\frac{1}{600}t + \frac{\pi}{80}$$

When $v = 0$

$$0 = -\frac{1}{600}t + \frac{\pi}{80}$$

Using $\arctan 0 = 0$.

$$t = \frac{\pi}{80}\times 600 = \frac{15\pi}{2} \approx 23.6$$

The time taken after the cable breaks for the truck to come to rest is 23.6 s (3 s.f.).

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Review Exercise 1
Exercise A, Question 53

Question:

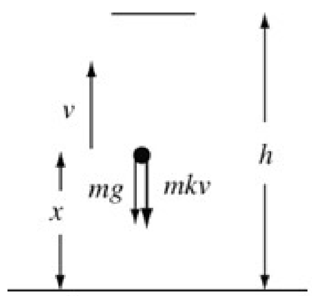
A particle P of mass m moves in a medium which produces a resistance of magnitude mkv, where v is the speed of P and k is a constant. The particle P is projected vertically upwards in this medium with speed $\frac{g}{k}$.

a Show that P comes instantaneously to rest after time $\frac{\ln 2}{k}$.

b Find, in terms of k and g, the greatest height above the point of projection reached by P. **[E]**

Solution:

a



$R(\uparrow)\quad \mathbf{F} = m\mathbf{a}$

$-mg - mkv = ma$

Both the weight of the particle (mg) and the resistance (mkv) act in the direction of x decreasing and so the both these terms are negative in the equation of motion.

$-\not{m}g - \not{m}kv = \not{m}\dfrac{\mathrm{d}v}{\mathrm{d}t}$

Separating the variables

$$\int 1\,\mathrm{d}t = -\int \frac{1}{g+kv}\,\mathrm{d}t$$

$t = -\dfrac{1}{k}\ln\left(g+kv\right) + A$

As $\displaystyle\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\mathrm{d}x = \ln \mathrm{f}(x)$, ignoring the arbitrary constant, $\displaystyle\int \frac{k}{g+kv}\mathrm{d}v = \ln(g+kv)$. So $\displaystyle\int \frac{1}{g+kv}\mathrm{d}v = \frac{1}{k}\ln(g+kv)$.

When $t = 0, v = \dfrac{g}{k}$

$0 = -\dfrac{1}{k}\ln\left(g + k \times \dfrac{g}{k}\right) + A \Rightarrow A = \dfrac{1}{k}\ln 2g$

Hence

$t = \dfrac{1}{k}\ln 2g - \dfrac{1}{k}\ln(g+kv) = \dfrac{1}{k}(\ln 2g - \ln(g+kv))$

$= \dfrac{1}{k}\ln\left(\dfrac{2g}{g+kv}\right)$

Using $\ln a - \ln b = \ln\left(\dfrac{a}{b}\right)$.

When $v = 0$

$t = \dfrac{1}{k}\ln\left(\dfrac{2g}{g}\right) = \dfrac{\ln 2}{k}$, as required

b R(↑) $\mathbf{F} = m\mathbf{a}$

$-mg - mkv = ma$

$-mg - mkv = mv\frac{dv}{dx}$

> In part **a**, where you are asked for a time, you use $a = \frac{dv}{dt}$.
>
> In part **b**, where you are asked for a distance, you use $a = v\frac{dv}{dx}$.

Separating the variables

$-\int 1\, dx = \int \frac{v}{g+kv}\, dv$

Let $\frac{v}{g+kv} = A + \frac{B}{g+kv}$

> In the fraction $\frac{v}{g+kv}$, the degree of the numerator is equal to the degree of the denominator, so, before integrating, you must reduce the improper fraction to a proper one. You can use any method to do this.

Multiply throughout by $g + kv$

$v = A(g+kv) + B$

Equating coefficients of v

$1 = Ak \Rightarrow A = \frac{1}{k}$

Equating constant coefficients

$0 = Ag + B \Rightarrow B = -gA = -\frac{g}{k}$

Hence

$-\int 1\, dx = \int \left(\frac{1}{k} - \frac{g}{k(g+kv)} \right) dv$

> As in part **a**, $\int \frac{1}{g+kv}\, dv = \frac{1}{k}\ln(g+kv)$.

$-x = \frac{1}{k}v - \frac{g}{k^2}\ln(g+kv) + C$

When $x = 0, v = \frac{g}{k}$

$0 = \frac{g}{k^2} - \frac{g}{k^2}\ln 2g + C \Rightarrow C = \frac{g}{k^2}\ln 2g - \frac{g}{k^2}$

Hence

$x = -\frac{1}{k}v + \frac{g}{k^2}\ln(g+kv) + \frac{g}{k^2} - \frac{g}{k^2}\ln 2g$

Let the greatest height above the point of projection reached by P be h.

When $x = h, v = 0$

> At the greatest height the velocity of P is 0.

$h = \frac{g}{k^2}\ln g + \frac{g}{k^2} - \frac{g}{k^2}\ln 2g$

$= \frac{g}{k^2} - \frac{g}{k^2}(\ln 2g - \ln g) = \frac{g}{k^2} - \frac{g}{k^2}\ln\left(\frac{2g}{g}\right)$

> Using $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.

$= \frac{g}{k^2}(1 - \ln 2)$

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Review Exercise 1
Exercise A, Question 54

Question:

A particle of mass m moves under gravity down a line of greatest slope of a smooth plane inclined at an angle α to the horizontal. When the speed of the particle is v the resistance to motion of the particle is mkv, where k is a positive constant.

a Show that the limiting speed c of the particle is given by $c = \frac{g \sin \alpha}{k}$.

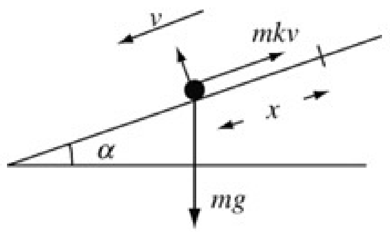
The particle starts from rest.

b Show that the time T taken to reach the speed of $\frac{1}{2}c$ is given by $T = \frac{1}{k} \ln 2$.

c Find, in terms of c and k, the distance travelled by the particle in attaining the speed of $\frac{1}{2}c$. **[E]**

Solution:

a



R(↗) $\mathbf{F} = m\mathbf{a}$

$mg\sin\alpha - mkv = ma$

Dividing throughout by m

$g\sin\alpha - kv = a$ ①

The component of the weight acts down the plane and the resistance acts up the plane.

At the limiting speed c, $a = 0$

$g\sin\alpha - kc = 0$

Hence

$c = \frac{g\sin\alpha}{k}$, as required

The limiting speed cannot be exceeded so, at the limiting speed, the acceleration is 0.

b From part a, $g\sin\alpha = kc$

Equation ① in part a can be written as

$kc - kv = a$ ②

Replacing $g\sin\alpha$ by kc simplifies the algebra considerably.

$k(c - v) = \frac{dv}{dt}$

Separating the variables

$\int k\,dt = \int \frac{1}{c - v}\,dt$

$kt = -\ln(c - v) + A$

To find a time, you use $a = \frac{dv}{dt}$. In part c, you are asked for distance and there you will use $a = v\frac{dv}{dx}$.

When $t = 0, v = 0$

$0 = -\ln c + A \Rightarrow A = \ln c$

$kt = \ln c - \ln(c - v) = \ln\left(\frac{c}{c - v}\right)$

$t = \frac{1}{k}\ln\left(\frac{c}{c - v}\right)$

When $v = \frac{1}{2}c$

$t = \frac{1}{k}\ln\left(\frac{c}{c - \frac{1}{2}c}\right) = \frac{1}{k}\ln\left(\frac{c}{\frac{1}{2}c}\right)$

$= \frac{1}{k}\ln 2$, as required

c Writing $a = v\frac{dv}{dx}$ equation ② in part **b** becomes

$kc - kv = v\frac{dv}{dx}$

Separating the variables

$\int k \, dx = \int \frac{v}{c-v} \, dv$

Let $\frac{v}{c-v} = A + \frac{B}{c-v}$

> $\frac{v}{c-v}$ is an improper fraction and, before integrating, you must express it as a constant + a proper fraction. If preferred, you could use long division rather than the method shown here.

Multiply throughout by $c - v$

$v = A(c-v) + B$

Equating coefficients of v

$1 = -A \Rightarrow A = -1$

Let $v \to c$

$c = B$

Hence

$\int k \, dx = \int \left(-1 + \frac{c}{c-v}\right) dv$

$kx = -v - c\ln(c-v) + D$

At $x = 0, v = 0$

$0 = -c\ln c + D \Rightarrow D = c\ln c$

Hence

$kx = c\ln c - c\ln(c-v) - v$

$= c(\ln c - \ln(c-v)) - v = c\ln\left(\frac{c}{c-v}\right) - v$

> Using $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.

$x = \frac{c}{k}\ln\left(\frac{c}{c-v}\right) - \frac{v}{k}$

When $v = \frac{1}{2}c$

$x = \frac{c}{k}\ln\left(\frac{c}{c - \frac{1}{2}c}\right) - \frac{\frac{1}{2}c}{k}$

$= \frac{c}{k}\left(\ln 2 - \frac{1}{2}\right)$

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Review Exercise 1
Exercise A, Question 55

Question:

A particle of mass m is falling vertically under gravity in a resisting medium. The particle is released from rest. The speed v of the particle at a distance x from rest is given by $v^2 = 2kg\left[1 - e^{-\frac{x}{k}}\right]$, where k is a positive constant.

a Show that the magnitude of the resistance is $\frac{mv^2}{2k}$.

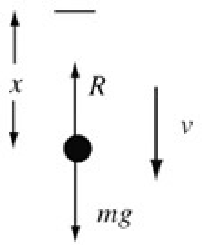
The particle is projected upwards in the same medium with speed $\sqrt{(2kg)}$.

b Show that the maximum height reached by the particle above the point of projection is $k\ln 2$.

c Find the time taken to reach the maximum height above the point of projection. **[E]**

Solution:

a



Let the resistance be R.

R(↓) $\mathbf{F} = m\mathbf{a}$

$$mg - R = ma = mv\frac{dv}{dx}$$

$$R = mg - mv\frac{dv}{dx} \quad ①$$

$$v^2 = 2kg\left[1 - e^{-\frac{x}{k}}\right] \quad ②$$

Differentiate ② with respect to x

$$2v\frac{dv}{dx} = 2kg \times \left(\frac{1}{k}\right)e^{-\frac{x}{k}}$$

$$v\frac{dv}{dx} = ge^{-\frac{x}{k}} \quad ③$$

You differentiate v^2 implicitly with respect to x.
$\frac{d}{dx}(v^2) = \frac{d}{dv}(v^2) \times \frac{dv}{dx} = 2v\frac{dv}{dx}$.

From ②

$$1 - e^{-\frac{x}{k}} = \frac{v^2}{2kg} \Rightarrow e^{-\frac{x}{k}} = 1 - \frac{v^2}{2kg} \quad ④$$

To complete this part, you use the equation given in the question to obtain $e^{-\frac{x}{k}}$ in terms of v.

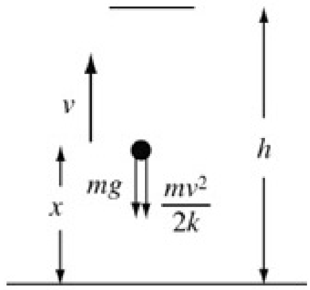
Substituting ④ into ③

$$v\frac{dv}{dx} = g\left(1 - \frac{v^2}{2kg}\right) = g - \frac{v^2}{2k}$$

Substituting for $v\frac{dv}{dx}$ into ①

$$R = mg - mg + \frac{mv^2}{2k} = \frac{mv^2}{2k}\text{, as required}$$

b



R(↑) $\mathbf{F} = ma$

$$-mg - \frac{mv^2}{2k} = ma = mv\frac{dv}{dx}$$

Dividing throughout by m

$$-g - \frac{v^2}{2k} = v\frac{dv}{dx}$$

$$-\frac{2kg + v^2}{2k} = v\frac{dv}{dx}$$

Before separating the variables, you need to put the left hand side of the equation over the common denominator $2k$.

Separating the variables

$$\int 1\,dx = -\int \frac{2kv}{2kg + v^2}\,dv$$

$$x = -k\ln\left(2kg + v^2\right) + A$$

As $\frac{d}{dv}\left(2kg + v^2\right) = 2v$,

$\int \frac{2v}{2kg + v^2}\,dv = \ln\left(2kg + v^2\right) +$ a constant.

At $x = 0, v = \sqrt{(2kg)}$

$$0 = -k\ln\left(2kg + 2kg\right) + A \Rightarrow A = k\ln\left(4kg\right)$$

When $x = 0$, $v = \sqrt{2kg}$

Hence

$$x = k\ln\left(4kg\right) - k\ln\left(2kg + v^2\right) = k\ln\left(\frac{4kg}{2kg + v^2}\right)$$

Let the greatest height above the point of projection reached by the particle be h.

At $x = h, v = 0$

$$h = k\ln\left(\frac{4kg}{2kg}\right) = k\ln 2, \text{ as required}$$

At the greatest height, the velocity of the particle is 0.

c From part **b**

$$-mg-\frac{mv^2}{2k}=ma=m\frac{dv}{dt}$$

Divide by m and put the left hand side of the equation over a common denominator.

$$-\frac{2kg+v^2}{2k}=\frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt=-2k\int\frac{1}{2kg+v^2}\,dv$$

$\int\frac{1}{a^2+v^2}\,dv=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+A$.
Here $a^2=2kg$, so $a=\sqrt{(2kg)}$.

$$t=-\frac{2k}{\sqrt{(2kg)}}\arctan\left(\frac{v}{\sqrt{(2kg)}}\right)+B$$

When $t=0, v=\sqrt{(2kg)}$

$$0=-\frac{2k}{\sqrt{(2kg)}}\arctan\left(\frac{\sqrt{(2kg)}}{\sqrt{(2kg)}}\right)+B$$

$$B=\frac{2k}{\sqrt{(2kg)}}\arctan 1=\frac{2k}{\sqrt{(2kg)}}\frac{\pi}{4}$$

Hence

$$t=\frac{2k}{\sqrt{(2kg)}}\left[\frac{\pi}{4}-\arctan\left(\frac{v}{\sqrt{(2kg)}}\right)\right]$$

At the maximum height, $v=0$

$$t=\frac{2k}{\sqrt{(2kg)}}\left[\frac{\pi}{4}-0\right]$$

$$=\frac{\pi}{2}\sqrt{\left(\frac{k}{2g}\right)}$$

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Review Exercise 1
Exercise A, Question 56

Question:

A ship of mass m is propelled in a straight line through the water by a propeller which develops a constant force of magnitude F. When the speed of the ship is v, the water causes a drag of magnitude kv, where k is a constant, to act on the ship. The ship starts from rest at time $t = 0$.

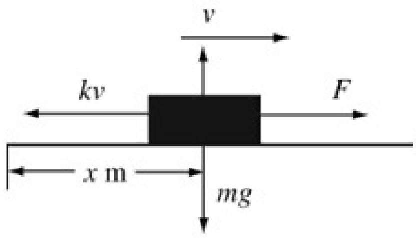
a Show that the ship reaches half of its theoretical maximum speed of $\frac{F}{k}$ when $t = \frac{m \ln 2}{k}$.

When the ship in moving with speed $\frac{F}{2k}$, an emergency occurs and the captain reverses the engines so that the propeller force, which remains of magnitude F, acts backwards.

b Show that the ship covers a further distance $\frac{mF}{k^2}\left[\frac{1}{2} - \ln\left(\frac{3}{2}\right)\right]$ on its original course, which may be assumed to remain unchanged, before being brought to rest. **[E]**

Solution:

a



R(→) $\mathbf{F} = m\mathbf{a}$

$$F - kv = ma = m\frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt = \int \frac{m}{F - kv}\,dv$$

$$t = -\frac{m}{k}\ln(F - kv) + A$$

As $\int \frac{-k}{F - kv}\,dv = \ln(F - kv) +$ a constant,
$\int \frac{1}{F - kv}\,dv = -\frac{1}{k}\ln(F - kv) +$ a constant.

When $t = 0, v = 0$

$$0 = -\frac{m}{k}\ln F + A \Rightarrow A = \frac{m}{k}\ln F$$

Hence

$$t = \frac{m}{k}\ln F - \frac{m}{k}\ln(F - kv)$$

$$t = \frac{m}{k}\ln\left(\frac{F}{F - kv}\right)$$

When $v = \frac{F}{2k}$

The limiting (or terminal) velocity is, in this case, the theoretical maximum speed of the ship. It is given by substituting $a = 0$ into $F - kv = ma$, which gives $v = \frac{F}{k}$. In this question, you are not asked to prove this result but you should know how to prove it. Half of the theoretical maximum speed is $\frac{F}{2k}$.

$$t = \frac{m}{k}\ln\left(\frac{F}{F - \frac{1}{2}F}\right) = \frac{m}{k}\ln\left(\frac{F}{\frac{1}{2}F}\right)$$

$$= \frac{m\ln 2}{k}, \text{ as required}$$

b $-F - kv = ma = mv\frac{dv}{dx}$

The force developed by the propeller is now reversed, so you change the sign of F in the equation of motion you found in part **a**.

Separating the variables

$$-\int 1\,dx = m\int \frac{v}{F + kv}\,dv$$

$$= \frac{m}{k}\int \frac{kv}{F + kv}\,dv = \frac{m}{k}\int \frac{F + kv - F}{F + kv}\,dv$$

$$-x = \frac{m}{k}\int\left(1 - \frac{F}{F + kv}\right)dv$$

$\frac{v}{F + kv}$ is an improper fraction and must be transformed into an expression involving a proper fraction before integration. You may use any appropriate method to do this.

$$= \frac{m}{k}\left(v - \frac{F}{k}\ln(F + kv)\right) + B$$

When $x = 0, v = \frac{F}{2k}$

$$0 = \frac{m}{k}\left(\frac{F}{2k} - \frac{F}{k}\ln\left(F + \frac{F}{2}\right)\right) + B$$

$$B = -\frac{mF}{k^2}\left(\frac{1}{2} - \ln\left(\frac{3F}{2}\right)\right)$$

The algebra here is complicated and it is worth taking the factor $\frac{mF}{k^2}$ outside the bracket as both this and $\ln\left(\frac{3}{2}\right)$ appear in the printed answer. Always keep in mind what you are aiming for and try to work towards it.

Hence

$$x = \frac{mF}{k^2}\left[\frac{1}{2} - \ln\left(\frac{3F}{2}\right)\right] - \frac{m}{k}\left[v - \frac{F}{k}\ln\left(F + kv\right)\right]$$

When $v = 0$

$$x = \frac{mF}{k^2}\left[\frac{1}{2} - \ln\left(\frac{3F}{2}\right)\right] + \frac{mF}{k^2}\ln F$$

$$= \frac{mF}{k^2}\left[\frac{1}{2} - \left(\ln\left(\frac{3F}{2}\right) - \ln F\right)\right] = \frac{mF}{k^2}\left[\frac{1}{2} - \ln\left(\frac{3F}{2F}\right)\right]$$

$$= \frac{mF}{k^2}\left[\frac{1}{2} - \ln\left(\frac{3}{2}\right)\right],$$ as required

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Review Exercise 2
Exercise A, Question 1

Question:

A particle P moves in a straight line. At time t seconds its displacement from a fixed point O on the line is x metres. The motion of P is modelled by the differential equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=12\cos 2t-6\sin 2t$.

When $t=0, P$ is at rest at O.

a Find, in terms of t, the displacement of P from O.

b Show that P comes to instantaneous rest when $t=\frac{\pi}{4}$.

c Find, in metres to 3 significant figures, the displacement of P from O when $t=\frac{\pi}{4}$.

d Find the approximate period of the motion for large values of t. **[E]**

Solution:

a $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=12\cos 2t-6\sin 2t$

Auxiliary equation: $m^2+2m+2=0$ ← An expression for x is needed. Refer to book FP2 Chapter 5 for the method of solving these equations.

$m=\frac{-2\pm\sqrt{(4-8)}}{2}=-1\pm i$

Complementary function is

$x=e^{-t}(A\cos t+B\sin t)$

Let $x=p\cos 2t+q\sin 2t$ ← Try this for a particular integrat.

$\dot{x}=-2p\sin 2t+2q\cos 2t$

$\ddot{x}=-4p\cos 2t-4q\sin 2t$

$\therefore -4p\cos 2t-4q\sin 2t$

$+2(-2p\sin 2t+2q\cos 2t)$ ← Substitute the expressions for $x, \dot{x}$ and $\ddot{x}$ into the differential equation.

$+2(p\cos 2t+q\sin 2t)$

$=12\cos 2t-6\sin 2t$

$\cos 2t(-4p+4q+2p)$

$+\sin 2t(-4q-4p+2q)$

$=12\cos 2t-6\sin 2t$

$-4p+4q+2p=12$ ← Equate coefficients of $\cos 2t$...

$-2p+4q=12$ ①

$-4q-4p+2q=-6$

$-4p-2q=-6$ ← ... and of $\sin 2t$.

$-2p-q=-3$ ②

$5q=15$ ← Solve ① and ②.

$q=3, p=0$

$\therefore x=e^{-t}(A\cos t+B\sin t)+3\sin 2t$

$t=0, x=0 \ \therefore 0=A$ ← Use the initial conditions given in the question to obtain values for A and B.

$\dot{x}=-e^{-t}B\sin t+e^{-t}B\cos t+6\cos 2t$

$t=0, \dot{x}=0 \ \therefore 0=B+6$

$B=-6$

$\therefore x=3\sin 2t-6e^{-t}\sin t$

b $\dot{x}=6e^{-t}\sin t-6e^{-t}\cos t+6\cos 2t$

$t=\frac{\pi}{4}, \dot{x}=6\left[e^{-\frac{\pi}{4}}\sin\frac{\pi}{4}-e^{-\frac{\pi}{4}}\cos\frac{\pi}{4}+\cos\frac{\pi}{2}\right]$

$=0$ ← $\sin\frac{\pi}{4}=\cos\frac{\pi}{4}\quad \cos\frac{\pi}{2}=0$

$\therefore P$ comes to instantaneous test when $t=\frac{\pi}{4}$

c $x = 3\sin 2t - 6e^{-t}\sin t$

$t = \frac{\pi}{4}$ $x = 3\sin\frac{\pi}{2} - 6e^{-\frac{\pi}{4}}\sin\frac{\pi}{4}$

$= 3 - 6e^{-\frac{\pi}{4}} \times \frac{1}{\sqrt{2}}$

$= 1.07$ (3 s.f.)

d $t \to \infty$ | Large values of t needed, so let $t \to \infty$.

$x \approx 3\sin 2t$ | $e^{-t} \to 0$ as $t \to \infty$

$\therefore$ approximate period is π

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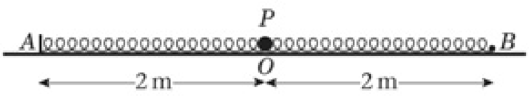
Review Exercise 2
Exercise A, Question 2

Question:



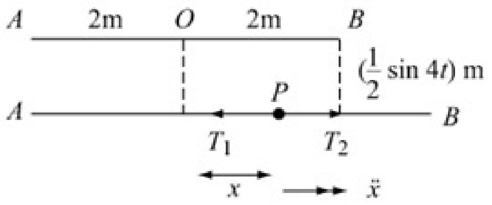
A particle P of mass 2 kg is attached to the mid-point of a light elastic spring of natural length 2 m and modulus of elasticity 4 N. One end A of the elastic spring is attached to a fixed point on a smooth horizontal table. The spring is then stretched until its length is 4 m and its other end B is held at a point on the table where $AB = 4$ m . At time $t = 0, P$ is at rest on the table at the point O where $AO = 2$ m, as shown. The end B is now moved on the table in such a way that AOB remains a straight line. At time t seconds, $AB = (4 + \frac{1}{2}\sin 4t)$ m and $AP = (2 + x)$ m.

a Show that $\frac{d^2x}{dt^2} + 4x = \sin 4t$.

b Hence find the time when P first comes to instantaneous rest. **[E]**

Solution:

a



$T_1 = 4(1+x)$

By Hooke's law. $AP = 2 + x \Rightarrow$ extension $= 1 + x$

$T_2 = 4\left(1+\frac{1}{2}\sin 4t - x\right)$

By Hooke's law. $AB = \left(4+\frac{1}{2}\sin 4t\right) \Rightarrow$ extension in PB $= \left(1+\frac{1}{2}\sin 4t - x\right)$

$T_2 - T_1 = 2\ddot{x}$,

Using $F = ma$

$4\left(1+\frac{1}{2}\sin 4t - x\right) - 4(1+x) = 2\ddot{x}$

Substitute for T_1 and T_2.

$2\sin 4t + 4 - 4x - 4 - 4x = 2\ddot{x}$

$\ddot{x} + 4x = \sin 4t$

b Auxiliary equation: $m^2 + 4 = 0$

$m = \pm 2\text{i}$

Now solve the differential equation (refer to book FP2 Chapter 5).

$\therefore$ Complementary function:

$x = A\sin 2t + B\cos 2t$

For particular integral, try

$x = P\sin 4t$

$\dot{x} = 4P\cos 4t \quad \ddot{x} = -16P\sin 4t$

$\therefore -16P\sin 4t + 4P\sin 4t = \sin 4t$

Substitute in the differential equation.

$-12P = 1$

$P = -\frac{1}{12}$

$\therefore x = A\sin 2t + B\cos 2t - \frac{1}{12}\sin 4t$

$t = 0, x = 0 \Rightarrow B = 0$

Use the initial conditions given in the question to obtain values for A and B.

$\dot{x} = 2A\cos 2t - \frac{1}{3}\cos 4t$

$t = 0, \dot{x} = 0 \Rightarrow 0 = 2A - \frac{1}{3}$

$A = \frac{1}{6}$

When $\dot{x} = 0$

$$0 = 2 \times \frac{1}{6}\cos 2t - \frac{1}{3}\cos 4t$$

$$\cos 4t = \cos 2t$$

$$\therefore 4t = 2t + 2\pi \text{ or } 2\pi - 2t$$

$$t = \pi \quad \text{or} \quad t = \frac{\pi}{3}$$

$\therefore P$ first comes to rest when $t = \frac{\pi}{3}$

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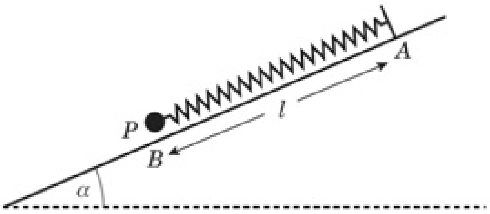
Review Exercise 2
Exercise A, Question 3

Question:



A light elastic spring has natural length l and modulus of elasticity $4mg$. One end of the spring is attached to a point A on a plane that is inclined to the horizontal at an angle α, where $\tan\alpha = \frac{3}{4}$. The other end of the spring is attached to a particle P of mass m. The plane is rough and the coefficient of friction between P and the plane is $\frac{1}{2}$. The particle P is held at a point B on the lane where B is below A and $AB = l$, with the spring lying along a line of greatest slope of the plane, as shown. At time $t = 0$, the particle is projected up the plane towards A with speed $\frac{1}{2}\sqrt{(gl)}$. At time t, the compression of the spring is x.

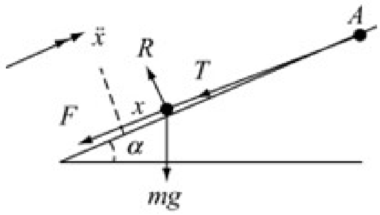
a Show that $\frac{d^2x}{dt^2} + 4\omega^2 x = -g$, where $\omega = \sqrt{\left(\frac{g}{l}\right)}$

b Find x in terms of l, ω and t.

c Find the distance that P travels up the plane before first coming to rest. **[E]**

Solution:

a



$\lambda = 4mg$

$\tan\alpha = \frac{3}{4}$

Hooke's law: $T = \frac{\lambda x}{l} = \frac{4mgx}{l}$

R($\perp$, plane): $R = mg\cos\alpha = \frac{4}{5}mg$

$F = \mu R = \frac{1}{2}\times\frac{4}{5}mg = \frac{2}{5}mg$

The magnitude of the frictional force must be obtained.

Equation of motion:

$$-F - T - mg\sin\alpha = m\ddot{x}$$

$$-\frac{2}{5}mg - \frac{4mgx}{l} - \frac{3}{5}mg = m\ddot{x}$$

$$\ddot{x} + \frac{4gx}{l} = -g$$

x is the compression in the spring. It is measured from B and increases as P travels up the plane.

Let $\frac{g}{l} = \omega^2$

This is given in the question.

$$\frac{d^2x}{dt^2} + 4\omega^2 x = -g$$

where $\omega = \sqrt{\left(\frac{g}{l}\right)}$

b $\frac{d^2x}{dt^2} + 4\omega^2 x = -g$

Now solve the differential equation using the methods of book FP2 chapter 5.

Auxiliary equation: $m^2 + 4\omega^2 = 0$

$m = \pm 2\mathrm{i}\omega$

Complementary function:

$x = A\sin 2\omega t + B\cos 2\omega t$

For the particular integral try

$x = p$

$0 + 4\omega^2 p = -g$

$$p = \frac{-g}{4\omega^2} = \frac{-g}{4}\times\frac{l}{g}$$

$$p = -\frac{l}{4}$$

$\therefore$ Complete solution is

$$x = A\sin 2\omega t + B\cos 2\omega t - \frac{l}{4}$$

$$t = 0, x = 0 \;\therefore\; 0 = B - \frac{l}{4}$$

$$B = \frac{l}{4}$$

$$\dot{x} = 2\omega A\cos 2\omega t - 2\omega B\sin 2\omega t$$

$$t = 0, \dot{x} = \frac{1}{2}\sqrt{(gl)}$$

$$\frac{1}{2}\sqrt{(gl)} = 2\omega A$$

$$A = \frac{1}{4\omega}\sqrt{(gl)}$$

$$= \frac{1}{4} \times \sqrt{\frac{l}{g}} \times \sqrt{gl} = \frac{1}{4}l$$

$$\therefore x = \frac{l}{4}(\sin 2\omega t + \cos 2\omega t - 1)$$

c $\dot{x} = 2\omega\frac{l}{4}(\cos 2\omega t - \sin 2\omega t)$

Use the expression for $\dot{x}$ obtained in **b** with the known values for A and B.

At rest $\dot{x} = 0$

$$\cos 2\omega t = \sin 2\omega t$$

$$\tan 2\omega t = 1$$

$$2\omega t = \frac{\pi}{4}$$

when $2\omega t = \frac{\pi}{4}$

You are finding the distance P travels before it first comes to rest. The value of t is not needed explicitly.

$$x = \frac{l}{4}\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - 1\right)$$

$$= \frac{l}{4}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{l}{4}\left(\frac{2}{\sqrt{2}} - 1\right)$$

P travels a distance $\frac{l}{4}(\sqrt{2} - 1)$ up the plane before first coming to rest.

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Review Exercise 2
Exercise A, Question 4

Question:

A particle P of mass m is suspended from a fixed point by a light elastic spring. The spring has natural length a and modulus of elasticity $2m\omega^2 a$, where ω is a positive constant. A time $t = 0$ the particle is projected vertically downwards with speed U from its equilibrium position. The motion of the particle is resisted by a force of magnitude $2m\omega v$, where v is the speed of the particle. At time t, the displacement of P downwards from its equilibrium position is x.

a Show that $\frac{d^2x}{dt^2} + 2\omega\frac{dx}{dt} + 2\omega^2 x = 0$.

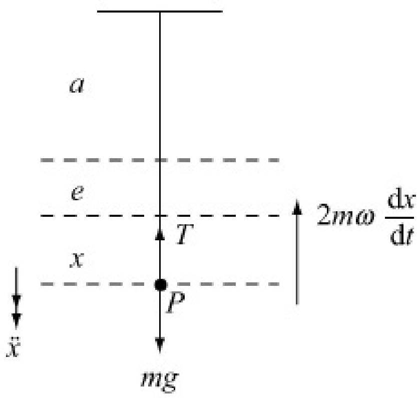
Given that the solution of this differential equation is $x = e^{-\omega t}(A\cos\omega t + B\sin\omega t)$, where A and B are constants,

b find A and B.

c Find an expression for the time at which P first comes to rest. **[E]**

Solution:

a



$l = a$
$\lambda = 2m\omega^2 a$

Hooke's law:

$$T = \frac{\lambda x}{l} = \frac{2m\omega^2 a(e+x)}{a}$$

$$F = ma$$

$$mg - T - 2m\omega\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

In equilibrium:

$$T_E = \frac{2m\omega^2 ae}{a} = mg$$

The equilibrium tension is equal to the weight of P.

$$\therefore \frac{2m\omega^2 ae}{a} - \frac{2m\omega^2 a(e+x)}{a} - 2m\omega\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -2\omega^2 x - 2\omega\frac{dx}{dt}$$

$$\therefore \frac{d^2x}{dt^2} + 2\omega\frac{dx}{dt} + 2\omega^2 x = 0$$

b

$$x = e^{-\omega t}(A\cos\omega t + B\sin\omega t)$$

The general solution of the differential equation was given in the question.

$$t = 0, x = 0 \Rightarrow 0 = A$$

$$\frac{dx}{dt} = -\omega e^{-\omega t}B\sin\omega t + \omega e^{-\omega t}B\cos\omega t$$

$$t = 0, \frac{dx}{dt} = U$$

$$\rightarrow U = B\omega, B = \frac{U}{\omega}$$

$$\therefore A = 0 \text{ and } B = \frac{U}{\omega}$$

c

$$\frac{dx}{dt} = 0$$

$v = \frac{dx}{dt} = 0$ when P is at rest.

$$\Rightarrow 0 = -\omega e^{-\omega t}\frac{U}{\omega}(\sin\omega t - \cos\omega t)$$

$$\therefore \sin\omega t = \cos\omega t$$

$$\tan\omega t = 1$$

$$t = \frac{\pi}{4\omega}$$

$\therefore$ P first comes to rest when $t = \frac{\pi}{4\omega}$

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Review Exercise 2
Exercise A, Question 5

Question:

A light elastic string, of natural length $2a$ and modulus of elasticity mg, has a particle P of mass m attached to its mid-point. One end of the string is attached to a fixed point A and the other end is attached to a fixed point B which is at a distance $4a$ vertically below A.

a Show that P hangs in equilibrium at the point E where $AE = \frac{5}{2}a$.

The particle P is held at a distance $3a$ vertically below A and is released from rest at time $t = 0$. When the speed of the particle is v, there is a resistance to motion of magnitude $2mkv$, where $k = \sqrt{\left(\frac{g}{a}\right)}$.

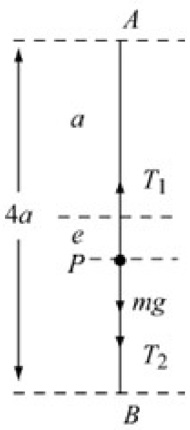
At time t the particle is at a distance $\left(\frac{5}{2}a + x\right)$ from A.

b Show that $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0$.

c Hence find x in terms of t. **[E]**

Solution:

a



$\lambda = mg$

$l = a$

Treat the two halves of the string as separate strings of natural length a.

R(↑) $T_1 = T_2 + mg$

Hooke's law:

$$T_1 = \frac{\lambda x}{l} = \frac{mge}{a}$$

$$T_2 = \frac{mg}{a}(2a - e)$$

The combined extensions equal $2a$.

$$\therefore \frac{mge}{a} = \frac{mg}{a}(2a - e) + mg$$

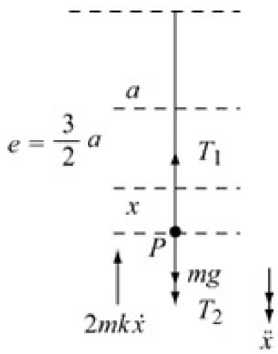
$$e = 2a - e + a$$

$$2e = 3a \quad e = \frac{3}{2}a$$

$$\therefore AE = \frac{5}{2}a$$

Remember to add the natural length of the upper 'half' string to the extension to obtain the required answer.

b



Hooke's law:

$$T_1 = \frac{mg}{a}\left(\frac{3}{2}a + x\right)$$

$$T_2 = \frac{mg}{a}\left(\frac{1}{2}a - x\right)$$

x is measured from the equilibrium level.

Equation of motion:

$$\frac{mg}{a}\left(\frac{1}{2}a - x\right) + mg - \frac{mg}{a}\left(\frac{3a}{2} + x\right) - 2mk\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \Rightarrow \frac{2\cancel{m}g}{a}x - 2\cancel{m}k\frac{\mathrm{d}x}{\mathrm{d}t} = \cancel{m}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + 2k^2x = 0 \quad \text{since } k^2 = \frac{g}{a}$$

Equation of motion:

$$\frac{mg}{a}\left(\frac{1}{2}a - x\right) + mg - \frac{mg}{a}\left(\frac{3a}{2} + x\right) - 2mk\frac{dx}{dt} = m\frac{d^2x}{dt^2} \Rightarrow \frac{2\not{m}g}{a}x - 2\not{m}k\frac{dx}{dt} = \not{m}\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0 \quad \text{since } k^2 = \frac{g}{a}$$

c Auxiliary equation:

$m^2 + 2km + 2k^2 = 0$

$m = -k \pm ki$

Complementary function:

$x = e^{-kt}(A\cos kt + B\sin kt)$

$t = 0, x = \frac{1}{2}a \Rightarrow A = \frac{1}{2}a$

$\frac{dx}{dt} = -ke^{-kt}(A\cos kt + B\sin kt)$

$+e^{-kt}(-kA\sin kt + kB\cos kt)$

$t = 0, \frac{dx}{dt} = 0$

$\Rightarrow O = -kA + kB$

$B = A = \frac{1}{2}a$

$\therefore x = \frac{1}{2}ae^{-kt}(\cos kt + \sin kt)$

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Review Exercise 2
Exercise A, Question 6

Question:

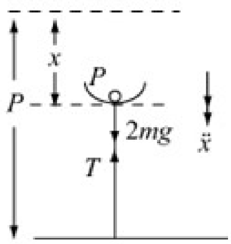
A light spring PQ is fixed at its lower end Q and is constrained to move in a vertical line. At its upper end P the spring is fixed to a small cup, of mass m, which contains a sugar lump of mass m. The spring has modulus of elasticity $8mg$ and natural length l. Given that the compression of the spring is x at time t,

a show that, while the sugar lump is in contact with the cup, $\frac{d^2x}{dt^2}+\frac{4gx}{l}=g$.

b Given that the system is released from rest when $x=\frac{3l}{4}$ and $t=0$, show that the lump will lose contact with the cup when $t=\frac{\pi}{3}\sqrt{\frac{l}{g}}$. **[E]**

Solution:

a



$\lambda = 8mg$

Hooke's law: $T = \dfrac{\lambda x}{l} = \dfrac{8mgx}{l}$

$F = ma$

$2mg - T = 2m\ddot{x}$

$2mg - \dfrac{8mg}{l}x = 2m\ddot{x}$

$g - \dfrac{4gx}{l} = \ddot{x}$

$\dfrac{d^2x}{dt^2} + \dfrac{4gx}{l} = g$

b Auxiliary equation:

$m^2 + \dfrac{4g}{l} = 0$

Solve the differential equation using the methods of book FP2 chapter 5.

$m = \pm 2\mathrm{i}\sqrt{\dfrac{g}{l}}$

Complementary function:

$x = A\cos\left(2\sqrt{\dfrac{g}{l}}t\right) + B\sin\left(2\sqrt{\dfrac{g}{l}}t\right)$

Particular integral: try $x = K$

$\dot{x} = 0, \ddot{x} = 0$

$\Rightarrow 4g\dfrac{K}{l} = g \quad K = \dfrac{l}{4}$

$\therefore x = A\cos\left(2\sqrt{\dfrac{g}{l}}t\right) + B\sin\left(2\sqrt{\dfrac{g}{l}}t\right) + \dfrac{l}{4}$

$t = 0, \ x = \dfrac{3l}{4}$

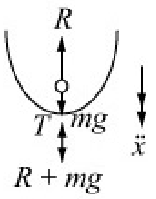
$\dfrac{3l}{4} = A + \dfrac{l}{4} \quad A = \dfrac{l}{2}$

$\dot{x} = 2\sqrt{\dfrac{g}{l}}\left[-A\sin\left(2\sqrt{\dfrac{g}{l}}t\right) + B\cos\left(2\sqrt{\dfrac{g}{l}}t\right)\right]$

$t = 0, \dot{x} = 0$

$\Rightarrow 0 = B$

$\therefore x = \dfrac{l}{2}\cos\left(2\sqrt{\dfrac{g}{l}}\,t\right) + \dfrac{l}{4}$



The sugar lump and the cup must be considered separately to find out when the lump leaves the cup. Be sure that you can identify the forces on each.

For sugar lump:

$$F = ma$$

$$mg - R = m\ddot{x} \quad ①$$

For cup:

$$mg + R - T = m\ddot{x}$$

$$mg + R - \frac{8mgx}{l} = m\ddot{x} \quad ②$$

When $R = 0$ — This is when the sugar lump loses contact with the cup.

$$\ddot{x} = g$$ From ①

$$g - \frac{8gx}{l} = g$$ From ②

$$\therefore x = 0$$

$$x = \frac{l}{2}\cos\left(2\sqrt{\frac{g}{l}}t\right) + \frac{l}{4}$$

From the solution of the differential equation.

$$x = 0 \quad \cos\left(2\sqrt{\frac{g}{l}}t\right) = -\frac{1}{2}$$

$$2\sqrt{\frac{g}{l}}t = \frac{2\pi}{3}$$

$$t = \frac{\pi}{3}\sqrt{\frac{l}{g}}$$

$\therefore$ The sugar lump loses contact with the cup when $t = \frac{\pi}{3}\sqrt{\frac{l}{g}}$

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Review Exercise 2
Exercise A, Question 7

Question:

A truck is towing a trailer of mass m along a straight horizontal road by means of a tow-rope. The truck and trailer are modelled as particles and the tow-rope is modelled as a light elastic string with modulus of elasticity $4mg$ and natural length $\frac{g}{n^2}$, where n is a positive constant. The effects of friction and air resistance on the trailer are ignored. Initially the trailer is at rest and the tow-rope is slack. The truck then accelerates until the tow-rope is taut and thereafter the truck travels in a straight line with constant speed u. At time t after the tow-rope becomes taut, its extension is x, and the trailer has moved a distance y.
Show that, whilst the rope remains taut,

a $y + x = ut$,

b $\frac{d^2x}{dt^2} + 4n^2x = 0$.

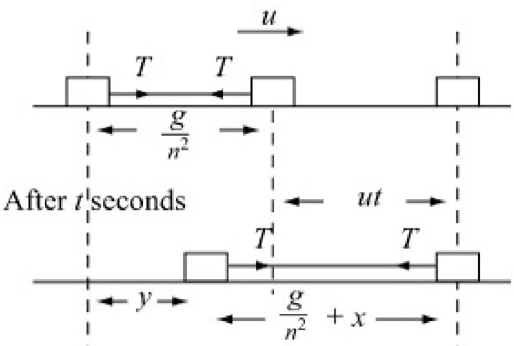
c Hence show that the tow-rope goes slack when $t = \frac{\pi}{2n}$.

d Find the speed of the trailer when $t = \frac{\pi}{3n}$.

e Find the value of t when the trailer first collides with the truck. **[E]**

Solution:

a



$\lambda = 4mg$

$l = \frac{g}{n^2}$

Draw diagrams to show the situation when the tow-rope is first taut and the situation t seconds later.

$y + \frac{g}{n^2} + x = \frac{g}{n^2} + ut$ ← Use the diagrams to equate distances.

$y + x = ut$

$y + \frac{g}{n^2} + x = \frac{g}{n^2} + ut$ ← Use the diagrams to equate distances.

$\therefore \quad y + x = ut$

b For the trailer:

$T = m\ddot{y}$ ① ← Using $F = ma$

From **a** $y + x = ut$

$\dot{y} + \dot{x} = u$

$\ddot{y} + \ddot{x} = 0$ ②

Hooke's law:

$T = \frac{\lambda x}{l}$

$T = 4mgx \times \frac{n^2}{g} = 4mn^2 x$

From ① and ②:

$T = m\ddot{y} = -m\ddot{x}$

$\therefore -m\ddot{x} = 4mn^2 x$

$\therefore \frac{d^2x}{dt^2} + 4n^2 x = 0$

c Auxiliary equation: $m^2 + 4n^2 = 0$

$m = \pm 2in$

General solution:

$x = A\cos 2nt + B\sin 2nt$

$t = 0 \quad x = 0 \Rightarrow A = 0$

Tow-rope slack when $x = 0$

$0 = B\sin 2nt$ ← Value of B not needed here.

$\sin 2nt = 0$

$nt = \pi$

$t = \frac{\pi}{2n}$

d $\dot{x} = 2nB\cos 2nt$

$\dot{y} = u - \dot{x}$

$\therefore \dot{y} = u - 2nB\cos 2nt$

$t = 0 \quad \dot{y} = 0$

$\therefore 2nB = u$

$B = \dfrac{u}{2n}$

$\therefore \dot{y} = u - u\cos 2nt$

$t = \dfrac{\pi}{3n} \quad \dot{y} = u\left(1 - \cos\dfrac{2\pi}{3}\right)$

$\dot{y} = u\left(1 + \dfrac{1}{2}\right) = \dfrac{3u}{2}$

The speed of the trailer is $\dot{y}$, not $\dot{x}$.

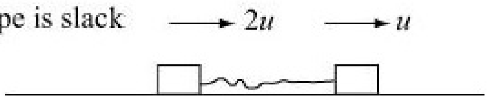
From differentiating $y + x = ut$ (done in **b**).

e Tow-rope slack when $t = \dfrac{\pi}{2n}$

From **c**.

$\dot{y} = u - u\cos\pi = 2u$

Once rope is slack



Closing speed $= 2u - u = u$

Relative distance travelled $= \dfrac{g}{n^2}$

Use your knowledge of relative motion for an efficient solution!

$\therefore \text{Time} = \dfrac{\frac{g}{n^2}}{u} = \dfrac{g}{un^2}$

Total time $(t) = \dfrac{\pi}{2n} + \dfrac{g}{un^2}$

The time from the moment the tow-rope becomes taut is required.

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Review Exercise 2
Exercise A, Question 8

Question:

Seats on a coach rest on stabilisers to enable the seats to return to their initial positions smoothly after the coach hits a bump in the road. In a mathematical model of the situation, the following assumptions are made: each stabiliser is a light elastic spring, enclosed in a viscous liquid and fixed in a vertical position; the spring exerts a force of 1.8 N for each cm by which it is extended or compressed; the seat, together with the person sitting on it, constitute a particle P attached to the upper end of the spring which is vertical, the lower end of the spring being fixed; the viscous liquid exerts a resistance to the motion of P of magnitude $240v$ N when the speed of P is v m s^{-1}.
Given that the mass of P is m kg, and the distance of P from its equilibrium position at time t seconds is x metres measured in a downwards direction,

a show that x satisfies the differential equation $m\frac{d^2x}{dt^2} + 240\frac{dx}{dt} + 180x = 0$.

b Show that, when P is disturbed from its equilibrium position, the resulting motion is oscillatory when $m > 80$.

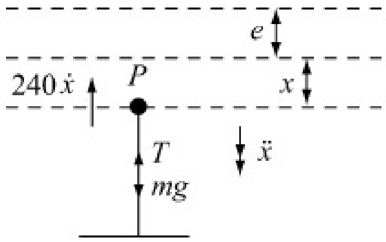
A man is sitting on the seat when the coach hits a bump in the road, giving the seat and initial upward speed of U m s^{-1}. The combined mass of the man and the seat is 80 kg.

c Find and expression for x in terms of t.

d Find the greatest displacement of the man from his equilibrium position in the subsequent motion. **[E]**

Solution:

a



When in equilibrium, $T = mg$.
Equilibrium compression $= e$ m

$\Rightarrow 1.8\times 100e = mg$

When x m below the equilibrium level,

$$T = 1.8(e + x)\times 100$$
$$= mg + 1.8\times 100x$$
$$F = ma$$
$$mg - T - 240\dot{x} = m\ddot{x}$$
$$mg - mg - 1.8\times 100x - 240\dot{x} = m\ddot{x}$$
$$\therefore m\ddot{x} + 240\dot{x} + 180x = 0$$
$$m\frac{d^2x}{dt^2} + 240\frac{dx}{dt} + 180x = 0$$

Be careful about the units. The tension / thrust is 1.8 N for each *cm* of extension or compression.

b The motion is oscillatory if the auxiliary equation has complex roots i.e. $240^2 < 4m\times 180$

$$m > \frac{240^2}{4\times 180}$$
$$m > 80$$

c $m = 80$

$$80\frac{d^2x}{dt^2} + 240\frac{dx}{dt} + 180x = 0$$
$$4\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 9x = 0$$

Auxiliary equation: $4m^2 + 12m + 9 = 0$

$$(2m+3)^2 = 0$$
$$m = \frac{-3}{2} \quad \text{(twice)}$$

Now solve the differential equation using the methods of book FP2 Chapter 5.

General solution:

$$x = (A + Bt)e^{-\frac{3}{2}t} + Be^{-\frac{3}{2}t}$$

$t = 0\ x = 0 \Rightarrow 0 = A$

$$x = Bt\,e^{\frac{-3}{2}t}$$

$$\dot{x} = -\frac{3}{2}Bte^{-\frac{3}{2}t}$$

$t = 0\ \dot{x} = -U \Rightarrow -U = B$

$$\therefore x = -Ute^{-\frac{3}{2}t}$$

d Maximum displacement $\Rightarrow \dot{x} = 0$

$$\therefore 0 = \frac{3}{2}Ute^{-\frac{3}{2}t} - Ue^{-\frac{3}{2}t}$$

$$Ue^{-\frac{3}{2}t}\left(\frac{3}{2}t - 1\right) = 0$$

$$t = \frac{2}{3}$$

$$\therefore x_{max} = -U \times \frac{2}{3}e^{-1}$$

i.e. maximum displacement of the man is $\frac{2U}{3e}$

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Review Exercise 2
Exercise A, Question 9

Question:

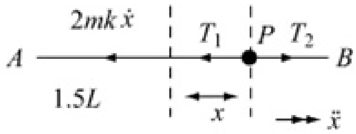
A particle P of mass m is attached to the mid-point of a light elastic string, of natural length $2L$ and modulus of elasticity $2mk^2L$, where k is a positive constant. The ends of the string are attached to points A and B on a smooth horizontal surface, where $AB = 3L$. The particle is released from rest at the point C, where $AC = 2L$ and ACB is a straight line. During the subsequent motion P experiences air resistance of magnitude $2mkv$, where v is the speed of P. At time t, $AP = 1.5L + x$.

a Show that $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 4k^2x = 0$.

b Find an expression, in terms of t, k and L, for the distance AP at time t. **[E]**

Solution:

a



$\lambda = 2mk^2L$

Hooke's law:

$$T_1 = \frac{\lambda x}{l} = \frac{2mk^2L(0.5L + x)}{L}$$
$$= 2mk^2(0.5L + x)$$
$$T_2 = 2mk^2(0.5L - x)$$

$$F = ma$$
$$T_2 - T_1 - 2mk\dot{x} = m\ddot{x}$$
$$2mk^2(0.5L - x) - 2mk^2(0.5L + x) - 2mk\dot{x} = m\ddot{x}$$
$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 4k^2x = 0$$

b Auxiliary equation: $m^2 + 2km + 4k^2 = 0$

$$m = \frac{-2k \pm \sqrt{(4k^2 - 16k^2)}}{2}$$
$$= -k \pm ki\sqrt{3}$$

General solution:

$$x = e^{-kt}(A\cos k\sqrt{3}t + B\sin k\sqrt{3}t)$$
$$t = 0,\ x = \frac{1}{2}L \Rightarrow A = \frac{1}{2}L$$
$$\dot{x} = -ke^{-kt}(A\cos k\sqrt{3}t + B\sin k\sqrt{3}t) + e^{-kt}(-k\sqrt{3}A\sin k\sqrt{3}t + k\sqrt{3}B\cos k\sqrt{3}t)$$
$$t = 0,\ \dot{x} = 0 \Rightarrow -kA + k\sqrt{3}B = 0$$
$$B = \frac{A}{\sqrt{3}} = \frac{1}{2\sqrt{3}}L$$
$$AP = 1.5L + x$$

Length AP is needed, not just x.

$$AP = 1.5L + e^{-kt}\left(\frac{1}{2}L\cos k\sqrt{3}t + \frac{1L}{2\sqrt{3}}\sin k\sqrt{3}t\right)$$
$$AP = 1.5L + \frac{Le^{-kt}}{2\sqrt{3}}(\sqrt{3}\cos k\sqrt{3}t + \sin k\sqrt{3}t)$$

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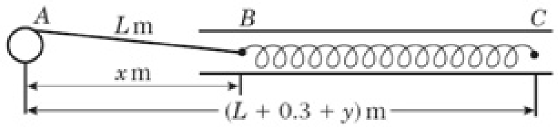
Review Exercise 2
Exercise A, Question 10

Question:



The diagram shows a sketch of a machine component consisting of a long rod AB of length L m. The end A is attached to the circumference of a flywheel centre O, radius 0.2 m, which rotates with constant angular speed 10 rad s^{-1}. The other end B is attached to a ring constrained to move in a smooth horizontal tube. The length of the rod is very much greater than the radius of the flywheel and it may be assumed that, at time t seconds, the distance x m, of B from O is given by the equation

$$x = L + 0.2\cos 10t.$$

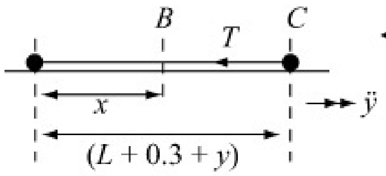
Attached to B is a light spring BC of modulus 3.75 N and natural length 0.3 m, at the other end of which is a particle C of mass 0.5 kg which is also constrained to move in the tube. When $t = 0$, the flywheel starts to rotate with B and C at rest and with the spring BC unextended.

a Show that, if the distance $OC = (L + 0.3 + y)$ m, then y satisfies the differential equation $\frac{d^2x}{dt^2} + 25y = 5\cos 10t$.

b Find an expression for y in terms of t. **[E]**

Solution:

a



Draw a diagram showing all the information. Be particularly careful about the lengths.

Length of spring $= L + 0.3 + y - x$

$= L + 0.3 + y - (L + 0.2\cos 10t)$

$= 0.3 + y - 0.2\cos 10t$

$\therefore$Extension $= y - 0.2\cos 10t$

Use the lengths in the diagram to obtain an expression for the extension.

Hooke's law:

$$T = \frac{\lambda x}{l} = \frac{3.75}{0.3}(y - 0.2\cos 10t)$$

$$T = 12.5(y - 0.2\cos 10t)$$

Consider particle C:

$$F = ma$$
$$-T = 0.5\ddot{y}$$
$$0.5\ddot{y} = -12.5y + 12.5 \times 0.2\cos 10t$$
$$\ddot{y} = -25y + 25 \times 0.2\cos 10t$$

$$\frac{d^2y}{dt^2} + 25y = 5\cos 10t$$

d Auxiliary equation: $m^2 + 25 = 0$

$$m = \pm 5i$$

Solve the differential equation using the methods of book FP2 Chapter 5.

Complementary function:

$y = A\cos 5t + B\sin 5t$

Particular integral:

try $y = p\cos 10t + q\sin 10t$

$\dot{y} = -10p\sin 10t + 10q\cos 10t$

$\ddot{y} = -100p\cos 10t - 100q\sin 10t$

$\therefore -100p\cos 10t - 100q\sin 10t + 25(p\cos 10t + q\sin 10t) = 5\cos 10t$

$-75p\cos 10t - 75q\sin 10t = 5\cos 10t$

$$\Rightarrow -75p = 5 \quad p = -\frac{1}{15}$$
$$q = 0$$

Complete solution:

$$y = A\cos 5t + B\sin 5t - \frac{1}{15}\cos 10t$$

From **a**.
extension $= y - 0.2\cos 10t$ and when $t = 0$, extension $= 0$.

$$t = 0\ y = 0.2 \Rightarrow 0.2 = A - \frac{1}{15}$$

$$A = \frac{1}{5} + \frac{1}{15} = \frac{4}{15}$$

$$\dot{y} = -5A\sin 5t + 5B\cos 5t + \frac{10}{15}\sin 10t$$

$$t = 0,\ \dot{y} = 0 \Rightarrow 0 = 5B, B = 0$$

$$\therefore y = \frac{4}{15}\cos 5t - \frac{1}{15}\cos 10t$$

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Review Exercise 2
Exercise A, Question 11

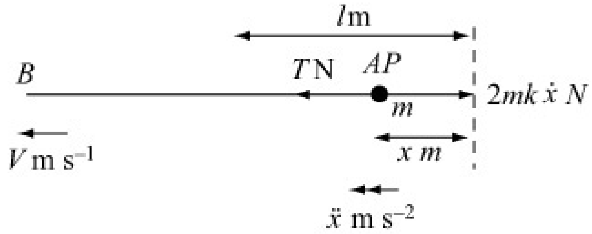
Question:

A particle P of mass m kg can move on a smooth horizontal table. It is attached to one end A of an elastic string AB, whose natural length is l metres, and whose modulus of elasticity is $10mlk^2$ newtons, where k is a positive constant. The string and particle are lying in equilibrium on the table, with $AB = l$ metres. At time $t = 0$, the end B of the string is forced to move horizontally with speed V m s^{-1} in the line of BA and in a direction away from P. The end B is forced to maintain this constant speed throughout the subsequent motion. As P moves, it experiences air resistance of magnitude $2mkv$ newtons, where v m s^{-1} is the speed of P. After t seconds, the distance of P from its initial position is x metres.
By considering the extension of the string at time t,

a show that x satisfies the differential equation $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 10k^2x = 10k^2Vt$.

b Find an expression for x in terms of t, k and V. **[E]**

Solution:



a At time t:
B has moved Vt m
P has moved x m

$\therefore$ length $AB = (Vt + l - x)$ m

$\therefore$ extension $= (Vt - x)$ m

Hooke's law $T = \frac{\lambda x}{l}$

$$T = 10mlk^2 \frac{(Vt - x)}{l}$$

$$T = 10mk^2 (Vt - x)$$

For P: $F = ma$

$$T - 2mk\dot{x} = m\ddot{x}$$

$$10mk^2 (Vt - x) - 2mk\dot{x} = m\ddot{x}$$

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 10k^2 x = 10k^2 Vt$$

b Auxiliary equation: $m^2 + 2km + 10k^2 = 0$

$$m = \frac{-2k \pm \sqrt{(4k^2 - 40k^2)}}{2}$$

Use the methods of book FP2 chapter 5 to solve the differential equation.

$$m = -k \pm 3k\text{i}$$

Complementary function:
$x = e^{-kt}(A\cos 3kt + B\sin 3kt)$

Particular integral: $x = at + b$

$$\Rightarrow \dot{x} = a, \ddot{x} = 0$$

$$\therefore 2ka + 10k^2(at + b) = 10k^2 Vt$$

$$2ka + 10k^2 b = 0$$

Equating constant terms.

$$10k^2 a = 10k^2 V$$

Equating coefficients of t.

$$\Rightarrow a = V$$

$$10k^2 b = -2kV$$

$$b = \frac{-V}{5k}$$

$\therefore$ Complete solution is

$$x = e^{-kt}(A\cos 3kt + B\sin 3kt) + Vt - \frac{V}{5k}$$

$$t = 0,\ x = 0 \Rightarrow 0 = A - \frac{V}{5k}$$

$$A = \frac{V}{5k}$$

$$\dot{x} = -ek^{-kt}\left(A\cos 3kt + B\sin 3kt\right) + e^{-kt}\left(-3kA\sin 3kt + 3kB\cos 3kt\right) + V$$

$$t = 0, \dot{x} = 0 \Rightarrow 0 = -kA + 3kB + V$$

$$3kB = V - \frac{V}{5}$$

$$B = \frac{-4V}{15k}$$

$$\therefore x = e^{-kt}\left(\frac{V}{5k}\cos 3kt - \frac{4V}{15k}\sin 3kt\right) + Vt - \frac{V}{5k}$$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 12

Question:

A particle P of mass m is attached to one end A of a light elastic string AB, of natural length l and modulus of elasticity mln^2, where n is a constant. The string is lying at rest on a smooth horizontal table, with $AB = l$. At time $t = 0$, the end B is forced to move with constant acceleration f in the direction AB away from A. After time t, the distance of P from its initial position is y, and the extension of the string is x.

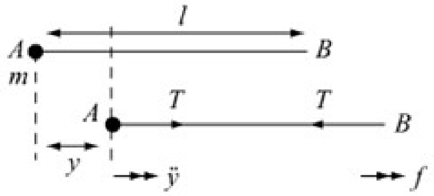
a By finding a relationship between x, y, f and t, show that, while the string remains taut, $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + n^2 x = f$.

b Hence express x and y as functions of t.

c Find the speed of P when the string is at its natural length for the first time in the ensuing motion.

d Show that the string never becomes slack. **[E]**

Solution:



$\lambda = mln^2$

Draw diagrams showing the initial situation and the situation after time t.

a At time t, distance moved by B

$= \frac{1}{2}ft^2$

$\Rightarrow$ length $AB = l + \frac{1}{2}ft^2 - y$

$\therefore$ extension $= x = \frac{1}{2}ft^2 - y$ ①

Use the lengths in the diagram to obtain the suggested relationship.

Consider particle P: $F = ma$

$T = m\ddot{y}$ ②

From ①

$y = \frac{1}{2}ft^2 - x$

$\dot{y} = ft - \dot{x}$

$\ddot{y} = f - \ddot{x}$

T will be a function of the extension, x, so use ① to obtain $\ddot{y}$ in terms of $\ddot{x}$.

Hooke's law:

$T = \frac{\lambda x}{l}$

$T = \frac{mln^2 x}{l} = mn^2 x$

In ②

$mn^2 x = m(f - \ddot{x})$

$\ddot{x} + n^2 x = f$

$\frac{d^2 x}{dt^2} + n^2 x = f$

b Auxiliary equation: $m^2 + n^2 = 0$

$m = \pm in$

Solve the differential equation using the methods of book FP2 Chapter 5.

Complementary function:

$x = A\cos nt + B\sin nt$

Particular integral: try $x = k$

$\dot{x} = \ddot{x} = 0$

$\Rightarrow n^2 k = f$

$k = \dfrac{f}{n^2}$

Complete solution is

$x = A\cos nt + B\sin nt + \dfrac{f}{n^2}$

$t = 0, x = 0 \Rightarrow 0 = A + \dfrac{f}{n^2}$

$A = \dfrac{-f}{n^2}$

$\dot{x} = -An\sin nt + Bn\cos nt$

$t = 0, \dot{x} = 0 \Rightarrow 0 = -Bn,\ B = 0$

$\therefore x = -\dfrac{f}{n^2}\cos nt + \dfrac{f}{n^2}$

and $y = \dfrac{1}{2}ft^2 - x$

$y = \dfrac{1}{2}ft^2 + \dfrac{f}{n^2}\cos nt - \dfrac{f}{n^2}$

c $x = 0 \Rightarrow \cos nt = 1$

$nt = 0, 2\pi$

↙

at start

Extension is zero when the string is at its natural length.

$\dot{y} = \dfrac{1}{2}ft \times 2 - \dfrac{f}{n^2} \times n\sin nt$

$\dot{x}$ is the rate of increase of the extension. The speed of P is $\dot{y}$.

$t = \dfrac{2\pi}{n}$ $\quad \dot{y} = f \times \dfrac{2\pi}{n} - \dfrac{f}{n}\sin 2\pi$

$\dot{y} = \dfrac{2f\pi}{n} - 0$

The speed of P is $\dfrac{2f\pi}{n}$

d extension $= x = -\dfrac{f}{n^2}\cos nt + \dfrac{f}{n^2}$

$-1 \le \cos nt \le 1$

$\Rightarrow x = \dfrac{f}{n^2}(1 - \cos nt)$ is never negative

$\therefore$ string never becomes slack.

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 13

Question:

A particle P of mass 0.2 kg is attached to one end of a light elastic string of natural length 0.6 m and modulus of elasticity 0.96 N. The other end of the string is fixed to a point which is 0.6 m above the surface of a liquid. The particle is held on the surface of the liquid, with the string vertical, and then released from rest. The liquid exerts a constant upward force on P of magnitude 1.48 N, and also a resistive force of magnitude $1.2v$ N, when the speed of P is v m s^{-1}. At time t seconds, the distance travelled down by P is x metres.

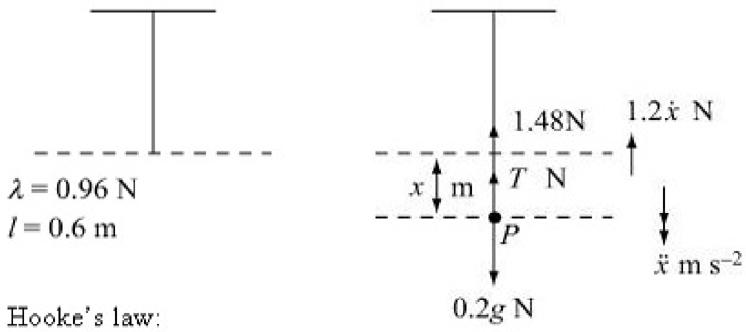
a Show that, during the time when P is moving downwards, $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+8x=2.4$.

b Find x in terms of t.

c Show that the particle continues to move down through the liquid throughout the motion. **[E]**

Solution:

a



Hooke's law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{0.96x}{0.6} = 1.6x$$

$$F = ma$$

$$0.2g - 1.6x - 1.48 - 1.2\dot{x} = 0.2\ddot{x}$$

$$\ddot{x} = 9.8 - 8x - 7.4 - 6\dot{x}$$

Use $g = 9.8$ m s^{-2}.

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 2.4$$

b Auxiliary equation:

$$m^2 + 6m + 8 = 0$$

$$(m+4)(m+2) = 0$$

$$m = -4 \quad \text{or} \quad m = -2$$

Complementary function:

$$x = Ae^{-4t} + Be^{-2t}$$

Particular integral:

$$x = a$$

$$\dot{x} = \ddot{x} = 0$$

$$\Rightarrow 8a = 2.4$$

$$a = 0.3$$

$$\therefore x = Ae^{-4t} + Be^{-2t} + 0.3$$

$$t = 0\ x = 0 \Rightarrow 0 = A + B + 0.3 \quad ①$$

$$\dot{x} = -4Ae^{-4t} - 2Be^{-2t}$$

$$t = 0\ \dot{x} = 0 \Rightarrow 0 = -4A - 2B$$

$$2A = -B \quad ②$$

$$\therefore 0 = A - 2A + 0.3$$

Solve ① and ② simultaneously.

$$A = 0.3, B = -0.6$$

$$\therefore x = 0.3e^{-4t} - 0.6e^{-2t} + 0.3$$

c

$$\dot{x} = -1.2e^{-4t} + 1.2e^{-2t}$$

$$= 1.2e^{-4t}\left(e^{2t} - 1\right)$$

For P to move downwards throughout the motion $\dot{x}$ must always be positive for

$e^{2t} > 1$ for all $t > 0$

$\therefore \dot{x} > 0$ throughout the motion (except for $t = 0$)

i.e. the particle continues to move down through the liquid throughout the motion.

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Review Exercise 2
Exercise A, Question 14

Question:

A particle P of mass m is attached to one end of a light elastic string, of natural length a and modulus of elasticity $2mak^2$, where k is a positive constant. The other end of the string is attached to a fixed point A. At time $t = 0$, P is released from rest from a point which is a distance $2a$ vertically below A. When P is moving with speed v, the air resistance has magnitude $2mkv$. At time t, the extension of the string is x.

a Show that, while the string is taut, $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = g$.

You are given that the general solution of this differential equation is

$x = e^{-kt}(C\sin kt + D\cos kt) + \frac{g}{2k^2}$, where C and D are constants.

b Find the value of C and the value of D.

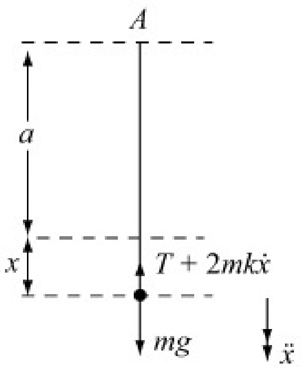
Assuming that the string remains taut,

c find the value of t when P first comes to rest,

d show that $2k^2a < g(1 + e^{\pi})$. **[E]**

Solution:

a



$l = a$
$\lambda = 2mak^2$

$F = ma$

$mg - T - 2mk\dot{x} = m\ddot{x}$

Hooke's law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{2mak^2 x}{a}$$

$$\therefore mg - 2mk^2 x - 2mk\dot{x} = m\ddot{x}$$

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2 x = g$$

b $x = e^{-kt}(C\sin kt + D\cos kt) + \frac{g}{2k^2}$ ← Given in the question.

$t = 0, x = a \Rightarrow a = D + \frac{g}{2k^2}$

$\therefore D = a - \frac{g}{2k^2}$

$\dot{x} = -ke^{-kt}(C\sin kt + D\cos kt)$
$+ e^{-kt}(Ck\cos kt - Dk\sin kt)$

$t = 0,\ \dot{x} = 0 \Rightarrow 0 = -kD + kC$

$\therefore C = D$

$\therefore C = D = a - \frac{g}{2k^2}$

c $\dot{x} = 0$

$\therefore -Ck\sin kt - Dk\cos kt$
$+Ck\cos kt - Dk\sin kt = 0$ ← Avoid substituting for C and D unless it becomes unavoidable.

$C = D \Rightarrow \sin kt = 0$

$kt = \pi$

$t = \frac{\pi}{k}$ ← Only the first non-zero value is required.

P first comes to rest when $t = \frac{\pi}{k}$.

d When $t = \frac{\pi}{k}$

$$x = e^{-\pi} \times D\cos\pi + \frac{g}{2k^2}$$

$$x = -De^{-\pi} + \frac{g}{2k^2}$$

$$xe^{\pi} = \frac{g}{2k^2}e^{\pi} - \left(a - \frac{g}{2k^2}\right)$$

The expression for D must be used now.

$$xe^{\pi} = \frac{g}{2k^2}\left(e^{\pi} + 1\right) - a$$

$$x > 0 \Rightarrow g\left(e^{\pi} + 1\right) > 2k^2 a$$

String remains taut so $x > 0$.

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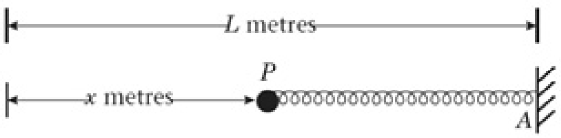
Review Exercise 2
Exercise A, Question 15

Question:



In a simple model of a shock absorber, a particle P of mass m kg is attached to one end of a light elastic horizontal spring. The other end of the spring is fixed at A and the motion of P takes place along a fixed horizontal line through A. The spring has natural length L metres and modulus of elasticity $2mL$ newtons. The whole system is immersed in a fluid which exerts a resistance on P of magnitude $3mv$ newtons, where v m s^{-1} is the speed of P at time t seconds. The compression of the spring at time t seconds is x metres, as shown in the diagram.

a Show that $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$.

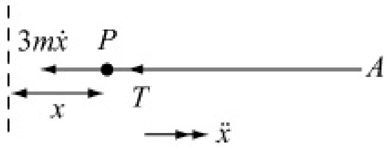
Given that when $t = 0, x = 2$ and $\frac{dx}{dt} = -4$,

b find x in terms of t.

c Sketch the graph of x against t.

d State, with a reason, whether the model is realistic. **[E]**

Solution:



$\lambda = 2mL$

$l = L$

The spring is compressed so it exerts a thrust on P.

a Hooke's law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{2mLx}{L} = 2mx$$

$F = ma$:

$$-T - 3m\dot{x} = m\ddot{x}$$

$$m\ddot{x} + 2mx + 3m\dot{x} = 0$$

$$\ddot{x} + 3\dot{x} + 2x = 0$$

or $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 2x = 0$

b Auxiliary equation:

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1, -2$$

Now solve the differential equation using the methods of book FP2. Chapter 5.

General solution:

$$x = Ae^{-t} + Be^{-2t}$$

$t = 0,\ x = 2 \Rightarrow 2 = A + B$ ①

$$\dot{x} = -Ae^{-t} - 2Be^{-2t}$$

$t = 0,\ \dot{x} = -4 \quad -4 = -A - 2B$

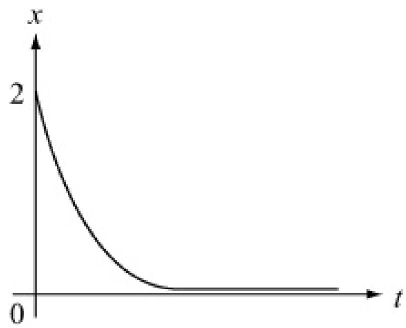
$4 = A + 2B$ ②

$\therefore 2 = B, A = 0$

$\therefore x = 2e^{-2t}$

Solving equations ① and ② simultaneously.

c



Remember t must be on the horizontal axis and you only draw the part of the curve for which $t \geq 0$.

d The model is not realistic as $\dot{x} = -4e^{-2t}$ and so P is always moving.
($\dot{x} = -4e^{-2t}$ is never zero.)

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Review Exercise 2
Exercise A, Question 16

Question:

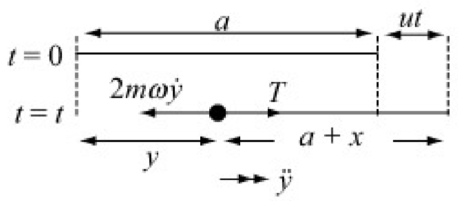
A light elastic spring, of natural length a and modulus of elasticity $5ma\omega^2$, lies unstretched along a straight line on a smooth horizontal plane. A particle of mass m is attached to one end of the spring. At time $t = 0$, the other end of the spring starts to move with constant speed U along the line of the spring and away from the particle. As the particle moves along the plane it is subject to a resistance of magnitude $2m\omega v$, where v is its speed. At time t, the extension of the spring is x and the displacement of the particle from its initial position is y. Show that

a $x + y = Ut$,

b $\frac{d^2x}{dt^2} + 2\omega\frac{dx}{dt} + 5\omega^2 x = 2\omega U$.

c Find x in terms of ω, U and t. **[E]**

Solution:



Draw a diagram to show clearly the situation when $t = 0$ and at time t.

$\lambda = 5ma\omega^2$

$l = a$

a $a + Ut = y + a + x$

$\Rightarrow \quad x + y = Ut$

Use your diagram to establish the required equation connecting x and y.

b Hooke's law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{5ma\omega^2}{a}x$$

$$T = 5m\omega^2 x$$

$$F = ma$$

$$T - 2m\omega\dot{y} = m\ddot{y}$$

$$5m\omega^2 x - 2m\omega\dot{y} = m\ddot{y}$$

Using:

$$x + y = Ut$$

$$\dot{x} + \dot{y} = U$$

and $\ddot{x} + \ddot{y} = 0$

You need to eliminate $\dot{y}$ and $\ddot{y}$ from the equation of motion.

$$\therefore 5\omega^2 x - 2\omega(U - \dot{x}) = -\ddot{x}$$

$$\ddot{x} + 2\omega\dot{x} + 5\omega^2 x = 2\omega U$$

or $\dfrac{d^2x}{dt^2} + 2\omega\dfrac{dx}{dt} + 5\omega^2 x = 2\omega U$

c Auxiliary equation:

$m^2 + 2m\omega + 5\omega^2 = 0$

$$m = \frac{-2\omega \pm \sqrt{(4\omega^2 - 20\omega^2)}}{2}$$

$$m = -\omega \pm 2i\,\omega$$

Now solve the differential equation using the methods of book FP2 Chapter 5.

Complementary function:

$x = e^{-\omega t}(A\cos 2\omega t + B\sin 2\omega t)$

Particular integral:

try $x = k$

$\dot{x} = \ddot{x} = 0$

$\therefore 5\omega^2 k = 2\omega U$

$$k = \frac{2U}{5\omega}$$

Complete solution:

$$x = e^{-\omega t}(A\cos 2\omega t + B\sin 2\omega t) + \frac{2U}{5\omega}$$

$$t = 0\ x = 0 \Rightarrow 0 = A + \frac{2U}{5\omega}$$

$$A = -\frac{2U}{5\omega}$$

$$\dot{x} = -\omega e^{-\omega t}(A\cos 2\omega t + B\sin 2\omega t)$$
$$+ e^{-\omega t}(-2\omega A\sin \omega t + 2\omega B\cos \omega t)$$

$$t = 0\ \dot{y} = 0 \Rightarrow \dot{x} = U$$

$$\therefore U = -\omega A + 2\omega B$$

$$U = -\omega \times \left(-\frac{2u}{5\omega}\right) + 2\omega B$$

When $t = 0$ P is at rest and $\dot{y}$ is the speed of P.

$$2\omega B = U - \frac{2}{5}U$$

$$B = \frac{3U}{10\omega}$$

$$\therefore x = e^{-\omega t}\left(\frac{3U}{10\omega}\sin 2\omega t - \frac{2U}{5\omega}\cos 2\omega t\right) + \frac{2U}{5\omega}$$

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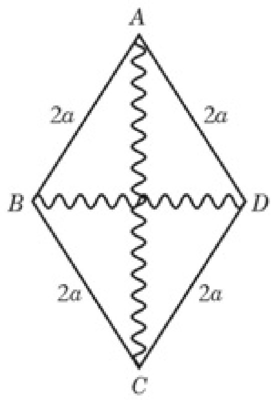
Review Exercise 2
Exercise A, Question 17

Question:



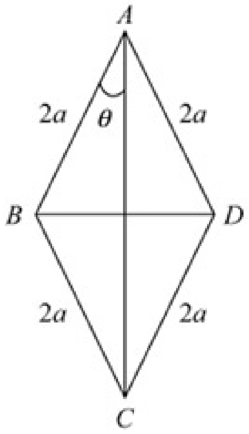
$ABCD$ is a rhombus consisting of four freely jointed uniform rods, each of mass m and length $2a$. The rhombus is freely suspended from A and is prevented from collapsing by two light springs, each of natural length a and modulus of elasticity $2mg$. One spring joins A and C and the other joins B and D, as shown in the diagram.

a Show that when AB makes an angle θ with the downward vertical, the potential energy V of the system is given by $V = -8mga(\sin\theta + 2\cos\theta) + \text{constant}$.

b Hence find the value of θ, in degrees to one decimal place, for which the system is in equilibrium.

c Determine whether this position of equilibrium is stable or unstable. **[E]**

Solution:



a length $BD = 2\times 2a\sin\theta$

$\therefore$Energy in $BD = \frac{1}{2}\frac{\lambda x^2}{l}$

$= \frac{1}{2}\times\frac{2mg}{a}(4a\sin\theta - a)^2$

$= mga(4\sin\theta - 1)^2$

BD is an elastic spring and so the elastic potential energy must be found.

length $AC = 2\times 2a\cos\theta$

$\therefore$Energy in $AC = mga(4\cos\theta - 1)^2$

AC is an identical elastic spring. Use the work done above to write down the E.P.E.

Gravitational P.E. of rhombus
$= -4mg\times 2a\cos\theta$

Take *A* as the zero level as *A* is fixed.

$\therefore V = mga(4\sin\theta - 1)^2 + mga(4\cos\theta - 1)^2$
$-8mga\cos\theta + \text{constant}$

V is the sum of the potential energies found above.
Including a 'constant' removes the need to specify a zero level.

$V = mga(16\sin^2\theta - 8\sin\theta + 1 + 16\cos^2\theta - 8\cos\theta + 1)$
$-8mga\cos\theta + \text{constant}$

$V = mga(18 - 8\sin\theta - 8\cos\theta)$
$-8mga\cos\theta + \text{constant}$

Use $\sin^2\theta + \cos^2\theta = 1$

$V = -mga(8\sin\theta + 16\cos\theta) + \text{constant}$

$V = -8mga(\sin\theta + 2\cos\theta) + \text{constant}$

$18mga$ can be absorbed into the 'constant'.

b $\frac{dV}{d\theta} = -8mga(\cos\theta - 2\sin\theta)$

$\frac{dV}{d\theta} = 0 \Rightarrow \cos\theta = 2\sin\theta$

$\tan\theta = \frac{1}{2}$

$\theta = 26.6°$

Equilibrium occurs when V has a maximum or minimum value.

c $\frac{d^2V}{d\theta^2} = -8mga(-\sin\theta - 2\cos\theta)$

Determine whether V is maximum or minimum when $\theta = 26.6°$

$\frac{d^2V}{d\theta^2} = 8mga(\sin\theta + 2\cos\theta)$

When $\theta = 26.6°, \frac{d^2V}{d\theta^2} > 0$

θ is acute so $\frac{d^2V}{d\theta^2}$ is positive. There is no need to evaluate $\frac{d^2V}{d\theta^2}$.

$\Rightarrow$ Equilibrium is stable

V has a minimum value when $\theta = 26.6°$ so equilibrium is stable.

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Review Exercise 2
Exercise A, Question 18

Question:

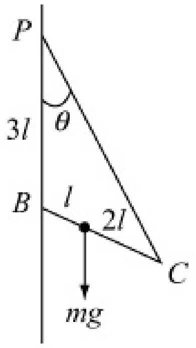
A non-uniform rod BC has mass m and length $3l$. The centre of mass of the rod is at distance l from B. The rod can turn freely about a fixed smooth horizontal axis through B. One end of a light elastic string, of natural length l and modulus of elasticity $\frac{mg}{6}$, is attached to C. The other end of the string is attached to a point P which is at a height $3l$ vertically above B.

a Show that, while the string is stretched, the potential energy of the system is $mgl(\cos^2\theta - \cos\theta) + \text{constant}$, where θ is the angle between the string and the downward vertical and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

b Find the values of θ for which the system is in equilibrium with the string stretched. **[E]**

Solution:

a



$\lambda = \dfrac{mg}{6}$

natural length $= l$

length $PC = 2 \times 3l\cos\theta$ ← ΔPBC is isosceles.

$$\text{E.P.E. in } PC = \frac{1}{2}\frac{\lambda x^2}{l}$$

$$= \frac{1}{2} \times \frac{mg}{6l}(6l\cos\theta - l)^2$$

$$= \frac{mgl}{12}(6\cos\theta - 1)^2$$

$$\text{G.P.E. of rod} = -mgl\cos 2\theta$$

← Take B as the zero level as B is fixed. ΔPBC is isosceles, so the angle between BC and the downward vertical is 2θ.

$$\therefore V = \frac{mgl}{12}\left(36\cos^2\theta - 12\cos\theta + 1\right)$$
$$-mgl\cos 2\theta + \text{constant}$$

$$V = mgl\left(3\cos^2\theta - \cos\theta + 1\right)$$
$$-mgl\left(2\cos^2\theta - 1\right) + \text{constant}$$

← The required answer does not contain 2θ, so use $\cos 2\theta = 2\cos^2\theta - 1$ to change to $\cos^2\theta$.

$$V = mgl\left(3\cos^2\theta - \cos\theta - 2\cos^2\theta\right)$$
$$+mgl + mgl + \text{constant}$$

$$V = mgl\left(\cos^2\theta - \cos\theta\right) + \text{constant}$$

← $2mgl$ can be absorbed into the constant.

b

$$\frac{dV}{d\theta} = mgl\left(-2\cos\theta\sin\theta + \sin\theta\right)$$

← When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow \sin\theta\left(-2\cos\theta + 1\right) = 0$$

$\sin\theta = 0 \quad \theta = 0$

or $\quad 2\cos\theta = 1$

$$\cos\theta = \frac{1}{2} \quad \theta = \pm\frac{\pi}{3}$$

← If $\theta = \pm\dfrac{\pi}{3}$ ΔPBC is equilateral, so the string has length $3l$ and is stretched.

$\therefore$ The values of θ are O and $\pm\dfrac{\pi}{3}$

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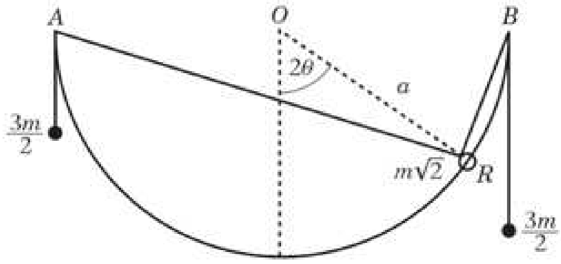
Review Exercise 2
Exercise A, Question 19

Question:



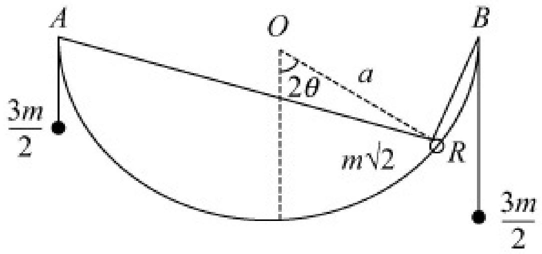
A smooth wire with ends A and B is in the shape of a semi-circle of radius a. The mid-point of AB is O. The wire is fixed in a vertical plane and hangs below AB which is horizontal. A small ring R, of mass $m\sqrt{2}$, is threaded on the wire and is attached to two light inextensible strings. The other end of each string is attached to a particle of mass $\frac{3m}{2}$. The particles hang vertically under gravity, as shown in the diagram.

a Show that, when the radius OR makes an angle 2θ with the vertical, the potential energy, V, of the system is given by $V = \sqrt{2}mga(3\cos\theta - \cos 2\theta) + \text{constant}$.

b Find the values of θ for which the system is in equilibrium.

c Determine the stability of the position of equilibrium for which $\theta > 0$. **[E]**

Solution:



a P.E. of $R = -\sqrt{2}mga\cos 2\theta$

P.E. of left hand mass

$= -\frac{3}{2}mg\left(2a - 2a\sin(45+\theta)\right)$

P.E. of right hand mass

$= -\frac{3}{2}mg\left(2a - 2a\sin(45-\theta)\right)$

Take the level of AB as the zero level for P.E. as AB is fixed.

$$\therefore V = -\sqrt{2}mga\cos 2\theta - \frac{3}{2}mga\left(2 - 2\sin(45+\theta)\right) - \frac{3}{2}mga\left(2 - 2\sin(45-\theta)\right) + \text{constant}$$

Including ' + constant' removes the need to specify a zero level.

$$V = -\sqrt{2}mga\cos 2\theta + 3mga\sin(45+\theta) + 3mga\sin(45-\theta) - 3mga - 3mga + \text{constant}$$

$$V = -\sqrt{2}mga\cos 2\theta + 3mga\left[\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right] + \text{constant}$$

Expand $\sin(45+\theta)$ and $\sin(45-\theta)$ and absorb $-6mga$ into the constant.

$$V = -\sqrt{2}mga\cos 2\theta + 3mga \times 2 \times \frac{1}{\sqrt{2}}\cos\theta + \text{constant}$$

$$V = \sqrt{2}mga\left[3\cos\theta - \cos 2\theta\right] + \text{constant}$$

b $$\frac{dV}{d\theta} = \sqrt{2}mga\left(-3\sin\theta + 2\sin 2\theta\right)$$

When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow 2\sin 2\theta = 3\sin\theta$$

$$\sin\theta\left(4\cos\theta - 3\right) = 0$$

Use $\sin 2\theta = 2\sin\theta\cos\theta$ and factorise.

$$\sin\theta = 0, \theta = 0$$

$$\text{or } \cos\theta = \frac{3}{4}, \theta = \pm\cos^{-1}\left(\frac{3}{4}\right)$$

You can give the exact answers or the decimal equivalents ($\pm 0.723^c$).

c $\dfrac{d^2V}{d\theta^2} = \sqrt{2}mga\left(-3\cos\theta + 4\cos 2\theta\right)$

$\cos\theta = \dfrac{3}{4} \quad \cos 2\theta = 2\cos^2\theta - 1$

$= 2 \times \dfrac{9}{16} - 1$

$= \dfrac{1}{8}$

$\dfrac{d^2V}{d\theta^2} = \sqrt{2}mga\left(-3 \times \dfrac{3}{4} + 4 \times \dfrac{1}{8}\right) < 0$

$\therefore$ unstable equilibrium.

V is a maximum, so equilibrium is unstable.

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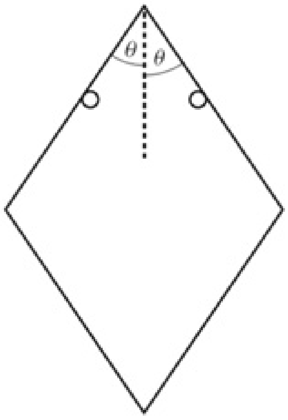
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 20

Question:



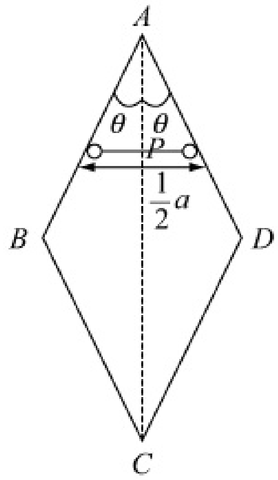
Four equal uniform rods each of length $2a$ and mass m are smoothly jointed to form a rhombus. This is used by a gardener to measure areas of lawn for treatment. When not in use it is stored resting on two smooth pegs, which are at the same level and a distance $\frac{1}{2}a$ apart, with the rhombus in a vertical plane, as shown in the diagram.

Given that each of the rods make an angle θ with the vertical,

a show that the potential energy of the system is $mga\cot\theta - 8mga\cos\theta + c$, where c is a constant.

b Hence find the value of θ when the system is in equilibrium. **[E]**

Solution:



a $AP = \frac{1}{4}a\cot\theta$

The level of the pegs must be used to calculate the P.E. as this level is fixed.

$$\text{P.E. of rod } AB = -mg\left(a\cos\theta - \frac{1}{4}a\cot\theta\right)$$

$$\text{P.E. of rod } BC = -mg\left(3a\cos\theta - \frac{1}{4}a\cot\theta\right)$$

$$\therefore \text{P.E. of system} = 2mga\left(-\cos\theta + \frac{1}{4}\cot\theta - 3\cos\theta + \frac{1}{4}\cot\theta\right) + \text{constant}$$

$$= 2mga\left(-4\cos\theta + \frac{1}{2}\cot\theta\right) + \text{constant}$$

$$= mga\cot\theta - 8mga\cos\theta + \text{constant}$$

b $\frac{dV}{d\theta} = -mga\,\text{cosec}^2\theta + 8mga\sin\theta$

When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow 8\sin\theta = \text{cosec}^2\theta$$

$$8\sin\theta = \frac{1}{\sin^2\theta}$$

$$\sin^3\theta = \frac{1}{8}$$

$$\sin\theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{6}$$

The system is in equilibrium when $\theta = \frac{\pi}{6}$.

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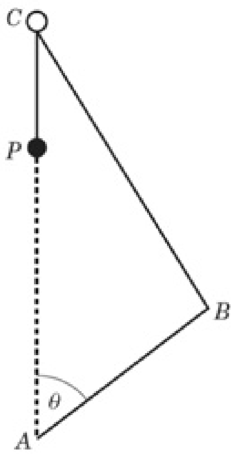
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 21

Question:

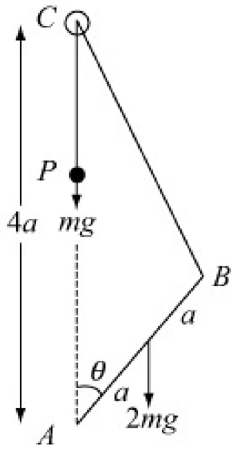


A uniform rod AB of mass $2m$ and length is freely hinged about a horizontal axis through A. The end B is attached to a light inextensible string of length b, $2a < b < 6a$, which passes through a small, smooth ring at C. A particle P, of mass m, is attached to the other end of the string and hangs freely. The point C is vertically above the point A and $AC = 4a$. The angle CAB is denoted by θ, as shown in the diagram.

a Show that the total potential energy of the system is given by $2mga(\cos\theta + \sqrt{[5 - 4\cos\theta]} + k)$, where k is a constant.

b Find, in degrees to 1 decimal place, a value of $\theta, 0 < \theta < 180°$, for which the system is in equilibrium. **[E]**

Solution:



a

$$CB^2 = (4a^2) + (2a)^2 - 2 \times 4a \times 2a\cos\theta$$

$$CB^2 = 20a^2 - 16a^2\cos\theta$$

$$CB = 2a\sqrt{(5 - 4\cos\theta)}$$

The length of CB is needed to obtain the length of CP.

$$\therefore AP = 4a - \left[b - 2a\sqrt{(5-4\cos\theta)}\right]$$

CP = total length of the string $- CB$
$= b - 2a\sqrt{(5-4\cos\theta)}$

$$\therefore \text{P.E. of } P = mg\left(4a - \left[b - 2a\sqrt{(5-4\cos\theta)}\right]\right)$$

$$\text{P.E. of rod} = 2mga\cos\theta$$

Using the level of A as the zero level for P.E. as A is fixed.

$$\therefore \text{P.E. of system} = 4mga - mgb + 2mga\sqrt{(5-4\cos\theta)} + 2mga\cos\theta + \text{constant}$$

$$= 2mga\left(\cos\theta + \sqrt{[5 - 4\cos\theta]}\right) + \text{constant}$$

Absorb $4mga - mgb$ into the constant

b

$$V = 2mga\left(\cos\theta + (5-4\cos\theta)^{\frac{1}{2}}\right) + \text{constant}$$

$$\frac{dV}{d\theta} = 2mga\left(-\sin\theta + \frac{1}{2}(5-4\cos\theta)^{-\frac{1}{2}} \times 4\sin\theta\right)$$

When the system is in equilibrium V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow -\sin\theta + \frac{2\sin\theta}{\sqrt{(5-4\cos\theta)}} = 0$$

$$\sin\theta\left(1 - \frac{2}{\sqrt{(5-4\cos\theta)}}\right) = 0$$

$\sin\theta = 0$ − no solution in range $0 < \theta < 180°$

or $\dfrac{2}{\sqrt{(5-4\cos\theta)}} = 1$

$$2 = \sqrt{(5-4\cos\theta)}$$

$$4 = 5 - 4\cos\theta$$

$$4\cos\theta = 1$$

$$\cos\theta = \frac{1}{4}$$

$$\theta = 75.52$$

Only *one* value of θ is required.

$$\therefore \theta = 75.5°$$

The system is in equilibrium when $\theta = 75.5°$.

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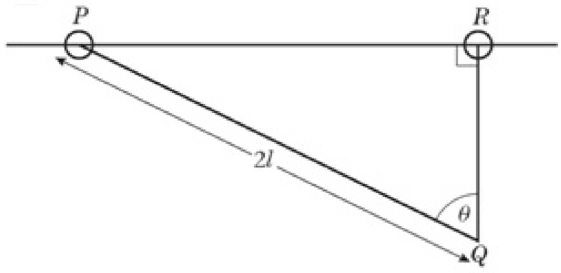
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 22

Question:



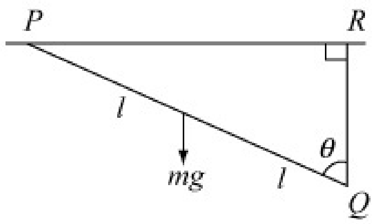
A uniform rod PQ has mass m and length $2l$. A smooth light ring is fixed to the end P of the rod. This ring is threaded on to a fixed horizontal smooth straight wire. A second small smooth light ring R is threaded on to the wire and is attached by a light elastic string, of natural length l and modulus of elasticity kmg, to the end Q of the rod, where k is a constant.

a Show that, when the rod PQ makes an angle θ with the vertical, where $0 < \theta \leq \frac{\pi}{3}$, and Q is vertically below R, as shown in the diagram, the potential energy of the system is $mgl[2k\cos^2\theta - (2k+1)\cos\theta] + \text{constant}$.

Given that there is a position of equilibrium with $\theta > 0$,

b show that $k > \frac{1}{2}$. **[E]**

Solution:



$\lambda = kmg$

a length $RQ = 2l\cos\theta$

$$\text{E.P.E.} = \frac{1}{2}\frac{\lambda x^2}{l}$$

$$= \frac{1}{2}\times\frac{kmg}{l}(2l\cos\theta - l)^2$$

$$= \frac{kmgl}{2}(2\cos\theta - 1)^2$$

$$\text{G.P.E. of rod} = -mgl\cos\theta$$

Take level of the wire as the zero level since this is fixed.

$$\text{Total P.E. of the system} = \frac{kmgl}{2}(4\cos^2\theta - 4\cos\theta + 1) - mgl\cos\theta + \text{constant}$$

$$= mgl(2k\cos^2\theta - 2k\cos\theta - \cos\theta) + \frac{kmgl}{2} + \text{constant}$$

$$= mgl(2k\cos^2\theta - (2k+1)\cos\theta) + \text{constant}$$

Absorb $\frac{kmgl}{2}$ into the constant.

b $V = mgl(2k\cos^2\theta - (2k+1)\cos\theta) + \text{constant}$

$$\frac{dV}{d\theta} = mgl[-4k\cos\theta\sin\theta + (2k+1)\sin\theta]$$

When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow \sin\theta[-4k\cos\theta + (2k+1)] = 0$$

$\sin\theta = 0 \quad \theta = 0$ not applicable

Outside the given range

$$\text{or } \cos\theta = \frac{2k+1}{4k}$$

$$0 < \theta \le \frac{\pi}{3} \Rightarrow 1 > \cos\theta \ge \frac{1}{2}$$

Use the condition on θ given in the question.

$$\therefore \frac{2k+1}{4k} < 1$$

$$2k+1 < 4k$$

$$1 < 2k$$

$$k > \frac{1}{2}$$

$$\text{and } \frac{2k+1}{4k} \ge \frac{1}{2}$$

$$2k+1 \ge 2k$$

always true for any k.

Check that the other inequality does not give rise to any problem.

$$\therefore k > \frac{1}{2}$$

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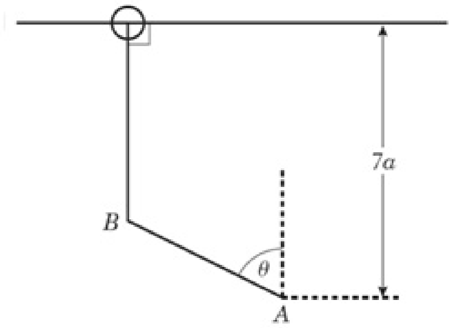
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 23

Question:



A uniform rod AB, of length $2a$ and mass $8m$, is free to rotate in a vertical plane about a fixed smooth horizontal axis through A. One end of a light elastic string, of natural length a and modulus of elasticity $\frac{4}{5}mg$, is fixed to B. The other end of the string is attached to a small ring which is free to slide on a smooth straight horizontal wire which is fixed in the same vertical plane as AB at a height $7a$ vertically above A. The rod AB makes an angle θ with the upward vertical at A, as shown in the diagram.

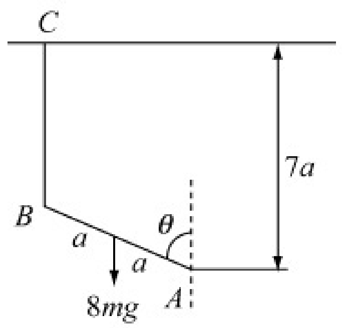
a Show that the potential energy V of the system is given by

$V = \frac{8}{5}mga(\cos^2\theta - \cos\theta) + \text{constant}$.

b Hence find the values of $\theta, 0 \le \theta \le \pi$, for which the system is in equilibrium.

c Determine the nature of these positions of equilibrium. **[E]**

Solution:



BC:

$\lambda = \frac{4}{5}mg$

$l = a$

a length of string $= 7a - 2a\cos\theta$

$\therefore$extension $= 6a - 2a\cos\theta$

$= 2a(3 - \cos\theta)$

$$\text{E.P.E.} = \frac{1}{2}\frac{\lambda x^2}{l} = \frac{1}{2}\times\frac{4}{5}mg\times\frac{[2a(3-\cos\theta)]^2}{a}$$

$$= \frac{8}{5}mga(3-\cos\theta)^2$$

G.P.E. of rod $= 8mga\cos\theta$

Using level of A as zero level as A is fixed.

$$\therefore \text{Total P.E.} = 8mga\cos\theta + \frac{8}{5}mga\,(9 - 6\cos\theta + \cos^2\theta) + \text{constant}$$

$$V = \frac{8}{5}mga(9 - 6\cos\theta + \cos^2\theta + 5\cos\theta)$$
$$+\text{constant}$$

$$V = \frac{8}{5}mga(\cos^2\theta - \cos\theta) + \text{constant}$$

Absorb $\frac{8}{5}mga\times 9$ into the constant.

b

$$\frac{dV}{d\theta} = \frac{8mga}{5}\left(-2\cos\theta\sin\theta + \sin\theta\right)$$

$$\frac{dV}{d\theta} = 0 \Rightarrow -2\cos\theta\sin\theta + \sin\theta = 0$$

When the system is in equilibrium, V has a maximum or minimum value.

$$\sin\theta\left(1 - 2\cos\theta\right) = 0$$

$$\sin\theta = 0 \quad \theta = 0, \pi$$

$$\text{or } \cos\theta = \frac{1}{2} \quad \theta = \frac{\pi}{3}$$

c

$$\frac{d^2V}{d\theta^2} = \frac{8mga}{5}\left(-2\cos^2\theta + 2\sin^2\theta + \cos\theta\right)$$

$$\theta = 0 \quad \frac{d^2V}{d\theta^2} = -\frac{8mga}{5} < 0$$

$\therefore$ V is maximum and equilibrium is unstable

$$\theta = \frac{\pi}{2} \quad \frac{d^2V}{d\theta^2} = -3\times\frac{8mga}{5} < 0$$

$\therefore$ unstable

$$\theta = \frac{\pi}{3} \quad \frac{d^2V}{d\theta^2} = \frac{8mga}{5}\left(-2\times\frac{1}{4} + 2\times\left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2}\right)$$

$$= \frac{8mga}{5}\times\frac{3}{2} > 0$$

$\therefore$ V is a minimum and the equilibrium is stable.

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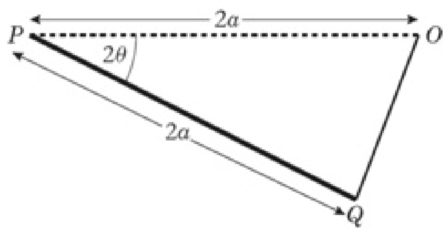
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 24

Question:



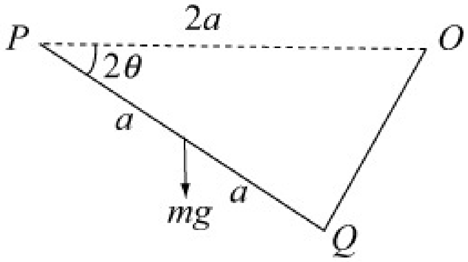
A uniform rod PQ, of length $2a$ and mass m, is free to rotate in a vertical plane about a fixed smooth horizontal axis through the end P. The end Q is attached to one end of a light elastic string, of natural length a and modulus of elasticity $\frac{mg}{2\sqrt{3}}$. The other end of the string is attached to a fixed point O, where OP is horizontal and $OP = 2a$, as shown in the diagram. $\angle OPQ$ is denoted by 2θ.

a Show that, when the string is taut, the potential energy of the system is
$$-\frac{mga}{\sqrt{3}}(2\cos 2\theta + \sqrt{3}\sin 2\theta + 2\sin\theta) + \text{constant}.$$

b Verify that there is a position of equilibrium at $\theta = \frac{\pi}{6}$.

c Determine whether this is a position of stable equilibrium. **[E]**

Solution:



OQ:

$\lambda = \frac{mg}{2\sqrt{3}}$

$l = a$

a length $OQ = 2 \times 2a\sin\theta$ since ΔOPQ is isoceles

$\therefore$extension $= a(4\sin\theta - 1)$

E.P.E $= \frac{1}{2}\frac{\lambda x^2}{l} = \frac{1}{2}\frac{mg}{2\sqrt{3}}\frac{a^2(4\sin\theta-1)^2}{a}$

$= \frac{mga}{4\sqrt{3}}(4\sin\theta - 1)^2$

G.P.E. of $PQ = -mga\sin 2\theta$ ← Using the level of OP as the zero level as this is fixed.

$\therefore$Total P.E. $= \frac{mga}{4\sqrt{3}}(16\sin^2\theta - 8\sin\theta + 1)$
$-mga\sin 2\theta + \text{constant}$

$= \frac{mga}{\sqrt{3}}(4\sin^2\theta - 2\sin\theta)\ -mga\sin 2\theta + \text{constant}$ ← $\frac{mga}{4\sqrt{3}} \times 1$ can be absorbed into the constant.

$= \frac{mga}{\sqrt{3}}[2(1-\cos 2\theta) - 2\sin\theta]$
$-mga\sin 2\theta + \text{constant}$ ← Use $\cos 2\theta = 1 - 2\sin^2\theta$ to remove $\sin^2\theta$ from the expression.

$= -\frac{mga}{\sqrt{3}}[2\cos 2\theta + 2\sin\theta + \sqrt{3}\sin 2\theta]$
$+ \text{constant}$ ← $\frac{2mga}{\sqrt{3}}$ can be absorbed into the constant.

b $\frac{dV}{d\theta} = \frac{-mga}{\sqrt{3}}\left[-4\sin 2\theta + 2\cos\theta + 2\sqrt{3}\cos 2\theta\right]$ ← When the system is in equilibrium, V has a maximum or minimum value.

$\theta = \frac{\pi}{6}$

$\frac{dV}{d\theta} = \frac{-mga}{\sqrt{3}}\left[-4\sin\frac{\pi}{3} + 2\cos\frac{\pi}{6} + 2\sqrt{3}\cos\frac{\pi}{3}\right]$

$= \frac{-mga}{\sqrt{3}}\left[-4 \times \frac{\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} + 2\sqrt{3} \times \frac{1}{2}\right]$

$= 0$

$\therefore$ There is a position of equilibrium when $\theta = \frac{\pi}{6}$

c $\frac{d^2V}{d\theta^2} = \frac{-mga}{\sqrt{3}}\left[-8\cos 2\theta - 2\sin\theta - 4\sqrt{3}\sin 2\theta\right]$

$\theta = \frac{\pi}{6}$

$\frac{d^2V}{d\theta^2} = \frac{-mga}{\sqrt{3}}\left[-8\cos\frac{\pi}{3} - 2\sin\frac{\pi}{6} - 4\sqrt{3}\sin\frac{\pi}{3}\right]$

$= \frac{-mga}{\sqrt{3}}\left[-8\times\frac{1}{2} - 2\times\frac{1}{2} - 4\sqrt{3}\times\frac{\sqrt{3}}{2}\right]$

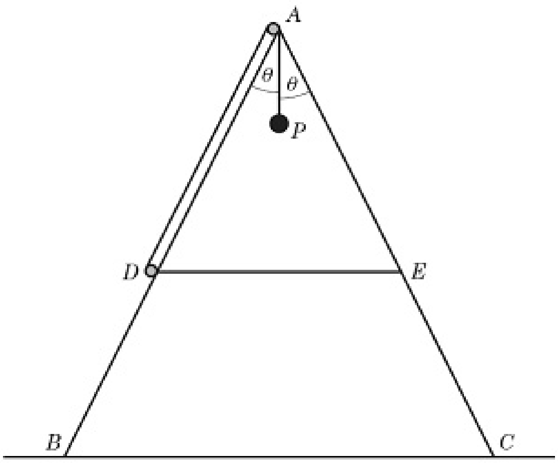
$= \frac{-mga}{\sqrt{3}}\left[-4 - 1 - 6\right] = \frac{11mga}{\sqrt{3}}$

$\frac{d^2V}{d\theta^2} > 0 \therefore V$ is a minimum and equilibrium is stable.

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 25

Question:



Two uniform rods AB and AC, each of mass $2m$ and length $2L$, are freely jointed at A. The mid-points of the rods are D and E respectively. A light inextensible string of length s is fixed to E and passes round small, smooth light pulleys at D and A. A particle P of mass m is attached to the other end of the string and hangs vertically. The points A, B and C lie in the same vertical plane with B and C on a smooth horizontal surface. The angles PAB and PAC are each equal to θ $(\theta > 0)$, as shown in the diagram.

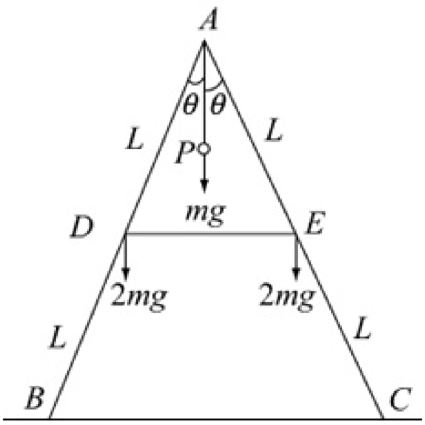
a Find the length of AP in terms of s, L and θ.

b Show that the potential energy V of the system is given by
$V = 2mgL(3\cos\theta + \sin\theta) + \text{constant}$.

c Hence find the value of θ for which the system is in equilibrium.

d Determine whether this position of equilibrium is stable or unstable. **[E]**

Solution:



a $AP = s - (AD + DE)$

$= s - (L + 2L\sin\theta)$

b $V = 2 \times 2mg \times L\cos\theta + mg(2L\cos\theta - AP) + \text{constant}$

$V = 4mgL\cos\theta + 2mgL\cos\theta - mg(s - L - 2L\sin\theta) + \text{constant}$

$V = 4mgL\cos\theta + 2mgL\cos\theta + 2mgL\sin\theta - mg(s - L) + \text{constant}$

$V = 2mgL(3\cos\theta + \sin\theta) + \text{constant}$

Using the level of BC as zero level as this is fixed.

$-mg(s-L)$ can be absorbed into the constant.

c $\dfrac{dV}{d\theta} = 2mgL(-3\sin\theta + \cos\theta)$

$\dfrac{dV}{d\theta} = 0 \Rightarrow 3\sin\theta = \cos\theta$

$\tan\theta = \dfrac{1}{3}$

$\therefore \theta = 0.322^c$ (3 s.f.)

When the system is in equilibrium, V has a maximum or minimum value.

d $\dfrac{d^2V}{d\theta^2} = 2mgL(-3\cos\theta - \sin\theta)$

$\theta = 0.322 \Rightarrow \dfrac{d^2V}{d\theta^2} < 0$

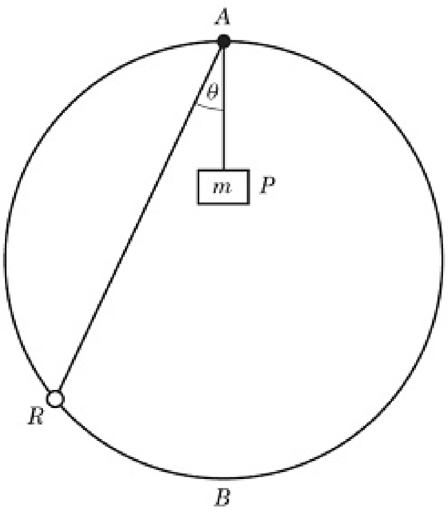
θ is acute so $\cos\theta > 0$ and $\sin\theta > 0$.

$\therefore V$ is maximum and the equilibrium is unstable.

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 26

Question:



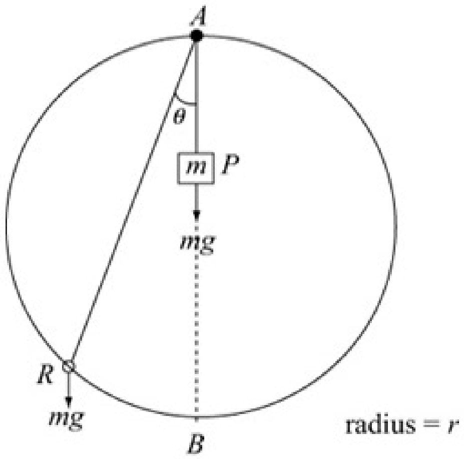
A smooth wire AB, in the shape of a circle of radius r, is fixed in a vertical plane with AB vertical. A small smooth ring R of mass m is threaded on the wire and is connected by a light inextensible string to a particle P of mass m. The length of the string is greater than the diameter of the circle. The string passes over a small smooth pulley which is fixed at the highest point A of the wire and angle $R\hat{A}P = \theta$, as shown in the diagram.

a Show that the potential energy of the system is given by $2mgr(\cos\theta - \cos^2\theta) + \text{constant}$.

b Hence determine the values of $\theta, \theta \geq 0$, for which the system is in equilibrium.

c Determine the stability of each position of equilibrium. **[E]**

Solution:



a length $AR = 2r\cos\theta$

$\angle ARB = 90°$ - angle in a semicircle

P.E. of $P = -mg\left(L - 2r\cos\theta\right)$

where L is a constant

The length of the string is constant - call it L.

P.E. of $R = -mgAR\cos\theta$

$= -mg \times 2r\cos^2\theta$

Take the level of A as the zero level as A is fixed.

$\therefore$ P.E. of the system

$= -mgL + 2mgr\cos\theta - 2mgr\cos^2\theta + \text{constant}$

$= 2mgr\left(\cos\theta - \cos^2\theta\right) + \text{constant}$

mgL is constant, so it can be absorbed into the constant.

b $V = 2mgr\left(\cos\theta - \cos^2\theta\right) + \text{constant}$

$$\frac{dV}{d\theta} = 2mgr\left(-\sin\theta + 2\cos\theta\sin\theta\right)$$

When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow \sin\theta\left(2\cos\theta - 1\right) = 0$$

$\sin\theta = 0 \quad \theta = 0$

or $\cos\theta = \frac{1}{2} \quad \theta = \frac{\pi}{3}$

c $$\frac{d^2V}{d\theta^2} = 2mgr\left(-\cos\theta - 2\sin^2\theta + 2\cos^2\theta\right)$$

$\theta = 0 \quad \frac{d^2V}{d\theta^2} = 2mgr\left(-1+2\right) > 0$

$\Rightarrow V$ is a minimum and equilibrium is stable.

$$\theta = \frac{\pi}{3} \quad \frac{d^2V}{d\theta^2} = 2mgr\left(-\frac{1}{2} - 2\left(\frac{\sqrt{3}}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2\right)$$

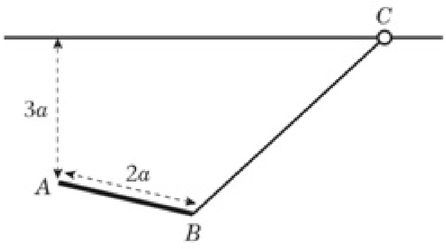
$$= 2mgr\left(-\frac{1}{2} - \frac{3}{2} + \frac{1}{2}\right) = -3mgr < 0$$

$\Rightarrow V$ is a maximum and equilibrium is unstable.

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 27

Question:



A uniform rod AB has mass m and length $2a$. One end A is freely hinged to a fixed point. One end of a light elastic string, of natural length a and modulus $\frac{1}{2}mg$, is attached to the other end B of the rod. The other end of the string is attached to a small ring C which can move freely on a smooth horizontal wire fixed at a height of $3a$ above A and in the vertical plane through A, as shown in the diagram.

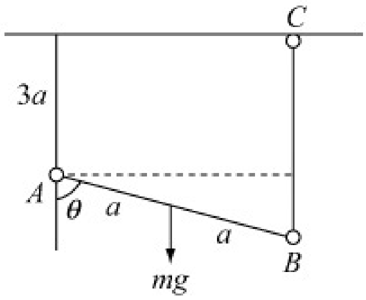
a Explain why, when the system is in equilibrium, the elastic string is vertical.

b Show that, when BC is vertical and the rod AB makes an angle θ with the downward vertical, the potential energy, V, of the system is given by
$V = mga(\cos^2\theta + \cos\theta) + \text{constant}$.

c Hence find the values of $\theta, 0 \le \theta \le \pi$, for which the system is in equilibrium.

d Determine whether each position of equilibrium is stable or unstable. **[E]**

Solution:



BC: $l = a$

$\lambda = \frac{1}{2}mg$

a Wire is smooth, so the reaction from the wire on the ring is vertical. If the ring is in equilibrium the tension in the string must be vertical as the third force on the ring is its weight.

b Length $BC = 3a + 2a\cos\theta$

$$\text{E.P.E. in } BC = \frac{1}{2}\frac{\lambda x^2}{l}$$

$$= \frac{1}{2}\times\frac{1}{2}\frac{mg}{a}(2a + 2a\cos\theta)^2$$

Extension is $BC - a$

$$= mga(1+\cos\theta)^2$$

$$\text{G.P.E. of rod} = -mga\cos\theta$$

Using level of A as the zero level as A is fixed.

$$\therefore V = mga\left(1 + 2\cos\theta + \cos^2\theta\right) - mga\cos\theta + \text{constant}$$

$$V = mga\left(\cos^2\theta + \cos\theta\right) + \text{constant}$$

Absorb mga into the constant.

c $$\frac{dV}{d\theta} = mga\left(-2\cos\theta\sin\theta - \sin\theta\right)$$

When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow \sin\theta\left(2\cos\theta + 1\right) = 0$$

$\sin\theta = 0 \quad \theta = 0, \pi$

$\cos\theta = -\frac{1}{2} \quad \theta = \frac{2\pi}{3}$

;

d $$\frac{d^2V}{d\theta^2} = mga\left(2\sin^2\theta - 2\cos^2\theta - \cos\theta\right)$$

$\theta = 0 \quad \frac{d^2V}{d\theta^2} = mga \times -3 < 0$

$\Rightarrow V$ is a maximum and equilibrium is unstable

$\theta = \pi \quad \frac{d^2V}{d\theta^2} = mga \times -1 < 0$

$\Rightarrow V$ is a maximum and equilibrium is unstable.

$$\theta = \frac{2\pi}{3} \quad \frac{d^2V}{d\theta^2} = mga\left(2\times\left(\frac{\sqrt{3}}{2}\right)^2 - 2\times\left(\frac{-1}{2}\right)^2 - \left(-\frac{1}{2}\right)\right)$$

$$= mga\left(\frac{3}{2} - \frac{1}{2} + \frac{1}{2}\right) = \frac{3mga}{2} > 0$$

$\Rightarrow V$ is a minimum and equilibrium is stable.

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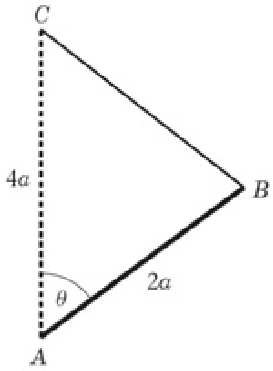
Review Exercise 2
Exercise A, Question 28

Question:

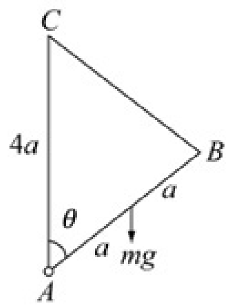


A uniform rod AB, of mass m and length $2a$, can rotate freely in a vertical plane about a fixed smooth horizontal axis through A. The fixed point C is vertically above A and $AC = 4a$. A light elastic string, of natural length $2a$ and modulus of elasticity $\frac{1}{2}mg$, joins B to C. The rod AB makes an angle θ with the upward vertical at A, as shown in the diagram.

a Show that the potential energy of the system is
$-mga[\cos\theta + \sqrt{(5 - 4\cos\theta)} + \text{constant}$.

b Hence determine the values of θ for which the system is in equilibrium. **[E]**

Solution:



$BC: \ l = 2a$

$\lambda = \frac{1}{2}mg$

a length $BC = \sqrt{\left[(4a)^2 + (2a)^2 - 2\times 4a\times 2a\cos\theta\right]}$

$= \sqrt{\left[20a^2 - 16a^2\cos\theta\right]}$

$= 2a\sqrt{[5 - 4\cos\theta]}$

Use the cosine rule to obtain BC.

$\therefore$ E.P.E. in BC

$= \frac{1}{2}\frac{\lambda x^2}{l} = \frac{1}{2}\times\frac{1}{2}\frac{mg}{2a}\left[2a\sqrt{(5-4\cos\theta)} - 2a\right]^2$

$= \frac{mga}{2}\left[\sqrt{(5-4\cos\theta)} - 1\right]^2$

G.P.E. of rod $= mga\cos\theta$

Taking level of A as zero level since A is fixed.

$\therefore$P.E. of system $= mga\cos\theta + \frac{mga}{2}\left[(5-4\cos\theta) - 2\sqrt{(5-4\cos\theta)} + 1\right]$

+ constant

$= mga\left[\cos\theta + \frac{5}{2} - 2\cos\theta - \sqrt{(5-4\cos\theta)} + 1\right] + \text{constant}$

$= -mga\left[\cos\theta + \sqrt{(5-4\cos\theta)}\right] + \text{constant}$

Absorb $mga\times\frac{7}{2}$ into the constant.

b $V = -mga\left[\cos\theta + (5-4\cos\theta)^{\frac{1}{2}}\right] + \text{constant}$

$\frac{dV}{d\theta} = -mga\left[-\sin\theta + \frac{1}{2}(5-4\cos\theta)^{-\frac{1}{2}}\times 4\sin\theta\right]$

When the system is in equilibrium, V has a maximum or minimum value.

$\frac{dV}{d\theta} = 0$

$\Rightarrow \sin\theta\left[1 - \frac{2}{\sqrt{(5-4\cos\theta)}}\right] = 0$

$\sin\theta = 0, \pi$

or $\frac{2}{\sqrt{(5-4\cos\theta)}} = 1$

$4 = 5 - 4\cos\theta$

$\cos\theta = \frac{1}{4}$

$\theta = \cos^{-1}\left(\frac{1}{4}\right) = 1.32^\circ$ (3 s.f.)

You may give the exact answer if accuracy has not been specified.

The system is in equilibrium when $\theta = 0, 1.32^\circ, \pi$

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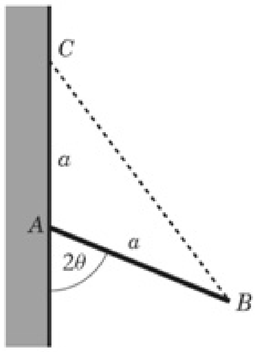
Review Exercise 2
Exercise A, Question 29

Question:



A uniform rod AB, of mass m and length a, can rotate in a vertical plane about a smooth hinge fixed at A on a vertical wall. A point C on the wall is at a height a vertically above A. One end of an elastic string, of natural length a and modulus of elasticity $2mg$, is attached to C and the other end is attached to the end B of the rod, as shown in the diagram.

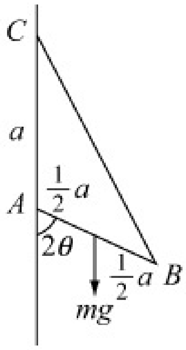
a Show that, when the rod AB makes an angle $2\theta, \theta > 0$, with the downward vertical, and the string is taut, the potential energy, V, of the system is given by

$V = -\frac{1}{2}mga\cos 2\theta + mga(2\cos\theta - 1)^2 + \text{constant}.$

b Hence determine the value of θ for which the system is in equilibrium.

c Determine whether this position of equilibrium is stable or unstable. **[E]**

Solution:



BC: $l = a$
$\lambda = 2mg$

a length $BC = 2a\cos\theta$ ← ΔABC is isosceles with $\hat{C} = \hat{B} = \theta$

$$\text{E.P.E. in } BC = \frac{1}{2}\frac{\lambda x^2}{l}$$
$$= \frac{1}{2}\times\frac{2mg}{a}(2a\cos\theta - a)^2$$
$$= mga(2\cos\theta - 1)^2$$

$$\text{G.P.E. of } AB = -mg\times\frac{1}{2}a\cos 2\theta$$ ← Using level of A as the zero level since A is fixed.

$$\therefore V = -\frac{1}{2}mga\cos 2\theta + mga(2\cos\theta - 1)^2 + \text{constant}$$

b $$\frac{dV}{d\theta} = 2\times\frac{1}{2}mga\sin 2\theta + 2mga(2\cos\theta - 1)\times(-2\sin\theta)$$ ← When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0$$
$$\sin 2\theta - 4\sin\theta(2\cos\theta - 1) = 0$$
$$2\sin\theta\cos\theta - 4\sin\theta(2\cos\theta - 1) = 0$$
$$\sin\theta(\cos\theta - 2(2\cos\theta - 1)) = 0$$
$$\sin\theta(2 - 3\cos\theta) = 0$$
$$\sin\theta = 0 \quad \theta = 0, \pi \text{ not applicable}$$ ← $\theta > 0$ and if $\theta = \pi$, $2\theta = 2\pi$ and the situation is the same as when $\theta = 0$
$$\cos\theta = \frac{2}{3}$$

$\therefore$ Equilibrium occurs when

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 0.841^c$$ ← You can give the exact answer as accuracy is not specified.

c $\frac{dV}{d\theta} = mga \sin 2\theta - 8mga \sin\theta \cos\theta + 4mga \sin\theta$

$= mga(-3\sin 2\theta + 4\sin\theta)$

$\frac{d^2V}{d\theta^2} = mga(-6\cos 2\theta + 4\cos\theta)$

Simplify $\frac{dV}{d\theta}$ to make the necessary differentiation easier.

$\cos\theta = \frac{2}{3}$ $\cos 2\theta = 2\cos^2\theta - 1$

$= 2\times\frac{4}{9} - 1$

$= -\frac{1}{9}$

$\therefore \frac{d^2V}{d\theta^2} = mga\left(-6\times\frac{-1}{9} + 4\times\frac{2}{3}\right)$

$= \frac{10}{3}mga > 0$

$\therefore V$ a minimum and equilibrium is stable.

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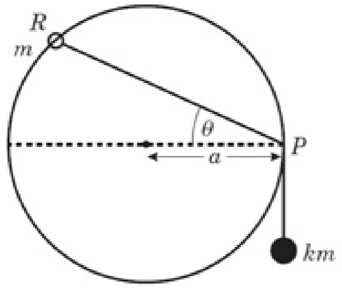
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 30

Question:



A small ring R, of mass m, is free to slide on a smooth wire in the shape of a circle with radius a. The wire is fixed in a vertical plane. A light inextensible string has one end attached to R and passes over a small smooth pulley at P, where P is one end of the horizontal diameter of the wire. The other end of the string is attached to a mass km $(k < 1)$ which hangs freely, as shown in the diagram. PR makes an angle θ with the horizontal.

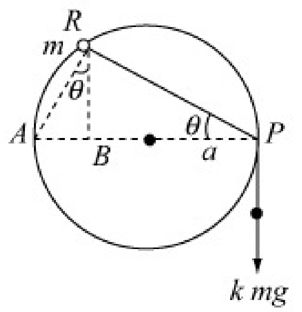
a Show that the potential energy of the system, V, is given by
$V = mga(\sin 2\theta + 2k\cos\theta) + \text{constant}$.

Given that $k = \frac{1}{2}$,

b find, in radians to 3 decimal places, the values of $\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, for which the system is in equilibrium.

c Determine whether each of the positions of equilibrium is stable or unstable. **[E]**

Solution:



a $AR = 2a\sin\theta$

$\therefore RB = AR\cos\theta$

$= 2a\sin\theta\cos\theta$

$= a\sin 2\theta$

$\angle ARP = 90°$ ($\angle$ in semicircle)

$\therefore$ P.E. of $R = mga\sin 2\theta$

Using AP as zero level as this level is fixed.

PE of hanging mass

$= -kmg\,(L - RP)$

where L is the length of the string.

$RP = 2a\cos\theta$

$\therefore V = mga\sin 2\theta - kmg\,(L - 2a\cos\theta) + \text{constant}$

$V = mga\,(\sin 2\theta + 2k\cos\theta) + \text{constant}$

Absorb $-kmg\,L$ into the constant.

b $k = \frac{1}{2}$

$\Rightarrow V = mga(\sin 2\theta + \cos\theta) + \text{constant}$

$\frac{dV}{d\theta} = mga(2\cos 2\theta - \sin\theta)$

When the system is in equilibrium, V has a maximum or minimum value.

$\frac{dV}{d\theta} = 0 \quad 2\cos 2\theta - \sin\theta = 0$

$2(1 - 2\sin^2\theta) - \sin\theta = 0$

$4\sin^2\theta + \sin\theta - 2 = 0$

$\sin\theta = \frac{-1 \pm \sqrt{(1+32)}}{8}$

$\sin\theta = \frac{-1 \pm \sqrt{33}}{8}$

$\theta = 0.6348\ldots$

or $\theta = -1.0029\ldots$

$\therefore$ Equilibrium occurs when $\theta = 0.635^c$ or -1.003^c (3 d.p.)

c $\frac{d^2V}{d\theta^2} = mga\,(-4\sin 2\theta - \cos\theta)$

$\theta = 0.6348 \quad \frac{d^2V}{d\theta^2} = mga \times (-4.63\ldots) < 0$

V is a maximum $\Rightarrow$ Unstable equilibrium when $\theta = 0.635^c$

$\theta = -1.003 \quad \frac{d^2V}{d\theta^2} = mga \times 3.089\ldots > 0$

V is a minimum $\Rightarrow$ Stable equilibrium when $\theta = -1.003^c$

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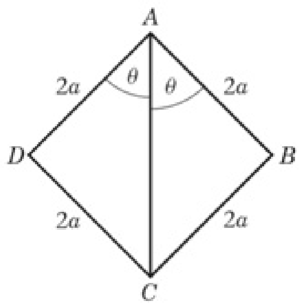
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 31

Question:



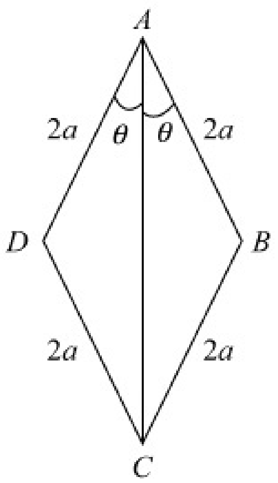
Four identical uniform rods, each of mass m and length $2a$, are freely jointed to form a rhombus $ABCD$. The rhombus is suspended from A and is prevented from collapsing by an elastic string which joins A to C, with $\angle BAD = 2\theta, 0 \le \theta \le \frac{1}{3}\pi$, as shown in the diagram. The natural length of the elastic string is $2a$ and its modulus of elasticity is $4mg$.

a Show that the potential energy, V, of the system is given by
$V = 4mga[(2\cos\theta - 1)^2 - 2\cos\theta] + \text{constant}$.

b Hence find the non-zero value of θ for which the system is in equilibrium.

c Determine whether this position of equilibrium is stable or unstable. **[E]**

Solution:



AC:
$\lambda = 4mg$
$l = 2a$

a length $AC = 2 \times 2a\cos\theta$

$$\text{E.P.E.} = \frac{1}{2}\frac{\lambda x^2}{l}$$
$$= \frac{1}{2} \times \frac{4mg}{2a}(4a\cos\theta - 2a)^2$$
$$= mga(4\cos\theta - 2)^2$$

$$\text{G.P.E. of rods} = -2 \times mga\cos\theta - 2 \times mg \times 3a\cos\theta = -8mga\cos\theta$$

Take A as zero level for P.E. as A is fixed.

$$\therefore V = -8mga\cos\theta + 4mga(2\cos\theta - 1)^2 + \text{constant}$$
$$= 4mga\left[(2\cos\theta - 1)^2 - 2\cos\theta\right] + \text{constant}$$

b $$\frac{dV}{d\theta} = 4mga\left[2(2\cos\theta - 1) \times (-2\sin\theta) + 2\sin\theta\right]$$

When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0$$
$$\Rightarrow -4\sin\theta(2\cos\theta - 1) + 2\sin\theta = 0$$
$$2\sin\theta\left[1 - 2(2\cos\theta - 1)\right] = 0$$
$\sin\theta = 0 \quad \theta = 0$ (not required answer)

$0 \le \theta \le \frac{1}{3}\pi$ but a non-zero value is required (see question).

or $1 - 4\cos\theta + 2 = 0$
$$\cos\theta = \frac{3}{4}$$
$$\therefore \theta = \cos^{-1} 0.75 \text{ or } 0.723^c \text{ (3 s.f.)}$$

c $\dfrac{dV}{d\theta} = 4mga\left[-8\sin\theta\cos\theta + 4\sin\theta + 2\sin\theta\right]$

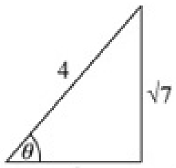
$\dfrac{d^2V}{d\theta^2} = 4mga\left[-8\cos^2\theta + 8\sin^2\theta + 6\cos\theta\right]$

$\cos\theta = \dfrac{3}{4} \Rightarrow \sin^2\theta = \dfrac{7}{16}$

$\therefore \dfrac{d^2V}{d\theta^2} = 4mga\left[-8\times\dfrac{9}{16} + 8\times\dfrac{7}{16} + 6\times\dfrac{3}{4}\right]$

$= 14mga > 0$

$\therefore V$ is a minimum and equilibrium is stable.

Use 

to obtain the value of $\sin^2\theta$.

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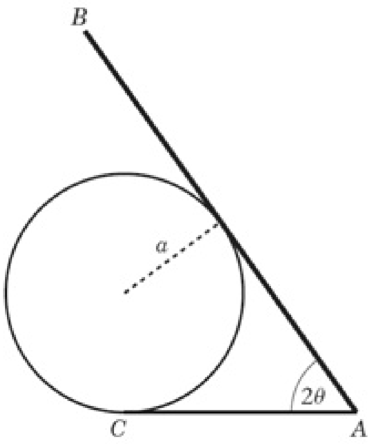
Review Exercise 2
Exercise A, Question 32

Question:



The diagram shows a uniform rod AB, of mass m and length $4a$, resting on a smooth fixed sphere of radius a. A light elastic string, of natural length a and modulus of elasticity $\frac{3}{4}mg$, has one end attached to the lowest point C of the sphere and the other end attached to A. The points A, B and C lie in a vertical plane with $\angle BAC = 2\theta$, where $\theta < \frac{\pi}{4}$.

Given that AC is always horizontal,

a show that the potential energy of the system is

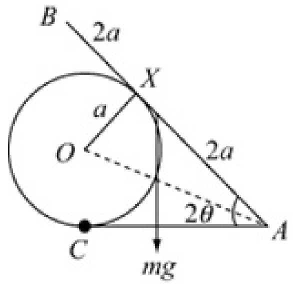
$$\frac{mga}{8}(16\sin 2\theta + 3\cot^2\theta - 6\cot\theta) + \text{constant},$$

b show that there is a value for θ for which the system is in equilibrium such that $0.535 < \theta < 0.545$.

c Determine whether this position of equilibrium is stable or unstable. **[E]**

Solution:

a



AC:

$\lambda = \frac{3}{4}mg$

$l = a$

P.E. of rod $= mg \times 2a \sin 2\theta$ ← Using AC as the zero level as this is a fixed horizontal level.

length $AC = a \cot\theta$ ← $AC = AX$ and AO bisects $\angle CAX$.

$$\therefore \text{E.P.E. in } AC = \frac{1}{2}\frac{\lambda x^2}{l}$$

$$= \frac{1}{2} \times \frac{3}{4}mg\frac{(a\cot\theta - a)^2}{a}$$

$$= \frac{3mga}{8}(\cot\theta - 1)^2$$

$\therefore$ Total P.E.

$$= 2mga\sin 2\theta + \frac{3mga}{8}\left(\cot^2\theta - 2\cot\theta + 1\right) + \text{constant}$$

$$= \frac{mga}{8}\left(16\sin 2\theta + 3\cot^2\theta - 6\cot\theta\right) + \text{constant}$$

← Absorb $\frac{3}{8}mga$ into the constant.

b $V = \frac{mga}{8}(16\sin 2\theta + 3\cot^2\theta - 6\cot\theta) + \text{constant}$

$$\frac{dV}{d\theta} = \frac{mga}{8}(32\cos 2\theta - 6\cot\theta\,\text{cosec}^2\theta + 6\text{cosec}^2\theta)$$

← When the system is in equilibrium, V has a maximum or minimum value.

$\theta = 0.535$

$$\frac{dV}{d\theta} = \frac{mga}{8} \times (-0.501..) < 0$$

$\theta = 0.545$

$$\frac{dV}{d\theta} = \frac{mga}{8} \times (0.299) > 0$$

← Investigate the sign of $\frac{dV}{d\theta}$ at the end points of the given interval.

Change of sign $\Rightarrow \frac{dV}{d\theta} = 0$ in the given interval and so there is a position of equilibrium.

c At $\theta = 0.535$ $\frac{dV}{d\theta} < 0$

At $\theta = 0.545$ $\frac{dV}{d\theta} > 0$

← There is no need to differentiate again as you know the signs of $\frac{dV}{d\theta}$ on either side of the turning point.

$\therefore V$ is a minimum

$\Rightarrow$ This position of equilibrium is stable