Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8 \text{ m s}^{-2}$.

A particle is projected with speed 35 m s^{-1} at an angle of elevation of $60°$. Find the time the particle takes to reach its greatest height.

Solution:

Resolving the initial velocity vertically

$$\begin{aligned} \text{R}(\uparrow) \quad u_y &= 35 \sin 60° \\ u &= 35 \sin 60°, v = 0, a = -9.8, t = ? \\ v &= u + at \\ 0 &= 35 \sin 60° - 9.8t \\ t &= \frac{35 \sin 60°}{9.8} = 3.092\ldots \approx 3.1 \end{aligned}$$

The time the particle takes to reach its greatest height is 3.1 (2 s.f.).

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Exercise A, Question 2

Question:

A ball is projected from a point 5 m above horizontal ground with speed 18 m s^{-1} at an angle of elevation of $40°$. Find the height of the ball above the ground 2 s after projection.

Solution:

Resolving the initial velocity vertically

$$
\begin{aligned}
\text{R}(\uparrow) \quad u_y &= 18\sin 40° \\
u &= 18\sin 40°, a = -9.8, t = 2, s = ? \\
s &= ut + \frac{1}{2}at^2 \\
&= 18\sin 40° \times 2 - 4.9 \times 2^2 \\
&= 3.540\ldots \approx 3.5
\end{aligned}
$$

The height of the ball above the ground 2 s after projection is $(5 + 3.5)\text{ m} = 8.5\text{ m}$ (2 s.f.).

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Exercise A, Question 3

Question:

A stone is projected horizontally from a point above horizontal ground with speed 32 m s^{-1}. The stone takes 2.5 s to reach the ground. Find

a the height of the point of projection above the ground,

b the distance from the point on the ground vertically below the point of projection to the point where the stone reached the ground.

Solution:

Resolving the initial velocity horizontally and vertically

$$R(\rightarrow) \quad u_x = 32$$
$$R(\downarrow) \quad u_y = 0$$

a

$$R(\downarrow) \quad u = 0, a = 9.8, t = 2.5, s = ?$$
$$s = ut + \frac{1}{2}at^2$$
$$= 0 + 4.9 \times 2.5^2 = 30.625 \approx 31$$

The height of the point of projection above the ground is 31 m (2 s.f.).

b

$$R(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$$
$$= 32 \times 2.5 = 80$$

The horizontal distance moved is 80 m.

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Exercise A, Question 4

Question:

A projectile is launched from a point on horizontal ground with speed 150 m s^{-1} at an angle of $10°$ to the horizontal. Find

a the time the projective takes to reach its highest point above the ground,

b the range of the projectile.

Solution:

Resolving the initial velocity horizontally and vertically

$$R(\rightarrow) \quad u_x = 150\cos 10°$$
$$R(\uparrow) \quad u_y = 150\sin 10°$$

a

$$R(\uparrow) \quad u = 150\sin 10°, v = 0, a = -9.8, t = ?$$
$$v = u + at$$
$$0 = 150\sin 10° - 9.8t$$
$$t = \frac{150\sin 10°}{9.8} = 2.657\ldots \approx 2.7$$

The time taken to reach the projectile's highest point is 2.7 s (2 s.f.).

b By symmetry, the time of flight is $(2.657\ldots \times 2)\text{s} = 5.315\ldots \text{s}$.

$$R(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$$
$$= 150\cos 10° \times 5.315\ldots$$
$$= 785.250\ldots \approx 790$$

The range of the projectile is 790 m (2 s.f.).

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Exercise A, Question 5

Question:

A particle is projected from a point O on a horizontal plane with speed 20 m s^{-1} at an angle of elevation of $45°$. The particle moves freely under gravity until it strikes the ground at a point X. Find

a the greatest height above the plane reached by the particle,

b the distance OX.

Solution:

Resolving the initial velocity horizontally and vertically

$$\text{R}(\rightarrow) \quad u_x = 20\cos 45° = 10\sqrt{2}$$
$$\text{R}(\uparrow) \quad u_y = 20\sin 45° = 10\sqrt{2}$$

a

$$\text{R}(\uparrow) \quad u = 10\sqrt{2}, v = 0, a = -9.8, s = ?$$
$$v^2 = u^2 + 2as$$
$$0 = 200 - 19.6s$$
$$s = \frac{200}{19.6} = 10.204\ldots \approx 10$$

The greatest height above the plane reached by the particle is 10 m (2 s.f.).

b To find the time taken to move from O to X

$$\text{R}(\uparrow) \quad s = 0, u = 10\sqrt{2}, a = -9.8, t = ?$$
$$s = ut + \frac{1}{2}at^2$$
$$0 = 10\sqrt{2}t - 4.9t^2 = t(10\sqrt{2} - 4.9t)$$
($t = 0$ corresponds to the point of projection.)
$$t = \frac{10\sqrt{2}}{4.9} = 2.886\ldots$$

$$\text{R}(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$$
$$= 10\sqrt{2} \times 2.886\ldots = 40.816\ldots \approx 41$$
$$OX = 41 \text{ m} \quad (2 \text{ s.f.})$$

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Exercise A, Question 6

Question:

A ball is projected from a point A on level ground with speed 24 m s^{-1}. The ball is projected at an angle θ to the horizontal where $\sin\theta = \frac{4}{5}$. The ball moves freely under gravity until it strikes the ground at a point B. Find

a the time of flight of the ball,

b the distance from A to B.

Solution:

$\sin\theta = \frac{4}{5} \Rightarrow \cos\theta = \frac{3}{5}$

Resolving the initial velocity horizontally and vertically

$\text{R}(\rightarrow) \quad u_x = 24\cos\theta = 14.4$

$\text{R}(\uparrow) \quad u_y = 24\sin\theta = 19.2$

a

$\text{R}(\uparrow) \quad u = 19.2, s = 0, a = -9.8, t = ?$

$s = ut + \frac{1}{2}at^2$

$0 = 19.2t - 4.9t^2 = t(19.2 - 4.9t)$

($t = 0$ corresponds to the point of projection.)

$t = \frac{19.2}{4.9} = 3.918\ldots \approx 3.9$

The time of flight of the ball is 3.9 s (2 s.f.)

b

$\text{R}(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$

$= 14.4 \times 3.918\ldots = 56.424\ldots \approx 56$

$AB = 56 \text{ m} \quad (2 \text{ s.f.})$

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Exercise A, Question 7

Question:

A particle is projected with speed 21 m s^{-1} at an angle of elevation α. Given that the greatest height reached above the point of projection is 15 m, find the value of α, giving your answer to the nearest degree.

Solution:

Resolving the initial velocity vertically and angle of elevation $= \alpha$

$$R(\uparrow) \quad u_y = 21\sin\alpha$$

$$u = 21\sin\alpha, v = 0, s = 15, a = -9.8$$

$$v^2 = u^2 + 2as$$

$$0 = (21\sin\alpha)^2 - 2\times 9.8\times 15$$

$$441\sin^2\alpha = 294$$

$$\sin^2\alpha = \frac{294}{441} = \frac{2}{3} \Rightarrow \sin\alpha = \sqrt{\left(\frac{2}{3}\right)} = 0.816\ldots$$

$\alpha \approx 54.736^\circ \approx 55^\circ$ (nearest degree)

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Exercise A, Question 8

Question:

A particle is projected horizontally from a point A which is 16 m above horizontal ground. The projectile strikes the ground at a point B which is at a horizontal distance of 140 m from A. Find the speed of projection of the particle.

Solution:

$\mathrm{R}(\downarrow)\quad u = 0, s = 16, a = 9.8, t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$16 = 0 + 4.9t^2$$

$$t^2 = \frac{16}{4.9} = 3.265\ldots \Rightarrow t = 1.807\ldots$$

Let the speed of projection be u m s^{-1}.

$\mathrm{R}(\rightarrow)\quad \text{distance} = \text{speed} \times \text{time}$

$$140 = u \times 1.807\ldots$$

$$u = \frac{140}{1.807\ldots} = 77.475\ldots \approx 77$$

The speed of projection of the particle is 77 m s^{-1} (2 s.f.).

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Exercise A, Question 9

Question:

A particle P is projected from the origin with velocity $(12\mathbf{i} + 24\mathbf{j})$ m s^{-1}, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors respectively. The particle moves freely under gravity. Find

a the position vector of P after 3 s,

b the speed of P after 3 s.

Solution:

a

$$R(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$$
$$= 12 \times 3 = 36$$

$$R(\uparrow) \quad s = ut + \frac{1}{2}at^2$$
$$= 24 \times 3 - 4.9 \times 9 = 27.9$$

The position vector of P after 3 s is $(36\mathbf{i} + 27.9\mathbf{j})$m .

b

$$R(\rightarrow) \quad u_x = 12, \text{throughout the motion}$$
$$R(\uparrow) \quad v = u + at$$
$$v_y = 24 - 9.8 \times 3 = -5.4$$

Let the speed of P after 3 s be V m s^{-1}.

$$V^2 = u_x^2 + v_y^2 = 12^2 + (-5.4)^2 = 173.16$$
$$V = \sqrt{173.16} = 13.159\ldots \approx 13$$

The speed of P after 3 s is 13 m s^{-1} (2 s.f.).

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Exercise A, Question 10

Question:

A stone is thrown with speed 30 m s^{-1} from a window which is 20 m above horizontal ground. The stone hits the ground 3.5 s later. Find

a the angle of projection of the stone,

b the horizontal distance from the window to the point where the stone hits the ground.

Solution:

Let α be the angle of projection above the horizontal.

Resolving the initial velocity horizontally and vertically

$$\text{R}(\rightarrow) \quad u_x = 30\cos\alpha$$
$$\text{R}(\uparrow) \quad u_y = 30\sin\alpha$$

a

$$\text{R}(\uparrow) \quad u = 30\sin\alpha, s = -20, a = -9.8, t = 3.5$$
$$s = ut + \frac{1}{2}at^2$$
$$-20 = 30\sin\alpha \times 3.5 - 4.9 \times 3.5^2$$
$$\sin\alpha = \frac{4.9 \times 3.5^2 - 20}{30 \times 3.5} = 0.381190\ldots$$
$$\alpha = 22.407\ldots^\circ = 22^\circ$$

The angle of projection of the stone is 22° (2 s.f.) above the horizontal.

b

$$\text{R}(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$$
$$= 30\cos\alpha \times 3.5 = 97.072\ldots$$

The horizontal distance from the window to the point where the stone hits the ground is 97 m (2 s.f.).

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Exercise A, Question 11

Question:

A ball is thrown from a point O on horizontal ground with speed u m s^{-1} at an angle of elevation of θ, where $\tan\theta = \frac{3}{4}$. The ball strikes a vertical wall which is 20 m from O at a point which is 3 m above the ground. Find

a the value of u,

b the time from the instant the ball is thrown to the instant that it strikes the wall.

Solution:

$\tan\theta = \frac{3}{4} \Rightarrow \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$

Resolving the initial velocity horizontally and vertically

$R(\rightarrow) \quad u_x = u\cos\theta = \frac{4u}{5}$

$R(\uparrow) \quad u_y = u\sin\theta = \frac{3u}{5}$

a

$R(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$

$20 = \frac{4u}{5} \times t \Rightarrow t = \frac{25}{u}$

$R(\uparrow) \quad s = ut + \frac{1}{2}at^2$

$3 = \frac{3u}{5}t - 4.9t^2 \qquad (1)$

Substituting $t = \frac{25}{u}$ into (1)

$3 = \frac{3u}{5} \times \frac{25}{u} - 4.9 \times \frac{25^2}{u^2}$

$3 = 15 - \frac{3062.5}{u^2} \Rightarrow u^2 = \frac{3062.5}{12} = 255.208\ldots$

$u = \sqrt{255.208\ldots} = 15.975\ldots \approx 16$

$u = 16$ (2 s.f.)

b $t = \frac{25}{u} = \frac{25}{15.975\ldots} = 1.5649\ldots \approx 1.6$

The time from the instant the ball is thrown to the instant that it strikes the wall is 1.6 s (2 s.f.).

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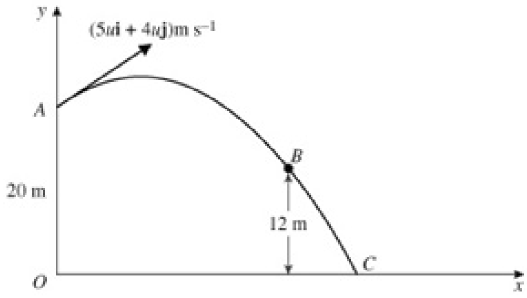
Exercise A, Question 12

Question:



[*In this question, the unit vectors* **i** *and* **j** *are in a vertical plane*, **i** *being horizontal and* **j** *being vertical.*]

A particle P is projected from a point A with position vector $20\mathbf{j}$ m with respect to a fixed origin O. The velocity of projection is $(5u\mathbf{i} + 4u\mathbf{j})$ m s^{-1}. The particle moves freely under gravity, passing through a point B, which has position vector $(k\mathbf{i} + 12\mathbf{j})$ m , where k is a constant, before reaching the point C on the x-axis, as shown in the figure above. The particle takes 4 s to move from A to B. Find

a the value of u,

b the value of k,

c the angle the velocity of P makes with the x-axis as it reaches C.

Solution:

a

$R(\uparrow) \quad s = ut + \frac{1}{2}at^2$

$-8 = 4u \times 4 - 4.9 \times 4^2$

$u = \frac{4.9 \times 4^2 - 8}{16} = 4.4$

b

$R(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$

$k = 5u \times t = 5 \times 4.4 \times 4 = 88$

c $u_x = 5u = 5 \times 4.4 = 22$, throughout the motion.

At C

$R(\uparrow) \quad v^2 = u^2 + 2as$

$v_y^2 = (4u)^2 + 2 \times (-9.8) \times (-20)$

$= 16 \times 4.4^2 + 392 = 701.76$

Let θ be angle the velocity of P makes with Ox as it reaches C.

$\tan\theta = \frac{v_y}{u_x} = \frac{\sqrt{701.76}}{22} = 1.204\ldots$

$\theta = 50.129\ldots \approx 50^\circ$

The angle the velocity of P makes with Ox as it reaches C is 50° (2 s.f.).

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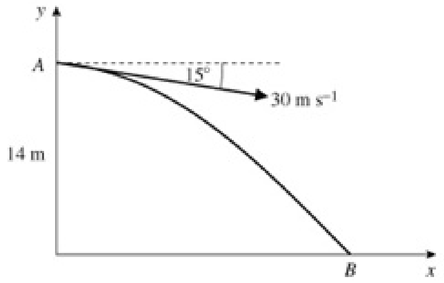
Exercise A, Question 13

Question:



A stone is thrown from a point A with speed 30 m s^{-1} at an angle of $15°$ below the horizontal. The point A is 14 m above horizontal ground. The stone strikes the ground at the point B, as shown in the figure above. Find

a the time the stone takes to travel from A to B,

b the distance AB.

Solution:

Resolving the initial velocity horizontally and vertically

$\mathrm{R}(\rightarrow) \quad u_x = 30\cos 15^\circ$

$\mathrm{R}(\downarrow) \quad u_y = 30\sin 15^\circ$

a

$\mathrm{R}(\downarrow) \quad u = 30\sin 15^\circ, s = 14, a = 9.8, t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$14 = 30\sin 15^\circ t + 4.9t^2$$

$$4.9t^2 + 30\sin 15^\circ t - 14 = 0$$

Using the formula for solving the quadratic, (the negative solution can be ignored)

$$t = \frac{-30\sin 15^\circ + \sqrt{(900\sin^2 15 + 4\times 14\times 4.9)}}{9.8}$$

$$= 1.074\ldots \approx 1.1$$

The time the particle takes to travel from A to B is 1.1 s (2 s.f.)

b

$\mathrm{R}(\rightarrow) \quad \text{distance} = \text{speed}\times\text{time}$

$$= 30\cos 15^\circ \times 1.074\ldots$$

$$= 31.136\ldots$$

$$AB^2 = 14^2 + (31.136\ldots)^2 = 1165.196\ldots$$

$$AB = 34.138\ldots \approx 34$$

The distance AB is 34 m (2 s.f.).

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Exercise A, Question 14

Question:

A particle is projected from a point with speed 21 m s^{-1} at an angle of elevation α and moves freely under gravity. When the particle has moved a horizontal distance x m, its height above the point of projection is y m.

a Show that $y = x\tan\alpha - \dfrac{x^2}{90\cos^2\alpha}$.

b Given that $y = 8.1$ when $x = 36$, find the value of $\tan\alpha$.

Solution:

Resolving the initial velocity horizontally and vertically

$\text{R}(\rightarrow)\quad u_x = 21\cos\alpha$

$\text{R}(\uparrow)\quad u_y = 21\sin\alpha$

a $\text{R}(\rightarrow)\quad \text{distance} = \text{speed} \times \text{time}$

$x = 21\cos\alpha \times t \Rightarrow t = \frac{x}{21\cos\alpha}$

$\text{R}(\uparrow)\quad s = ut + \frac{1}{2}at^2$

$$y = 21\sin\alpha t - \frac{g}{2}t^2$$

$$= 21\sin\alpha\left(\frac{x}{21\cos\alpha}\right) - 4.9\left(\frac{x}{21\cos\alpha}\right)^2$$

$$= x\tan\alpha - \frac{4.9x^2}{441\cos^2\alpha} = x\tan\alpha - \frac{x^2}{90\cos^2\alpha}, \text{ as required}$$

b $\frac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$

Using $y = 8.1, x = 36$ and the result in **a**

$8.1 = 36\tan\alpha - \frac{36^2}{90}(1 + \tan^2\alpha) = 36\tan\alpha - 14.4(1 + \tan^2\alpha)$

$\times 10$ and rearranging

$$144\tan^2\alpha - 360\tan\alpha + 225 = 0$$

$(\div 9)\ 16\tan^2\alpha - 40\tan\alpha + 25 = (4\tan\alpha - 5)^2 = 0$

$$\tan\alpha = \frac{5}{4}$$

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Exercise A, Question 15

Question:

A projectile is launched from a point on a horizontal plane with initial speed u m s^{-1} at an angle of elevation α. The particle moves freely under gravity until it strikes the plane. The range of the projectile is R m.

a Show that the time of flight of the particle is $\dfrac{2u\sin\alpha}{g}$ seconds.

b Show that $R = \dfrac{u^2\sin 2\alpha}{g}$.

c Deduce that, for a fixed u, the greatest possible range is when $\alpha = 45°$.

d Given that $R = \dfrac{2u^2}{5g}$, find the two possible values of the angle of elevation at which the projectile could have been launched.

Solution:

Resolving the initial velocity horizontally and vertically

$R(\rightarrow) \quad u_x = u\cos\alpha$

$R(\uparrow) \quad u_y = u\sin\alpha$

a

$R(\uparrow) \quad s = ut + \frac{1}{2}at^2$

$$0 = u\sin\alpha t - \frac{1}{2}gt^2 = t\left(u\sin\alpha - \frac{1}{2}gt\right)$$

($t = 0$ corresponds to the point of projection.)

$\frac{1}{2}gt = u\sin\alpha \Rightarrow t = \frac{2u\sin\alpha}{g}$, as required

b

$R(\rightarrow)$ distance = speed × time

$$R = u\cos\alpha \times \frac{2u\sin\alpha}{g} = \frac{u^2 \times 2\sin\alpha\cos\alpha}{g}$$

Using the trigonometric identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$R = \frac{u^2\sin 2\alpha}{g}$, as required

c The greatest possible value of $\sin 2\alpha$ is 1, which is when $2\alpha = 90° \Rightarrow \alpha = 45°$.

Hence, for a fixed u, the greatest possible range is when $\alpha = 45°$.

d

$$\frac{2u^2}{5g} = \frac{u^2\sin 2\alpha}{g} \Rightarrow \sin 2\alpha = \frac{2}{5}$$

$2\alpha \approx 23.578°, 156.422°$

$\alpha \approx 11.79°, 78.21°$

The two possible angles of elevation are $12°$ and $78°$ (nearest degree).

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Exercise A, Question 16

Question:

A particle is projected from a point on level ground with speed u m s^{-1} and angle of elevation α. The maximum height reached by the particle is 42 m above the ground and the particle hits the ground 196 m from its point of projection.

Find the value of α and the value of u.

Solution:

Resolving the initial velocity horizontally and vertically

$$R(\rightarrow) \quad u_x = u\cos\alpha$$
$$R(\uparrow) \quad u_y = u\sin\alpha$$

Using the maximum height is 42 m

$$R(\uparrow) \quad v^2 = u^2 + 2as$$
$$0 = u^2\sin^2\alpha - 2g \times 42$$
$$u^2\sin^2\alpha = 84g \qquad (1)$$

For the range

$$R(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$$
$$196 = u\cos\alpha \times t \Rightarrow t = \frac{196}{u\cos\alpha} \qquad (2)$$
$$R(\uparrow) \quad s = ut + \frac{1}{2}at^2$$
$$0 = u\sin\alpha t - \frac{1}{2}gt^2 = t\left(u\sin\alpha - \frac{1}{2}gt\right)$$
$$\frac{1}{2}gt = u\sin\alpha \Rightarrow t = \frac{2u\sin\alpha}{g} \qquad (3)$$

From (2) and (3)

$$\frac{196}{u\cos\alpha} = \frac{2u\sin\alpha}{g}$$
$$u^2\sin\alpha\cos\alpha = 98g \qquad (4)$$

Dividing (1) by (4)

$$\frac{u^2\sin^2\alpha}{u^2\sin\alpha\cos\alpha} = \frac{84g}{98g}$$
$$\tan\alpha = \frac{6}{7} \Rightarrow \alpha = 40.6^\circ \text{ (nearest } 0.1^\circ)$$

From (1)

$$u\sin\alpha = \sqrt{(84g)}$$
$$u = \frac{\sqrt{(84 \times 9.8)}}{\sin 40.6^\circ} = 44.08\ldots = 44 \text{ (2 s.f.)}$$

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Exercise B, Question 1

Question:

A particle is moving in a straight line. At time t seconds, its displacement, x m, from a fixed point O on the line is given by $x = 2t^3 - 8t$. Find

a the speed of the particle when $t = 3$,

b the magnitude of the acceleration of the particle when $t = 2$.

Solution:

a

$$x = 2t^3 - 8t$$

$$v = \frac{dx}{dt} = 6t^2 - 8$$

When $t = 3$

$$v = 6 \times 3^2 - 8 = 46$$

The speed of the particle when $t = 3$ is 46 m s^{-1}.

b $a = \frac{dv}{dt} = 12t$

When $t = 2$,

$$a = 12 \times 2 = 24$$

The magnitude of the acceleration of the particle when $t = 2$ is 24 m s^{-2}.

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Exercise B, Question 2

Question:

A particle P is moving on the x-axis. At time t seconds, the velocity of P is $(8+2t-3t^2)$ m s^{-1} in the direction of x increasing. At time $t=0$, P is at the point where $x=4$. Find

a the magnitude of the acceleration of P when $t=3$,

b the distance of P from O when $t=1$.

Solution:

a

$$v = 8+2t-3t^2$$
$$a = \frac{dv}{dt} = 2-6t$$

When $t=3$,

$2-6\times 3 = -16$

The magnitude of the acceleration of P when $t=3$ is 16 m s^{-2}.

b

$$x = \int v\,dt$$
$$= 8t+t^2-t^3+c, \text{ where } c \text{ is a constant of integration.}$$

When $t=0, x=4$

$$4 = 0+0-0+c \Rightarrow c=4$$
$$x = 4+8t+t^2-t^3$$

When $t=1$,

$$x = 4+8+1-1=12$$

The distance of P from O when $t=1$ is 12 m.

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Exercise B, Question 3

Question:

A particle P is moving on the x-axis. At time t seconds, the acceleration of P is $(16-2t)$ m s^{-2} in the direction of x increasing. The velocity of P at time t seconds is v m s^{-1}.

When $t=0, v=6$ and when $t=3, x=75$. Find

a v in terms of t,

b the value of x when $t=0$.

Solution:

a

$v=\int a\,\mathrm{d}t$

$=16t-t^2+c$, where c is a constant of integration.

When $t=0, v=6$

$6=0-0+c \Rightarrow c=6$

$v=6+16t-t^2$

b

$x=\int v\,\mathrm{d}t$

$=6t+8t^2-\frac{t^3}{3}+k$, where k is a constant of integration.

When $t=3, x=75$

$75=6\times3+8\times9-\frac{27}{3}+k$

$k=75-18-72+9=-6$

$x=6t+8t^2-\frac{t^3}{3}-6$

When $t=0$,

$x=0+0-0-6=-6$

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Exercise B, Question 4

Question:

A particle P is moving on the x-axis. At time t seconds (where $t \geq 0$), the velocity of P is v m s^{-1} in the direction of x increasing, where $v = 12 - t - t^2$.

Find the acceleration of P when P is instantaneously at rest.

Solution:

P is at rest when $v = 0$

$$0 = 12 - t - t^2$$

$$t^2 + t - 12 = (t+4)(t-3) = 0$$

$$t = -4, 3$$

As $t \geq 0, t = -4$ is rejected.

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -1 - 2t$$

When $t = 3$,

$$a = -1 - 2 \times 3 = -7$$

The acceleration of P when P comes to instantaneously to rest is 7 m s^{-2} in the direction of x decreasing.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 5

Question:

A particle is moving in a straight line. At time t seconds, its displacement, x m, from a fixed point O on the line is given by $x = 4t^3 - 39t^2 + 120t$.

Find the distance between the two points where P is instantaneously at rest.

Solution:

$x = 4t^3 - 39t^2 + 120t$

$v = \frac{dx}{dt} = 12t^2 - 78t + 120$

P is at rest when $v = 0$

$12t^2 - 78t + 120 = 6(2t^2 - 13t + 20) = 6(2t - 5)(t - 4) = 0$

$t = 2.5, 4$

When $t = 2.5$,

$x = 4(2.5)^3 - 39(2.5)^2 + 120 \times 2.5 = 118.75$

When $t = 4$,

$x = 4(4)^3 - 39(4)^2 + 120 \times 4 = 112$

The distance between the two points where P is instantaneously at rest is

$(118.75 - 112)\text{m} = 6.75\text{ m}.$

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 6

Question:

At time t seconds, where $t \geq 0$, the velocity v m s^{-1} of a particle moving in a straight line is given by $v = 12 + t - 6t^2$. When $t = 0$, P is at a point O on the line. Find

a the magnitude of the acceleration of P when $v = 0$,

b the distance of P from O when $v = 0$.

Solution:

a When $v = 0$,

$$12 + t - 6t^2 = 0$$
$$6t^2 - t - 12 = (2t - 3)(3t + 4) = 0$$
$$t = \frac{3}{2}, -\frac{4}{3}$$

As $t \geq 0, t = -\frac{4}{3}$ is rejected.

$$a = \frac{dv}{dt} = 1 - 12t$$

When $t = \frac{3}{2}$,

$$a = 1 - 12 \times \frac{3}{2} = -17$$

The magnitude of the acceleration of P when $v = 0$ is 17 m s^{-2}.

b

$$x = \int v \, dt$$
$$= 12t + \frac{1}{2}t^2 - \frac{6}{3}t^3 + c, \text{ where } c \text{ is a constant of integration.}$$

When $t = 0, x = 0$

$$0 = 0 + 0 - 0 + c \Rightarrow c = 0$$

When $t = \frac{3}{2}$,

$$x = 12 \times 1.5 + \frac{1.5^2}{2} - 2 \times 1.5^3 = 12.375$$

The distance of P from O when $v = 0$ is 12.375 m.

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Edexcel AS and A Level Modular Mathematics

Exercise B, Question 7

Question:

A particle P is moving on the x-axis. At time t seconds, the velocity of P is $(4t - t^2)$ m s^{-1} in the direction of x increasing. At time $t = 0$, P is at the origin O. Find

a the value of x at the instant when $t > 0$ and P is at rest,

b the total distance moved by P in the interval $0 \le t \le 5$.

Solution:

a P is at rest when $v = 0$

$$v = 4t - t^2 = 0$$
$$t(4 - t) = 0$$

As $t > 0$, $t = 4$

$$x = \int v\,dt$$
$$= 2t^2 - \frac{1}{3}t^3 + c$$

When $t = 0, x = 0$

$$0 = 0 - 0 + c = 0 \Rightarrow c = 0$$
$$x = 2t^2 - \frac{1}{3}t^3$$

When $t = 4$

$$x = 2 \times 4^2 - \frac{4^3}{3} = 10\frac{2}{3}$$

b When $t = 5$,

$$x = 2 \times 5^2 - \frac{5^3}{3} = 8\frac{1}{3}$$

In the interval $0 \le t \le 5$, moves to a point $10\frac{2}{3}$ m from O and then returns to a point $8\frac{1}{3}$ m from O.

The total distance moved is $10\frac{2}{3} + \left(10\frac{2}{3} - 8\frac{1}{3}\right) = 13$ m.

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Exercise B, Question 8

Question:

A particle P is moving on the x-axis. At time t seconds, the velocity of P is $(6t^2 - 26t + 15)$ m s^{-1} in the direction of x increasing. At time $t = 0$, P is at the origin O. In the subsequent motion P passes through O twice. Find

a the two non-zero values of t when P passes through O,

b the acceleration of P for these two values of t.

Solution:

a

$x = \int v \, dt$

$= 2t^3 - 13t^2 + 15t + c$, where c is a constant of integration.

When $t = 0, x = 0$

$0 = 0 - 0 + 0 + c \Rightarrow c = 0$

$x = 2t^3 - 13t^2 + 15t = t(2t - 3)(t - 5)$

When $x = 0$ and t is non-zero

$t = \frac{3}{2}, 5$

b

$$a = \frac{dv}{dt} = 12t - 26$$

When $t = \frac{3}{2}, a = 12 \times \frac{3}{2} - 26 = -8$

The acceleration of P is 8 m s^{-2} in the direction of x decreasing.

When $t = 5, a = 12 \times 5 - 26 = 34$

Then acceleration of P is 34 m s^{-2} in the direction of x increasing.

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Exercise B, Question 9

Question:

A particle P of mass 0.4 kg is moving in a straight line under the action of a single variable force $\mathbf{F}$ newtons. At time t seconds (where $t \geq 0$) the displacement x m of P from a fixed point O is given by $x = 2t + \frac{k}{t+1}$, where k is a constant. Given that when $t = 0$, the velocity of P is 6 m s^{-1}, find

a the value of k,

b the distance of P from O when $t = 0$,

c the magnitude of $\mathbf{F}$ when $t = 3$.

Solution:

a

$$x = 2t + k(t+1)^{-1}$$

$$v = \frac{dx}{dt} = 2 - k(t+1)^{-2} = 2 - \frac{k}{(t+1)^2}$$

When $t = 0, v = 6$

$$6 = 2 - \frac{k}{1^2} \Rightarrow k = -4$$

b With $k = -4$,

$$x = 2t - \frac{4}{t+1}$$

When $t = 0$,

$$x = 0 - \frac{4}{0+1} = -4$$

The distance of P from O when $t = 0$ is 4 m.

c

$$v = 2 - 4(t+1)^{-2}$$

$$a = \frac{dv}{dt} = 8(t+1)^{-3} = \frac{8}{(t+1)^3}$$

When $t = 3$

$$a = \frac{8}{4^3} = \frac{1}{8}$$

$$F = ma$$

$$= 0.4 \times \frac{1}{8} = 0.05$$

The magnitude of $\mathbf{F}$ when $t = 3$ is 0.05.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 10

Question:

A particle P moves along the x-axis. At time t seconds (where $t \geq 0$) the velocity of P is $(3t^2 - 12t + 5)$ m s^{-1} in the direction of x increasing. When $t = 0$, P is at the origin O. Find

a the velocity of P when its acceleration is zero,

b the values of t when P is again at O,

c the distance travelled by P in the interval $3 \leq t \leq 4$.

Solution:

a $a = \frac{dv}{dt} = 6t - 12 = 0 \Rightarrow t = 2$

When $t = 2$,

$v = 3 \times 2^2 - 12 \times 2 + 5 = -7$

The velocity of P when the acceleration is zero is 7 m s^{-1} in the direction of x decreasing.

b

$$s = \int (3t^2 - 12t + 5)\,dt$$
$$= t^3 - 6t^2 + 5t + C$$

When $t = 0, s = 0$

$0 = 0 - 0 + 0 + C \Rightarrow C = 0$

$s = t^3 - 6t^2 + 5t$

P returns to O when $s = 0$

$s = t^3 - 6t^2 + 5t = t(t-1)(t-5) = 0$

$t = 1, 5$

c When $t = 3, s = 3^3 - 6 \times 3^2 + 5 \times 3 = -12$

When $t = 4, s = 4^3 - 6 \times 4^2 + 5 \times 4 = -60$

The distance travelled by P in the interval $3 \leq t \leq 4$ is 48 m.

(The solutions of $v = 3t^2 - 12t + 5 = 0$ are approximately 7.79 and 0.21, so P does not turn round in the interval.)

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Exercise B, Question 11

Question:

A particle P moves in a straight line so that, at time t seconds, its velocity v m s^{-1} is given by

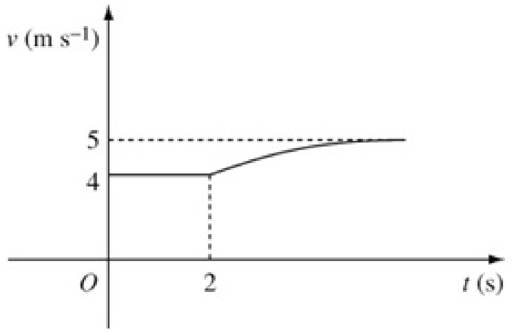
$$v = \begin{cases} 4, & 0 \le t \le 2 \\ 5 - \dfrac{4}{t^2}, & t > 2. \end{cases}$$

a Sketch a velocity-time graph to illustrate the motion of P.

b Find the distance moved by P in the interval $0 \le t \le 5$.

Solution:

a



b In the first two seconds P moves $2 \times 4 = 8$ m

$$s = \int v\,dt = \int (5 - 4t^{-2})\,dt$$

$$= 5t - \frac{4t^{-2}}{-1} + C = 5t + \frac{4}{t} + C$$

When $t = 2, s = 8$

$$8 = 5 \times 2 + \frac{4}{2} + C = 12 + C \Rightarrow C = -4$$

$$s = 5t + \frac{4}{t} - 4$$

When $t = 5$,

$$s = 5 \times 5 + \frac{4}{5} - 4 = 21.8$$

In the interval $0 \le t \le 5$, P moves 21.8 m.

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Exercise B, Question 12

Question:

A particle P moves in a straight line so that, at time t seconds, its acceleration, a m s^{-2}, is given by

$$a = \begin{cases} 6t - t^2, & 0 \le t \le 2 \\ 8 - t, & t > 2. \end{cases}$$

When $t = 0$ the particle is at rest at a fixed point O on the line. Find

a the speed of P when $t = 2$,

b the speed of P when $t = 4$,

c the distance from O to P when $t = 4$.

Solution:

a For $0 \le t \le 2$

$v = \int a \, dt = \int (6t - t^2) dt$

$= 3t^2 - \frac{1}{3}t^3 + c$, where c is a constant of integration.

When $t = 0, v = 0$

$0 = 0 - 0 + c \Rightarrow c = 0$

$v = 3t^2 - \frac{1}{3}t^3$

When $t = 2$,

$v = 3 \times 2^2 - \frac{2^3}{3} = \frac{28}{3}$

The speed of P when $t = 2$ is $\frac{28}{3}$ m s^{-1}.

b For $t > 2$,

$v = \int a dt = \int (8 - t) dt$

$= 8t - \frac{1}{2}t^2 + k$, where k is a constant of integration.

From **a**, when $t = 2, v = \frac{28}{3}$

$\frac{28}{3} = 16 - \frac{4}{2} + k \Rightarrow k = -\frac{14}{3}$

$v = 8t - \frac{1}{2}t^2 - \frac{14}{3}$

When $t = 4$,

$v = 32 - 8 - \frac{14}{3} = \frac{58}{3}$

The speed of P when $t = 4$ is $\frac{58}{3}$ m s^{-1}.

c For $0 \le t \le 2$,

$x = \int v\,dt = \int \left(3t^2 - \frac{1}{3}t^3\right) dt = t^3 - \frac{1}{12}t^4 + l$, where l is a constant of integration.

When $t = 0, x = 0$

$0 = 0 - 0 + l \Rightarrow l = 0$

When $t = 2$,

$x = 2^3 - \frac{2^4}{12} = \frac{20}{3}$ (1)

For $t > 2$,

$x = \int v\,dt = \int \left(8t - \frac{1}{2}t^2 - \frac{14}{3}\right) dt = 4t^2 - \frac{1}{6}t^3 - \frac{14}{3}t + m$,

where m is a constant of integration.

From (1) above

When $t = 2, x = \frac{20}{3}$

$\frac{20}{3} = 16 - \frac{8}{6} - \frac{28}{3} + m \Rightarrow m = \frac{4}{3}$

$x = 4t^2 - \frac{1}{6}t^3 - \frac{14}{3}t + \frac{4}{3}$

When $t = 4$.

$x = 64 - \frac{64}{6} - \frac{56}{3} + \frac{4}{3} = 36$

The distance from O to P when $t = 4$ is 36 m.

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Exercise C, Question 1

Question:

At time t seconds, a particle P has position vector $\mathbf{r}$ m with respect to a fixed origin O, where

$$\mathbf{r} = (3t - 4)\mathbf{i} + (t^3 - 4t)\mathbf{j}.$$

Find

a the velocity of P when $t = 3$,

b the acceleration of P when $t = 3$.

Solution:

a $\mathbf{v} = \dot{\mathbf{r}} = 3\mathbf{i} + (3t^2 - 4)\mathbf{j}$

When $t = 3$,

$\mathbf{v} = 3\mathbf{i} + 23\mathbf{j}$

The velocity of P when $t = 3$ is $(3\mathbf{i} + 23\mathbf{j})$ m s^{-1}.

b $\mathbf{a} = \dot{\mathbf{v}} = 6t\mathbf{j}$

When $t = 3$,

$\mathbf{a} = 18\mathbf{j}$

The acceleration of P when $t = 3$ is $18\mathbf{j}$ m s^{-2}.

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Exercise C, Question 2

Question:

A particle P is moving in a plane with velocity $\mathbf{v}$ m s^{-1} at time t seconds where

$$\mathbf{v} = t^2\mathbf{i} + (2t-3)\mathbf{j}.$$

When $t = 0$, P has position vector $(3\mathbf{i} + 4\mathbf{j})$ m with respect to a fixed origin O. Find

a the acceleration of P at time t seconds,

b the position vector of P when $t = 1$.

Solution:

a $\mathbf{a} = \dot{\mathbf{v}} = 2t\mathbf{i} + 2\mathbf{j}$

The acceleration of P at time t seconds $(2t\mathbf{i} + 2\mathbf{j})$ m s^{-2}.

b

$$\mathbf{r} = \int \mathbf{v}\, dt = \int (t^2\mathbf{i} + (2t-3)\mathbf{j})\, dt$$

$$= \frac{t^3}{3}\mathbf{i} + (t^2 - 3t)\mathbf{j} + \mathrm{C}$$

When $t = 0, \mathbf{r} = 3\mathbf{i} + 4\mathbf{j}$

$3\mathbf{i} + 4\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathrm{C} \Rightarrow \mathrm{C} = 3\mathbf{i} + 4\mathbf{j}$

Hence

$$\mathbf{r} = \left(\frac{t^3}{3} + 3\right)\mathbf{i} + (t^2 - 3t + 4)\mathbf{j}$$

When $t = 1$

$$\mathbf{r} = 3\tfrac{1}{3}\mathbf{i} + 2\mathbf{j}$$

The position vector of P when $t = 1$ is $\left(3\frac{1}{3}\mathbf{i} + 2\mathbf{j}\right)$m .

Solutionbank M2

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Exercise C, Question 3

Question:

A particle P starts from rest at a fixed origin O. The acceleration of P at time t seconds (where $t \geq 0$) is $(6t^2\mathbf{i} + (8-4t^3)\mathbf{j})$ m s^{-2}. Find

a the velocity of P when $t = 2$,

b the position vector of P when $t = 4$.

Solution:

a

$$\mathbf{v} = \int \mathbf{a}\,dt = \int (6t^2\mathbf{i} + (8-4t^3)\mathbf{j})\,dt$$
$$= 2t^3\mathbf{i} + (8t - t^4)\mathbf{j} + \mathbf{C}$$

When $t = 0, \mathbf{v} = 0\mathbf{i} + 0\mathbf{j}$

$$0\mathbf{i} + 0\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = 0\mathbf{i} + 0\mathbf{j}$$

Hence

$$\mathbf{v} = 2t^3\mathbf{i} + (8t - t^4)\mathbf{j}$$

When $t = 2$

$$\mathbf{v} = 16\mathbf{i} + (8 \times 2 - 2^4)\mathbf{j} = 16\mathbf{i}$$

The velocity of P when $t = 2$ is $16\mathbf{i}$ m s^{-1}.

b

$$\mathbf{r} = \int \mathbf{v}\,dt = \int (2t^3\mathbf{i} + (8t - t^4)\mathbf{j})\,dt$$
$$= \frac{1}{2}t^4\mathbf{i} + \left(4t^2 - \frac{1}{5}t^5\right)\mathbf{j} + \mathbf{D}$$

When $t = 0, \mathbf{r} = 0\mathbf{i} + 0\mathbf{j}$

$$0\mathbf{i} + 0\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{D} \Rightarrow \mathbf{D} = 0\mathbf{i} + 0\mathbf{j}$$

Hence

$$\mathbf{r} = \frac{t^4}{2}\mathbf{i} + \left(4t^2 - \frac{t^5}{5}\right)\mathbf{j}$$

When $t = 4$

$$\mathbf{r} = \frac{4^4}{2}\mathbf{i} + \left(4 \times 4^2 - \frac{4^5}{5}\right)\mathbf{j} = 128\mathbf{i} - 104.8\mathbf{j}$$

The position vector of P when $t = 4$ is $(128\mathbf{i} - 104.8\mathbf{j})$m.

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Exercise C, Question 4

Question:

At time t seconds, a particle P has position vector $\mathbf{r}$ m with respect to a fixed origin O, where

$$\mathbf{r} = 4t^2\mathbf{i} + (24t - 3t^2)\mathbf{j}.$$

a Find the speed of P when $t = 2$.

b Show that the acceleration of P is a constant and find the magnitude of this acceleration.

Solution:

a $\mathbf{v} = \dot{\mathbf{r}} = 8t\mathbf{i} + (24 - 6t)\mathbf{j}$

When $t = 2$

$\mathbf{v} = 16\mathbf{i} + 12\mathbf{j}$

$|\mathbf{v}|^2 = 16^2 + 12^2 = 400 \Rightarrow |\mathbf{v}| = \sqrt{400} = 20$

The speed of P when $t = 2$ is 20 m s^{-1}.

b $\mathbf{a} = \dot{\mathbf{v}} = 8\mathbf{i} - 6\mathbf{j}$

As there is no t in this expression, the acceleration is a constant.

$|\mathbf{a}|^2 = 8^2 + (-6)^2 = 100 \Rightarrow |\mathbf{a}| = \sqrt{100} = 10$

The magnitude of the acceleration is 10 m s^{-2}.

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Exercise C, Question 5

Question:

A particle P is initially at a fixed origin O. At time $t = 0$, P is projected from O and moves so that, at time t seconds after projection, its position vector $\mathbf{r}$ m relative to O is given by

$$\mathbf{r} = (t^3 - 12t)\mathbf{i} + (4t^2 - 6t)\mathbf{j},\ t \geq 0.$$

Find

a the speed of projection of P,

b the value of t at the instant when P is moving parallel to $\mathbf{j}$,

c the position vector of P at the instant when P is moving parallel to $\mathbf{j}$.

Solution:

a $\mathbf{v} = \dot{\mathbf{r}} = (3t^2 - 12)\mathbf{i} + (8t - 6)\mathbf{j}$

When $t = 0$,

$\mathbf{v} = -12\mathbf{i} - 6\mathbf{j}$

$|\mathbf{v}|^2 = (-12)^2 + (-6)^2 = 180 \Rightarrow |\mathbf{v}| = \sqrt{180} = 6\sqrt{5}$

The speed of projection is $6\sqrt{5}$ m s^{-1}.

b When P is moving parallel to $\mathbf{j}$ the velocity has no $\mathbf{i}$ component.

$3t^2 - 12 = 0 \Rightarrow t^2 = 4 \Rightarrow t = 2 \quad (t \geq 0)$

c When $t = 2$

$\mathbf{r} = (2^3 - 12 \times 2)\mathbf{i} + (4 \times 2^2 - 6 \times 2)\mathbf{j} = -16\mathbf{i} + 4\mathbf{j}$

The position vector of P at the instant when P is moving parallel to $\mathbf{j}$ is $(-16\mathbf{i} + 4\mathbf{j})$m .

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Exercise C, Question 6

Question:

At time t seconds, the force $\mathbf{F}$ newtons acting on a particle P, of mass 0.5 kg, is given by

$$\mathbf{F} = 3t\mathbf{i} + (4t - 5)\mathbf{j}.$$

When $t = 1$, the velocity of P is $12\mathbf{i}$ m s^{-1}. Find

a the velocity of P after t seconds,

b the angle the direction of motion of P makes with $\mathbf{i}$ when $t = 5$, giving your answer to the nearest degree.

Solution:

a

$$\begin{aligned}
\mathbf{F} &= m\mathbf{a} \\
3t\mathbf{i} + (4t-5)\mathbf{j} &= 0.5\mathbf{a} \\
\mathbf{a} &= 6t\mathbf{i} + (8t - 10)\mathbf{j} \\
\mathbf{v} &= \int \mathbf{a}\,\mathrm{d}t = \int (6t\mathbf{i} + (8t-10)\mathbf{j})\,\mathrm{d}t \\
&= 3t^2\mathbf{i} + (4t^2 - 10t)\mathbf{j} + \mathbf{C}
\end{aligned}$$

When $t = 1, \mathbf{v} = 12\mathbf{i}$

$$12\mathbf{i} = 3\mathbf{i} - 6\mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = 9\mathbf{i} + 6\mathbf{j}$$

Hence

$$\mathbf{v} = (3t^2 + 9)\mathbf{i} + (4t^2 - 10t + 6)\mathbf{j}$$

When $t = 5$

$$\mathbf{v} = (3 \times 5^2 + 9)\mathbf{i} + (4 \times 5^2 - 10 \times 5 + 6)\mathbf{j} = 84\mathbf{i} + 56\mathbf{j}$$

b The angle $\mathbf{v}$ makes with $\mathbf{i}$ is given by

$$\tan\theta = \frac{56}{84} \Rightarrow \theta \approx 34^\circ$$

The angle the direction of motion of P makes with $\mathbf{i}$ when $t = 5$ is 34° (nearest degree).

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 7

Question:

A particle P is moving in a plane with velocity $\mathbf{v}$ m s^{-1} at time t seconds where

$$\mathbf{v} = (3t^2 + 2)\mathbf{i} + (6t - 4)\mathbf{j}.$$

When $t = 2$, P has position vector $9\mathbf{j}$ m with respect to a fixed origin O. Find

a the distance of P from O when $t = 0$,

b the acceleration of P at the instant when it is moving parallel to the vector $\mathbf{i}$.

Solution:

a

$$\mathbf{r} = \int \mathbf{v}\,dt = \int ((3t^2 + 2)\mathbf{i} + (6t - 4)\mathbf{j})dt$$
$$= (t^3 + 2t)\mathbf{i} + (3t^2 - 4t)\mathbf{j} + \mathbf{A}$$

When $t = 2, \mathbf{v} = 9\mathbf{j}$

$$9\mathbf{j} = 12\mathbf{i} + 4\mathbf{j} + \mathbf{A} \Rightarrow \mathbf{A} = -12\mathbf{i} + 5\mathbf{j}$$

Hence

$$\mathbf{r} = (t^3 + 2t - 12)\mathbf{i} + (3t^2 - 4t + 5)\mathbf{j}$$

When $t = 0$,

$$\mathbf{r} \quad = -12\mathbf{i} + 5\mathbf{j}$$

$$|\mathbf{r}|^2 = (-12)^2 + 5^2 = 169 \Rightarrow |\mathbf{r}| = \sqrt{169} = 13$$

The distance of P from O when $t = 0$ is 13 m.

b When P is moving parallel to $\mathbf{i}$, $\mathbf{v}$ has no $\mathbf{j}$ component.

$$6t - 4 = 0 \Rightarrow t = \frac{2}{3}$$
$$\mathbf{a} = \dot{\mathbf{v}} = 6t\mathbf{i} + 6\mathbf{j}$$

When $t = \frac{2}{3}$,

$$\mathbf{a} = 4\mathbf{i} + 6\mathbf{j}$$

The acceleration of P at the instant when it is moving parallel to the vector $\mathbf{i}$ is $(4\mathbf{i} + 6\mathbf{j})$ m s^{-2}.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 8

Question:

At time t seconds, the particle P is moving in a plane with velocity $\mathbf{v}$ m s^{-1} and acceleration $\mathbf{a}$ m s^{-2}, where

$$\mathbf{a} = (2t - 4)\mathbf{i} + 6\mathbf{j}.$$

Given that P is instantaneously at rest when $t = 4$, find

a $\mathbf{v}$ in terms of t,

b the speed of P when $t = 5$.

Solution:

a $\mathbf{v} = \int \mathbf{a}\, dt = \int ((2t - 4)\mathbf{i} + 6\mathbf{j})\, dt = (t^2 - 4t)\mathbf{i} + 6t\mathbf{j} + \mathbf{C}$

When $t = 4$, $\mathbf{v} = 0\mathbf{i} + 0\mathbf{j}$

$0\mathbf{i} + 0\mathbf{j} = (4^2 - 4 \times 4)\mathbf{i} + 6 \times 4\mathbf{j} + \mathbf{C} = 24\mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = -24\mathbf{j}$

Hence

$\mathbf{v} = (t^2 - 4t)\mathbf{i} + (6t - 24)\mathbf{j}$

b When $t = 5$

$\mathbf{v} = 5\mathbf{i} + 6\mathbf{j}$

$|\mathbf{v}|^2 = 5^2 + 6^2 = 61 \Rightarrow |\mathbf{v}| = \sqrt{61} \approx 7.81$

The speed of P when $t = 5$ is 7.81 m s^{-1} (3 s.f.).

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 9

Question:

A particle P is moving in a plane. At time t seconds, the position vector of P, $\mathbf{r}$ m, is given by

$\mathbf{r} = (3t^2 - 6t + 4)\mathbf{i} + (t^3 + kt^2)\mathbf{j}$, where k is a constant.

When $t = 3$, the speed of P is $12\sqrt{5}$ m s^{-1}.

a Find the two possible values of k.

b For both of these values of k, find the magnitude of the acceleration of P when $t = 1.5$.

Solution:

a $\mathbf{v} = \dot{\mathbf{r}} = (6t - 6)\mathbf{i} + (3t^2 + 2kt)\mathbf{j}$

When $t = 3$

$\mathbf{v} = 12\mathbf{i} + (27 + 6k)\mathbf{j}$

$|\mathbf{v}|^2 = 12^2 + (27 + 6k)^2 = (12\sqrt{5})^2$

$144 + 729 + 324k + 36k^2 = 720$

$36k^2 + 324k + 153 = 0$

$(\div 9)$

$4k^2 + 36k + 17 = (2k + 1)(2k + 17) = 0$

$k = -0.5, -8.5$

b If $k = -0.5$

$\mathbf{v} = (6t - 6)\mathbf{i} + (3t^2 - t)\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 6\mathbf{i} + (6t - 1)\mathbf{j}$

When $t = 1.5$

$\mathbf{a} = 6\mathbf{i} + 8\mathbf{j}$

$|\mathbf{a}|^2 = 6^2 + 8^2 = 100 \Rightarrow |\mathbf{a}| = 10$

If $k = -8.5$

$\mathbf{v} = (6t - 6)\mathbf{i} + (3t^2 - 17t)\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 6\mathbf{i} + (6t - 17)\mathbf{j}$

When $t = 1.5$

$\mathbf{a} = 6\mathbf{i} - 8\mathbf{j}$

$|\mathbf{a}|^2 = 6^2 + (-8)^2 = 100 \Rightarrow |\mathbf{a}| = 10$

For both of the values of k the magnitude of the acceleration of P when $t = 1.5$ is 10 m s^{-2}.

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Exercise C, Question 10

Question:

At time t seconds (where $t \geq 0$), the particle P is moving in a plane with acceleration $\mathbf{a}$ m s^{-2}, where

$$\mathbf{a} = (5t - 3)\mathbf{i} + (8 - t)\mathbf{j}$$

When $t = 0$, the velocity of P is $(2\mathbf{i} - 5\mathbf{j})$ m s^{-1}. Find

a the velocity of P after t seconds,

b the value of t for which P is moving parallel to $\mathbf{i} - \mathbf{j}$,

c the speed of P when it is moving parallel to $\mathbf{i} - \mathbf{j}$.

Solution:

a

$$\mathbf{v} = \int \mathbf{a}\, dt = \int ((5t-3)\mathbf{i} + (8-t)\mathbf{j})dt$$

$$= \left(\frac{5}{2}t^2 - 3t\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2\right)\mathbf{j} + \mathbf{C}$$

When $t = 0, \mathbf{v} = 2\mathbf{i} - 5\mathbf{j}$

$2\mathbf{i} - 5\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = 2\mathbf{i} - 5\mathbf{j}$

Hence

$$\mathbf{v} = \left(\frac{5}{2}t^2 - 3t + 2\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2 - 5\right)\mathbf{j}$$

The velocity of P after t seconds is $\left(\left(\frac{5}{2}t^2 - 3t + 2\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2 - 5\right)\mathbf{j}\right)$ m s^{-1}.

b The gradients of $\mathbf{v}$ and $\mathbf{i} - \mathbf{j}$ are equal

$$\frac{8t - \frac{1}{2}t^2 - 5}{\frac{5}{2}t^2 - 3t + 2} = 1$$

$$8t - \frac{1}{2}t^2 - 5 = -\frac{5}{2}t^2 + 3t - 2$$

$$2t^2 + 5t - 3 = (2t - 1)(t + 3) = 0$$

$$t = \frac{1}{2}, -3$$

As $t \geq 0, t = \frac{1}{2}$

c When $t = \frac{1}{2}$

$$\mathbf{v} = \left(\frac{5}{8} - \frac{3}{2} + 2\right)\mathbf{i} + \left(4 - \frac{1}{8} - 5\right)\mathbf{j} = \frac{9}{8}\mathbf{i} - \frac{9}{8}\mathbf{j}$$

$$|\mathbf{v}|^2 = \left(\frac{9}{8}\right)^2 + \left(-\frac{9}{8}\right)^2 = 2 \times \left(\frac{9}{8}\right)^2 \Rightarrow |\mathbf{v}| = \frac{9\sqrt{2}}{8}$$

The speed of P when it is moving parallel to $\mathbf{i} - \mathbf{j}$ is $\frac{9\sqrt{2}}{8}$ m s^{-1}.

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Edexcel AS and A Level Modular Mathematics

Exercise C, Question 11

Question:

At time t seconds (where $t \geq 0$), a particle P is moving in a plane with acceleration $(2\mathbf{i} - 2t\mathbf{j})$ m s^{-2}. When $t = 0$, the velocity of P is $2\mathbf{j}$ m s^{-1} and the position vector of P is $6\mathbf{i}$ m with respect to a fixed origin P.

a Find the position vector of P at time t seconds.

At time t seconds (where $t \geq 0$), a second particle Q is moving in the plane with velocity $((3t^2 - 4)\mathbf{i} - 2t\mathbf{j})$ m s^{-1}. The particles collide when $t = 3$.

b Find the position vector of Q at time $t = 0$.

Solution:

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Exercise C, Question 12

Question:

A particle P of mass 0.2 kg is at rest at a fixed origin O. At time t seconds, where $0 \le t \le 3$, a force $(2t\mathbf{i} + 3\mathbf{j})$ N is applied to P.

a Find the position vector of P when $t = 3$.

When $t = 3$, the force acting on P changes to $(6\mathbf{i} + (12 - t^2)\mathbf{j})$ N, where $t \ge 3$.

b Find the velocity of P when $t = 6$.

Solution:

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Exercise D, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8\text{ m s}^{-2}$.

A particle P is projected from a point O on a horizontal plane with speed 42 m s^{-1} and with angle of elevation $45°$. After projection, the particle moves freely under gravity until it strikes the plane. Find

a the greatest height above the plane reached by P,

b the time of flight of P.

Solution:

a Resolving the initial velocity vertically

R(↑) $u_y = 42\sin 45° = 21\sqrt{2}$

R(↑) $u = 21\sqrt{2}, v = 0, a = -9.8, s = ?$

$$v^2 = u^2 + 2as$$

$$0^2 = (21\sqrt{2})^2 - 2 \times 9.8 \times s$$

$$s = \frac{(21\sqrt{2})^2}{2 \times 9.8} = \frac{882}{19.6} = 45$$

The greatest height above the plane reached by P is 45 m.

b

R(↑) $s = 0, u = 21\sqrt{2}, a = -9.8, t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = 21\sqrt{2}t - 4.9t^2$$

$t \neq 0$

$$t = \frac{21\sqrt{2}}{4.9} = 6.0609\ldots$$

The time of flight of P is 6.1 s (2 s.f.).

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Exercise D, Question 2

Question:

A stone is thrown horizontally with speed 21 m s^{-1} from a point P on the edge of a cliff h metres above sea level. The stone lands in the sea at a point Q, where the horizontal distance of Q from the cliff is 56 m.

Calculate the value of h.

Solution:

Resolving the initial velocity horizontally and vertically

$$\text{R}(\rightarrow) \quad u_x = 21$$
$$\text{R}(\downarrow) \quad u_y = 0$$

$\text{R}(\rightarrow)$ distance = speed × time

$$56 = 21 \times t \Rightarrow t = \frac{56}{21} = \frac{8}{3}$$

$$\text{R}(\downarrow) \quad s = h, u = 0, a = 9.8, t = \frac{8}{3}$$

$$s = ut + \frac{1}{2}at^2$$

$$h = 0 + 4.9 \times \left(\frac{8}{3}\right)^2 = 34.844\ldots$$

$$h = 35 \text{ (2 s.f.)}$$

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Exercise D, Question 3

Question:

A particle P moves in a horizontal straight line. At time t seconds (where $t \geq 0$) the velocity v m s^{-1} of P is given by $v = 15 - 3t$. Find

a the value of t when P is instantaneously at rest,

b the distance travelled by P between the time when $t = 0$ and the time when P is instantaneously at rest.

Solution:

a $v = 15 - 3t$

When P is at rest, $v = 0$

$0 = 15 - 3t \Rightarrow t = 5$

b

$$s = \int v \, dt = \int (15 - 3t) \, dt$$

$$= 15t - \frac{3}{2}t^2 + c$$

Let $s = 0$, when $t = 0$

$0 = 0 - 0 + c \Rightarrow c = 0$

$$s = 15t - \frac{3}{2}t^2$$

When $t = 5$

$$s = 15 \times 5 - \frac{3}{2}5^2 = 37.5$$

The distance travelled by P between the time when $t = 0$ and the time when P is instantaneously at rest is 37.5 m.

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Exercise D, Question 4

Question:

A particle P moves along the x-axis so that, at time t seconds, the displacement of P from O is x metres and the velocity of P is v m s^{-1}, where

$$v = 6t + \frac{1}{2}t^3.$$

a Find the acceleration of P when $t = 4$.

b Given also that $x = -5$ when $t = 0$, find the distance OP when $t = 4$.

Solution:

a $a = \frac{dv}{dt} = 6 + \frac{3}{2}t^2$

When $t = 4$

$$a = 6 + \frac{3}{2}4^2 = 30$$

The acceleration of P when $t = 4$ is 30 m s^{-2}.

b

$$x = \int v \, dt = \int \left(6t + \frac{1}{2}t^3\right) dt$$
$$= 3t^2 + \frac{1}{8}t^4 + c$$

When $t = 0, x = -5$

$$-5 = 0 + 0 + c \Rightarrow c = -5$$
$$x = 3t^2 + \frac{1}{8}t^4 - 5$$

When $t = 4$

$$x = 3 \times 4^2 + \frac{4^4}{8} - 5 = 75$$
$$OP = 75 \text{ m}$$

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Exercise D, Question 5

Question:

At time t seconds, a particle P has position vector $\mathbf{r}$ m with respect to a fixed origin O, where

$$\mathbf{r} = (3t^2 - 4)\mathbf{i} + (8 - 4t^2)\mathbf{j}.$$

a Show that the acceleration of P is a constant.

b Find the magnitude of the acceleration of P and the size of the angle which the acceleration makes with $\mathbf{j}$.

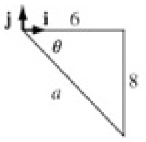
Solution:

a

$$\mathbf{v} = \dot{\mathbf{r}} = 6t\mathbf{i} - 8t\mathbf{j}$$
$$\mathbf{a} = \dot{\mathbf{v}} = 6\mathbf{i} - 8\mathbf{j}$$

Acceleration does not depend on t, hence the acceleration is a constant.

b



$$|\mathbf{a}| = 6^2 + (-8)^2 = 100 \Rightarrow |\mathbf{a}| = 10$$

The magnitude of the acceleration is 10 m s^{-2}.

$$\tan\theta = \frac{8}{6} \Rightarrow \theta \approx 53.1°$$

The angle the acceleration makes with $\mathbf{j}$ is $90° + 53.1° = 143.1°$ (nearest $0.1°$).

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Exercise D, Question 6

Question:

At time $t = 0$ a particle P is at rest at a point with position vector $(4\mathbf{i} - 6\mathbf{j})$ m with respect to a fixed origin O. The acceleration of P at time t seconds (where $t \geq 0$) is $((4t-3)\mathbf{i} - 6t^2\mathbf{j})$ m s^{-2}. Find

a the velocity of P when $t = \frac{1}{2}$,

b the position vector of P when $t = 6$.

Solution:

a

$$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int((4t-3)\mathbf{i} - 6t^2\mathbf{j})\,\mathrm{d}t$$
$$= (2t^2 - 3t)\mathbf{i} - 2t^3\mathbf{j} + \mathbf{A}$$

When $t = 0, \mathbf{v} = \mathbf{0}$

$$\mathbf{0} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{A} \Rightarrow \mathbf{A} = \mathbf{0}$$
$$\mathbf{v} = (2t^2 - 3t)\mathbf{i} - 2t^3\mathbf{j}$$

When $t = \frac{1}{2}$

$$\mathbf{v} = \left(2\left(\frac{1}{2}\right)^2 - 3\times\frac{1}{2}\right)\mathbf{i} - 2\left(\frac{1}{2}\right)^3\mathbf{j} = -\mathbf{i} - \frac{1}{4}\mathbf{j}$$

The velocity of P when $t = \frac{1}{2}$ is $\left(-\mathbf{i} - \frac{1}{4}\mathbf{j}\right)$ m s^{-1}.

b

$$\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \int\left(\left(2t^2 - 3t\right)\mathbf{i} - 2t^3\mathbf{j}\right)\mathrm{d}t$$
$$= \left(\frac{2}{3}t^3 - \frac{3}{2}t^2\right)\mathbf{i} - \frac{1}{2}t^4\mathbf{j} + \mathbf{B}$$

When $t = 0, \mathbf{r} = 4\mathbf{i} - 6\mathbf{j}$

$$4\mathbf{i} - 6\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{B} \Rightarrow \mathbf{B} = 4\mathbf{i} - 6\mathbf{j}$$
$$\mathbf{r} = \left(\frac{2}{3}t^3 - \frac{3}{2}t^2 + 4\right)\mathbf{i} - \left(\frac{1}{2}t^4 + 6\right)\mathbf{j}$$

When $t = 6$

$$\mathbf{r} = (144 - 54 + 4)\mathbf{i} - (648 + 6)\mathbf{j} = 94\mathbf{i} - 654\mathbf{j}$$

The position vector of P when $t = 6$ is $(94\mathbf{i} - 654\mathbf{j})$ m.

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Exercise D, Question 7

Question:

A ball is thrown from a window above a horizontal lawn. The velocity of projection is 15 m s^{-1} and the angle of elevation is α, where $\tan \alpha = \frac{4}{3}$. The ball takes 4 s to reach the lawn. Find

a the horizontal distance between the point of projection and the point where the ball hits the lawn,

b the vertical height above the lawn from which the ball was thrown.

Solution:

a $\tan \alpha = \frac{4}{3} \Rightarrow \sin \alpha = \frac{4}{5}, \cos \alpha = \frac{3}{5}$

Resolving the initial velocity horizontally and vertically

$\text{R}(\rightarrow)$ $u_x = 15\cos\alpha = 15 \times \frac{3}{5} = 9$

$\text{R}(\uparrow)$ $u_y = 15\sin\alpha = 15 \times \frac{4}{5} = 12$

$\text{R}(\rightarrow)$ distance $=$ speed $\times$ time

$= 9 \times 4 = 36$

The horizontal distance between the point of projection and the point where the ball hits the lawn is 36 m.

b Let the vertical height above the lawn from which the ball was thrown be h m

$\text{R}(\uparrow)$ $s = -h, u = 12, a = -9.8, t = 4$

$s = ut + \frac{1}{2}at^2$

$-h = 12 \times 4 - 4.9 \times 4^2 = -30.4 \Rightarrow h = 30.4$

The vertical height above the lawn from which the ball was thrown is 30 m (2 s.f.).

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Exercise D, Question 8

Question:

A projectile is fired with velocity 40 m s^{-1} at an angle of elevation of $30°$ from a point A on horizontal ground. The projectile moves freely under gravity until it reaches the ground at the point B. Find

a the distance AB,

b the speed of the projectile at the instants when it is 15 m above the plane.

Solution:

a Resolving the initial velocity horizontally and vertically

$$\text{R}(\rightarrow) \quad u_x = 40\cos 30° = 20\sqrt{3}$$
$$\text{R}(\uparrow) \quad u_y = 20\sin 30° = 10$$
$$\text{R}(\uparrow) \quad s = 0, u = 20, a = -9.8, t = ?$$
$$s = ut + \frac{1}{2}at^2$$
$$0 = 20t - 4.9t^2 = t(20 - 4.9t)$$

$$t \neq 0$$
$$t = \frac{20}{4.9}$$

$$\text{R}(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$$
$$= 20\sqrt{3} \times \frac{20}{4.9} = 141.39\ldots$$
$$AB = 140 \text{ (2 s.f.)}$$

b

$$\text{R}(\uparrow) \quad u = 20, a = -9.8, s = 15, v = v_y$$
$$v^2 = u^2 + 2as$$
$$v_y^2 = 20^2 - 2 \times 9.8 \times 15 = 106$$
$$V^2 = u_x^2 + v_y^2 = (20\sqrt{3})^2 + 106 = 1306$$
$$V = \sqrt{1306} = 36.138\ldots$$

The speed of the projectile at the instants when it is 15 m above the plane is 36 m s^{-1} (2 s.f.).

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Exercise D, Question 9

Question:

At time t seconds, a particle P has position vector $\mathbf{r}$ m with respect to a fixed origin O, where

$$\mathbf{r} = 2\cos 3t\mathbf{i} - 2\sin 3t\mathbf{j}.$$

a Find the velocity of P when $t = \frac{\pi}{6}$.

b Show that the magnitude of the acceleration of P is constant.

Solution:

a $\mathbf{v} = \dot{\mathbf{r}} = -6\sin 3t\mathbf{i} - 6\cos 3t\mathbf{j}$

When $t = \frac{\pi}{6}$

$$\mathbf{v} = \dot{\mathbf{r}} = -6\sin\frac{\pi}{2}\mathbf{i} - 6\cos\frac{\pi}{2}\mathbf{j} = -6\mathbf{i} - 0\mathbf{j}$$

The velocity of P when $t = \frac{\pi}{6}$ is $-6\mathbf{i}$ m s^{-1}.

b

$$\mathbf{a} = \dot{\mathbf{v}} = -18\cos 3t\mathbf{i} + 18\sin 3t\mathbf{j}$$
$$|\mathbf{a}|^2 = (-18\cos 3t)^2 + (18\sin 3t)^2$$
$$= 18^2(\cos^2 3t + \sin^2 3t) = 18^2$$
$$|\mathbf{a}| = 18$$

The magnitude of the acceleration is 18 m s^{-2}, a constant.

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Exercise D, Question 10

Question:

A particle P of mass 0.2 kg is moving in a straight line under the action of a single variable force $\mathbf{F}$ newtons. At time t seconds the displacement, s metres, of P from a fixed point A is given by $s = 3t + 4t^2 - \frac{1}{2}t^3$.

Find the magnitude of $\mathbf{F}$ when $t = 4$.

Solution:

$v = \frac{\mathrm{d}s}{\mathrm{d}t} = 3 + 4t - \frac{3}{2}t^2$

$a = \frac{\mathrm{d}v}{\mathrm{d}t} = 8 - 3t$

When $t = 4$

$a = 8 - 3 \times 4 = -4$

$\mathbf{F} = m\mathbf{a}$

$= 0.2 \times (-4) = -0.8$

The magnitude of $\mathbf{F}$ when $t = 4$ is 0.8 N.

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Exercise D, Question 11

Question:

At time t seconds (where $t \geq 0$) the particle P is moving in a plane with acceleration $\mathbf{a}$ m s^{-2}, where

$$\mathbf{a} = (8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j}.$$

When $t = 2$, the velocity of P is $(16\mathbf{i} + 3\mathbf{j})$ m s^{-1}. Find

a the velocity of P after t seconds,

b the value of t when P is moving parallel to $\mathbf{i}$.

Solution:

a

$$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int \left[(8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j}\right]\mathrm{d}t$$
$$= \left(2t^4 - 3t^2\right)\mathbf{i} + \left(4t^2 - 3t\right)\mathbf{j} + \mathbf{C}$$

When $t = 2, \mathbf{v} = 16\mathbf{i} + 3\mathbf{j}$

$$16\mathbf{i} + 3\mathbf{j} = 20\mathbf{i} + 10\mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = -4\mathbf{i} - 7\mathbf{j}$$
$$\mathbf{v} = (2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j}$$

The velocity of P after t seconds is $\left[\left(2t^4 - 3t^2 - 4\right)\mathbf{i} + \left(4t^2 - 3t - 7\right)\mathbf{j}\right]$ m s^{-1}.

b When P is moving parallel to $\mathbf{i}$, the $\mathbf{j}$ component of the velocity is zero.

$$4t^2 - 3t - 7 = 0$$
$$(t + 1)(4t - 7) = 0$$

$t \geq 0$

$t = \frac{7}{4}$

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Exercise D, Question 12

Question:

A particle of mass 0.5 kg is acted upon by a variable force $\mathbf{F}$. At time t seconds, the velocity $\mathbf{v}$ m s^{-1} is given by

$\mathbf{v} = (4ct - 6)\mathbf{i} + (7 - c)t^2\mathbf{j}$, where c is a constant.

a Show that $\mathbf{F} = [2c\mathbf{i} + (7 - c)t\mathbf{j}]$ N.

b Given that when $t = 5$ the magnitude of $\mathbf{F}$ is 17 N, find the possible values of c.

Solution:

a

$$\mathbf{a} = \dot{\mathbf{v}} = 4c\mathbf{i} + 2(7 - c)t\mathbf{j}$$

$$\mathbf{F} = m\mathbf{a}$$

$$= 0.5\left[4c\mathbf{i} + 2(7 - c)t\mathbf{j}\right] = 2c\mathbf{i} + (7 - c)t\mathbf{j}, \text{ as required}$$

b $t = 5 \Rightarrow \mathbf{F} = 2c\mathbf{i} + (7 - c)5\mathbf{j}$

$$|\mathbf{F}|^2 = 4c^2 + 25(7 - c)^2 = 17^2$$

$$4c^2 + 1225 - 350c + 25c^2 = 289$$

$$29c^2 - 350c + 936 = 0$$

$$(c - 4)(29c - 234) = 0$$

$$c = 4, \frac{234}{29} \approx 8.07$$

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Exercise D, Question 13

Question:

A ball, attached to the end of an elastic string, is moving in a vertical line. The motion of the ball is modelled as a particle B moving along a vertical axis so that its displacement, x m, from a fixed point O on the line at time t seconds is given by $x = 0.6\cos\left(\frac{\pi t}{3}\right)$. Find

a the distance of B from O when $t = \frac{1}{2}$,

b the smallest positive value of t for which B is instantaneously at rest,

c the magnitude of the acceleration of B when $t = 1$. Give your answer to 3 significant figures.

Solution:

a When $t = \frac{1}{2}$

$$x = 0.6\cos\left(\frac{\pi}{3} \times \frac{1}{2}\right) = 0.6\cos\frac{\pi}{6}$$

$$= 0.6 \times \frac{\sqrt{3}}{2} = 0.3\sqrt{3}$$

The distance of B from O when $t = \frac{1}{2}$ is $0.3\sqrt{3}$ m .

b $v = \frac{dx}{dt} = -0.6 \times \frac{\pi}{3}\sin\left(\frac{\pi t}{3}\right)$

The smallest positive value at which $v = 0$ is given by

$$\frac{\pi t}{3} = \frac{\pi}{2} \Rightarrow t = \frac{3}{2}$$

c $a = \frac{dv}{dt} = -0.6\left(\frac{\pi}{3}\right)^2 \cos\left(\frac{\pi t}{3}\right)$

When $t = 1$

$$a = -0.6\left(\frac{\pi}{3}\right)^2 \cos\left(\frac{\pi}{3}\right) = -0.3289\ldots$$

The magnitude of the acceleration of B when $t = 1$ is 0.329 m s^{-2} (3 s.f.)

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Exercise D, Question 14

Question:

A light spot S moves along a straight line on a screen. At time $t = 0$, S is at a point O. At time t seconds (where $t \geq 0$) the distance, x cm, of S from O is given by $x = 4te^{-0.5t}$. Find

a the acceleration of S when $t = \ln 4$,

b the greatest distance of S from O.

Solution:

a $v = \frac{dx}{dt} = 4\,e^{-0.5t} - 2t\,e^{-0.5t}$

$a = \frac{dv}{dt} = -2\,e^{-0.5t} - 2\,e^{-0.5t} + t\,e^{-0.5t} = (t-4)e^{-0.5t}$

When $t = \ln 4$

$a = (\ln 4 - 4)e^{-0.5\ln 4} = (2\ln 2 - 4)e^{\ln\frac{1}{2}}$

$= \frac{1}{2}(2\ln 2 - 4) = \ln 2 - 2$

The acceleration of S when $t = \ln 4$ is $(\ln 2 - 2)$ m s^{-2} in the direction of x increasing.

b For a maximum of x, $\frac{dx}{dt} = v = 0$

$v = (4 - 2t)e^{-0.5t} = 0 \Rightarrow t = 2$

When $t = 2$

$x = 4 \times 2\,e^{-0.5\times 2} = 8\,e^{-1}$

The greatest distance of S from O is $\frac{8}{e}$ m .

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Exercise D, Question 15

Question:

A particle P is projected with velocity $(3u\mathbf{i}+4u\mathbf{j})$ m s^{-1} from a fixed point O on a horizontal plane. Given that P strikes the plane at a point 750 m from O.

a show that $u=17.5$,

b calculate the greatest height above the plane reached by P,

c find the angle the direction of motion of P makes with $\mathbf{i}$ when $t=5$.

Solution:

a Taking components horizontally and vertically

$\text{R}(\rightarrow)\quad u_x = 3u$

$\text{R}(\uparrow)\quad u_y = 4u$

$\text{R}(\rightarrow)\quad$ distance = speed × time

$$750 = 3ut \Rightarrow t = \frac{250}{u}$$

$\text{R}(\uparrow)\quad s = ut + \frac{1}{2}at^2$

$$0 = 4ut - 4.9t^2$$

$$0 = \frac{4u\times 250}{u} - 4.9\left(\frac{250}{u}\right)^2 = 1000 - \frac{306\,250}{u^2}$$

$$u^2 = \frac{306\,250}{1000} = 306.25$$

$$u = \sqrt{306.25} = 17.5, \text{ as required}$$

b

$$u_y = 4u = 4\times 17.5 = 70$$

$\text{R}(\uparrow)\quad v^2 = u^2 + 2as$

$$0^2 = 70^2 - 2\times 9.8\times s$$

$$s = \frac{70^2}{2\times 9.8} = 250$$

The greatest height above the plane reached by P is 250 m.

c When $t=5$

$\text{R}(\uparrow)\quad v = u + at$

$$v_y = 70 - 9.8\times 5 = 21$$

$$\tan\theta = \frac{v_y}{u_x} = \frac{21}{3\times 17.5} = 0.4 \Rightarrow \theta = 21.8^\circ$$

The angle the direction of motion of P makes with $\mathbf{i}$ when $t=5$ is 22° (nearest degree).

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 16

Question:

A particle P is projected from a point on a horizontal plane with speed u at an angle of elevation θ.

a Show that the range of the projectile is $\frac{u^2 \sin 2\theta}{g}$.

b Hence find, as θ varies, the maximum range of the projectile.

c Given that the range of the projectile is $\frac{2u^2}{3g}$, find the two possible value of θ.

Give your answers to $0.1°$.

Solution:

a Taking components horizontally and vertically

$\mathrm{R}(\rightarrow)\quad u_x = u\cos\theta$

$\mathrm{R}(\uparrow)\quad u_y = u\sin\theta$

$\mathrm{R}(\uparrow)\quad s = ut + \frac{1}{2}at^2$

$$0 = u\sin\theta t - \frac{1}{2}gt^2 = t\left(u\sin\theta - \frac{1}{2}gt\right)$$

$t \neq 0$

$$t = \frac{2u\sin\theta}{g}$$

Let the range be R

distance $=$ speed $\times$ time

$$R = u\cos\theta \times \frac{2u\sin\theta}{g} = \frac{2u\sin\theta\cos\theta}{g}$$

Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$

$$R = \frac{u^2\sin 2\theta}{g}$$

b R is a maximum when $\sin 2\theta = 1$, that is when $\theta = 45°$.

The maximum range of the projectile is $\frac{u^2}{g}$.

c If $R = \frac{2u^2}{3g}$, $\frac{2u^2}{3g} = \frac{u^2\sin 2\theta}{g}$

$$\sin 2\theta = \frac{2}{3}$$

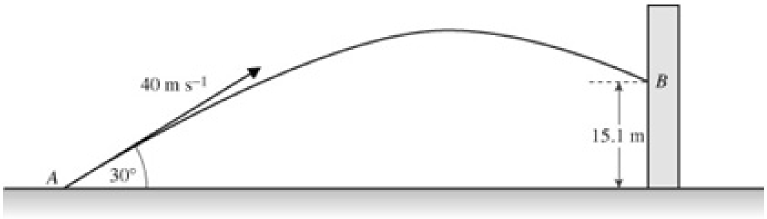
$$2\theta = 41.81°, (180 - 41.81)°$$

$$\theta = 20.9°, 69.1°, (\text{nearest } 0.1°)$$

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 17

Question:



A golf ball is driven from a point A with a speed of 40 m s^{-1} at an angle of elevation of $30°$. On its downward flight, the ball hits an advertising hoarding at a height 15.1 m above the level of A, as shown in the diagram above. Find

a the time taken by the ball to reach its greatest height above A,

b the time taken by the ball to travel from A to B,

c the speed with which the ball hits the hoarding.

Solution:

Taking components horizontally and vertically

$R(\rightarrow) \quad u_x = 40\cos 30^\circ = 20\sqrt{3}$

$R(\uparrow) \quad u_y = 40\sin 30^\circ = 20$

a

$R(\uparrow) \quad v = u + at$

$0 = 20 - 9.8t \Rightarrow t = \frac{20}{9.8} = 2.0408\ldots$

The time taken by the ball to reach its greatest height above A is 2.0 s (2 s.f.)

b

$R(\uparrow) \; s = ut + \frac{1}{2}at^2$

$15.1 = 20t - 4.9t^2$

$4.9t^2 - 20t + 15.1 = 0$

$(t-1)(4.9t - 15.1) = 0$

On the way down the time must be greater than the result in part **a**, so $t \neq 1$.

$t = \frac{15.1}{4.9} = 3.0816\ldots$

The time taken for the ball to travel from A to B is 3.1 s (2 s.f.).

c

$R(\uparrow) \quad v_y = u + at$

$= 20 - 9.8 \times \frac{15.1}{4.9} = -10.2$

$V^2 = u_x^2 + v_y^2 = (20\sqrt{3})^2 + (-10.2)^2 = 1304.04$

$V = \sqrt{1304.04} = 36.111\ldots$

The speed with which the ball hits the hoarding is 36 m s^{-1} (2 s.f.).

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 18

Question:

A particle P passes through a point O and moves in a straight line. The displacement, s metres, of P from O, t seconds after passing through O is given by

$$s = -t^3 + 11t^2 - 24t$$

a Find an expression for the velocity, v m s^{-1}, of P at time t seconds.

b Calculate the values of t at which P is instantaneously at rest.

c Find the value of t at which the acceleration is zero.

d Sketch a velocity-time graph to illustrate the motion of P in the interval $0 \le t \le 6$, showing on your sketch the coordinates of the points at which the graph crosses the axes.

e Calculate the values of t in the interval $0 \le t \le 6$ between which the speed of P is greater than 16 m s^{-1}.

Solution:

a $v = \frac{ds}{dt} = -3t^2 + 22t - 24$

The velocity of P after t seconds is $(-3t^2 + 22t - 24)$ m s^{-1}.

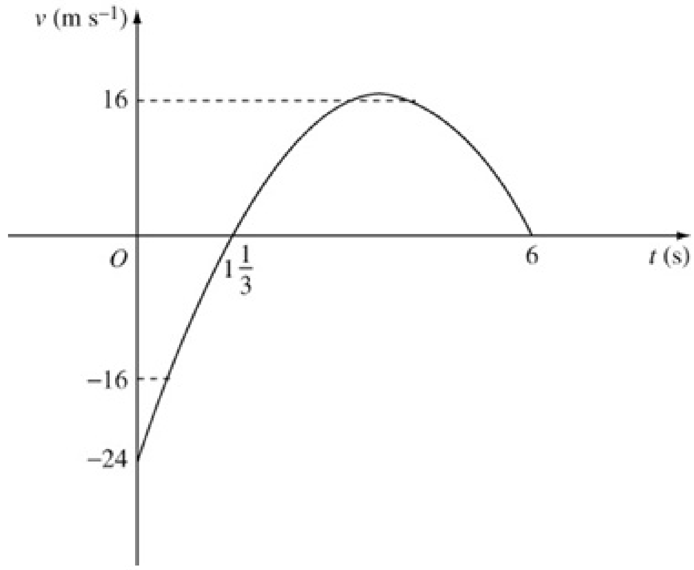
b When $v = 0$

$$3t^2 - 22t + 24 = (3t-4)(t-6) = 0$$
$$t = \frac{4}{3}, 6$$

c

$$a = \frac{dv}{dt} = -6t + 22 = 0$$
$$t = \frac{22}{6} = \frac{11}{3}$$

d



e The speed of P is 16 when $v = 16$ and $v = -16$.

When $v = 16$

$$-3t^2 + 22t - 24 = 16$$
$$3t^2 - 22t + 40 = 0$$
$$(3t-10)(t-4) = 0$$

$$t = \frac{10}{3}, 4$$

When $v = -16$

$$-3t^2 + 22t - 24 = -16$$

$$3t^2 - 22t + 8 = 0$$

$$t = \frac{22 \pm \sqrt{(484-96)}}{6} = 0.38, 6.95 \text{ (2 d.p.)}$$

From the diagram in part **d**, the required values are

$$0 \le t < 0.38, \frac{10}{3} < t < 4$$

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 19

Question:

A point P moves in a straight line so that, at time t seconds, its displacement from a fixed point O on the line is given by

$$s = \begin{cases} 4t^2, & 0 \le t \le 3 \\ 24t - 36, & 3 < t \le 6 \\ -252 + 96t - 6t^2, & t > 6. \end{cases}$$

Find

a the velocity of P when $t = 4$,

b the velocity of P when $t = 10$,

c the greatest positive displacement of P from O,

d the values of s when the speed of P is 18 m s^{-1}.

Solution:

a When $t = 4$, t is in the range $3 < t \le 6$, so $s = 24t - 36$

$v = \frac{ds}{dt} = 24$

The velocity of P when $t = 4$ is 24 m s^{-1} in the direction of s increasing.

b When $t = 10$, t is in the range $t > 6$, so $s = -252 + 96t - 6t^2$

$v = \frac{ds}{dt} = 96 - 12t$

When $t = 10$

$v = 96 - 12 \times 10 = -24$

The velocity of P when $t = 10$ is 24 m s^{-1} in the direction of s decreasing.

c The maximum displacement is when $\frac{ds}{dt} = v = 0$

$96 - 12t = 0 \Rightarrow t = 8$

When $t = 8$

$s = -252 + 96 \times 8 - 6 \times 8^2 = 132$

The greatest positive displacement of P from O is 132 m.

d The speed of P is 18 m s^{-1} when $v = \pm 18$

In the range $0 \le t \le 3$

$v = \frac{ds}{dt} = 8t = 18 \Rightarrow t = \frac{9}{4}$

When $t = \frac{9}{4}, s = 4 \times \left(\frac{9}{4}\right)^2 = 20.25$

In the range $t > 6$

$v = 96 - 12t = 18 \Rightarrow t = \frac{96-18}{12} = 6.5$

$s = -252 + 96 \times 6.5 - 6 \times 6.5^2 = 118.5$

$v = 96 - 12t = -18 \Rightarrow t = \frac{96+18}{12} = 9.5$

$s = -252 + 96 \times 9.5 - 6 \times 9.5^2 = 118.5$, the same result as for $v = 18$

The values of s when the speed of P is 18 m s^{-1} are 20.25 and 118.5.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 20

Question:

The position vector of a particle P, with respect to a fixed origin O, at time t seconds (where $t \geq 0$) is $\left[\left(6t - \frac{1}{2}t^3\right)\mathbf{i} + (3t^2 - 8t)\mathbf{j}\right]$ m. At time t seconds, the velocity of a second particle Q, moving in the same plane as P, is $(-8\mathbf{i} + 3t\mathbf{j})$ m s^{-1}.

a Find the value of t at the instant when the direction of motion of P is perpendicular to the direction of motion of Q.

b Given that P and Q collide when $t = 4$, find the position vector of Q with respect to O when $t = 0$.

Solution:

a For P

$$\mathbf{v} = \dot{\mathbf{p}} = \left(6 - \frac{3}{2}t\right)\mathbf{i} + (6t - 8)\mathbf{j}$$

The tangent the angle the direction of P makes with $\mathbf{i}$ is given by

$$m = \frac{6t - 8}{6 - \frac{3}{2}t}$$

The tangent the angle the direction of Q makes with $\mathbf{i}$ is given by

$$m' = -\frac{3t}{8}$$

Using $mm' = -1$

$$\frac{6t - 8}{6 - \frac{3}{2}t} \times -\frac{3}{8}t = -1$$

$$3t(6t - 8) = 8\left(6 - \frac{3}{2}t\right)$$

$$18t^2 - 24t = 48 - 12t$$

$$18t^2 - 12t - 48 = 0$$

$$3(t - 2)(3t + 4) = 0$$

$t \geq 0$

$t = 2$

b When $t = 4$

$$\mathbf{p} = \left(6 \times 24 - \frac{1}{2} \times 4^3\right)\mathbf{i} + \left(3 \times 4^2 - 8 \times 4\right)\mathbf{j} = -8\mathbf{i} + 16\mathbf{j}$$

For Q

$$\mathbf{q} = \int(-8\mathbf{i} + 3t\mathbf{j})\,\mathrm{d}t = -8t\mathbf{i} + \frac{3}{2}t^2\mathbf{j} + \mathbf{A}$$

When $t = 4, \mathbf{p} = \mathbf{q} = -8\mathbf{i} + 16\mathbf{j}$

$$-8\mathbf{i} + 16\mathbf{j} = -32\mathbf{i} + 24\mathbf{j} + \mathbf{A} \Rightarrow \mathbf{A} = 24\mathbf{i} - 8\mathbf{j}$$

$$\mathbf{q} = (24 - 8t)\mathbf{i} + \left(\frac{3}{2}t^2 - 8\right)\mathbf{j}$$

When $t = 0$

$$\mathbf{q} = 24\mathbf{i} - 8\mathbf{j}$$

The position vector of Q with respect to O when $t = 0$ is $(24\mathbf{i} - 8\mathbf{j})$ m .

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

Find the position of the centre of mass of four particles of masses 1 kg, 4 kg, 3 kg and 2 kg placed on the x-axis at the points (6, 0), (3, 0) (2, 0) and (4, 0) respectively.

Solution:

$$(1\times 6)+(4\times 3)+(3\times 2)+(2\times 4)=\bar{x}(1+4+3+2)$$

Use $\Sigma m_i x_i = \bar{x}\Sigma m_i$

$$6+12+6+8=10\bar{x}$$

Simplify.

$$32=10\bar{x}$$

$$3.2=\bar{x}$$

Solve for $\bar{x}$.

Centre of mass is at (3.2, 0).

Give both coordinates.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 2

Question:

Three masses 1 kg, 2 kg and 3 kg, are placed at the points with coordinates (0, 2), (0, 5) and (0, 1) respectively. Find the coordinates of G, the centre of mass of the three masses.

Solution:

$$(1\times 2)+(2\times 5)+(3\times 1)=\bar{y}(1+2+3)$$

Use $\Sigma m_i y_i = \bar{y}\Sigma m_i$.

$$2+10+3=6\bar{y}$$

Simplify.

$$15=6\bar{y}$$

$$2.5=\bar{y}$$

Solve for $\bar{y}$.

Centre of mass is at (0, 2.5).

Give both coordinates.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 3

Question:

Three particles of mass 2 kg, 3 kg and 5 kg, are placed at the points $(-1, 0), (-4, 0)$ and (5, 0) respectively. Find the coordinates of the centre of mass of the three particles.

Solution:

$$(2\times-1)+(3\times-4)+(5\times5)=\bar{x}(2+3+5)$$

Use $\Sigma m_i x_i = \bar{x}\Sigma m_i$.

$$-2+-12+25=10\bar{x}$$

$$11=10\bar{x}$$

Simplify.

$$1.1=\bar{x}$$

Solve for $\bar{x}$.

Centre of mass is at (1.1, 0).

Give both coordinates.

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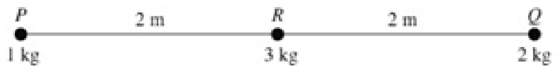
Exercise A, Question 4

Question:

A light rod PQ of length 4 m has particles of mass 1 kg, 2 kg and 3 kg attached to it at the points P, Q and R respectively, where $PR = 2\text{ m}$. The centre of mass of the loaded rod is at the point G. Find the distance PG.

Solution:



Draw a diagram.
The rod has no mass.

$$(1\times 0)+(3\times 2)+(2\times 4) = \bar{x}(1+3+2)$$

Take P as the origin and use $\Sigma m_i x_i = \bar{x}\Sigma m_i$.

$$0+6+8 = 6\bar{x}$$

Simplify.

$$\frac{7}{3} = \bar{x}$$

Solve for $\bar{x}$.

$$PG = 2\frac{1}{3}\text{ m}$$

Solutionbank M2

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Exercise A, Question 5

Question:

Three particles of mass 5 kg, 3 kg and m kg lie on the y-axis at the points (0, 4), (0, 2) and (0, 5) respectively. The centre of mass of the system is at the point (0, 4). Find the value of m.

Solution:

$$(5\times 4)+(3\times 2)+(m\times 5)=4(5+3+m)$$ Use $\Sigma m_i y_i = \bar{y}\Sigma m_i$.

$$20+6+5m=32+4m$$ Simplify

$$m=6$$ Solve for m.

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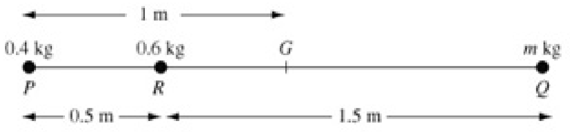
Exercise A, Question 6

Question:

A light rod PQ of length 2 m has particles of masses 0.4 kg and 0.6 kg fixed to it at the points P and R respectively, where $PR = 0.5\text{ m}$. Find the mass of the particle which must be fixed at Q so that the centre of mass of the loaded rod is at its mid-point.

Solution:



The rod, being light, has no mass.

Draw a diagram showing all the information.
G is the centre of mass.
Assume the mass of the particle required is m kg.

Take P as the origin.

Use $\Sigma m_i x_i = \bar{x}\Sigma m_i$.

$$(0.4 \times 0) + (0.6 \times 0.5) + (m \times 2) = 1 \times (0.4 + 0.6 + m)$$

Simplify.

$$0.3 + 2m = 1.0 + m$$

Solve for m.

$$m = 0.7$$

The mass of the particle is 0.7 kg.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 7

Question:

The centre of mass of four particles of masses $2m$, $3m$, $7m$ and $8m$, which are positioned at the points $(0, a)$, $(0, 2)$, $(0, -1)$ and $(0, 1)$ respectively, is the point G. Given that the coordinates of G are $(0, 1)$, find the value of a.

Solution:

$$(2m \times a) + (3m \times 2) + (7m \times -1) + (8m \times 1) = 1 \times (2m + 3m + 7m + 8m)$$

Use $\Sigma m_i y_i = \bar{y} \Sigma m_i$.

$$2ma + 6m - 7m + 8m = 20m$$

Divide by m.

$$2a + 7 = 20$$

$$a = 6\frac{1}{2}$$

Solve for a.

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Exercise A, Question 8

Question:

Particles of mass 3 kg, 2 kg and 1 kg lie on the y-axis at the points with coordinates $(0,-2)$, $(0, 7)$ and $(0, 4)$ respectively. Another particle of mass 6 kg is added to the system so that the centre of mass of all four particles is at the origin. Find the position of this particle.

Solution:

Suppose the particle is placed at $(0, y)$.

$$(3\times-2)+(2\times7)+(1\times4)+(6\times y)=0\times(3+2+1+6)$$ Use $\Sigma m_i y_i = \bar{y}\Sigma m_i$.

$$-6+14+4+6y=0$$ Simplify.

$$6y=-12$$

$$y=-2$$ Solve for y.

The particle must be placed at $(0,-2)$. Give both coordinates.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 9

Question:

Three particles A, B and C are placed along the x-axis. Particle A has mass 5 kg and is at the point (2, 0). Particle B has mass m_1 kg and is at the point (3, 0) and particle C has mass m_2 kg and is at the point $(-2, 0)$. The centre of mass of the three particles is at the point $G(1, 0)$. Given that the total mass of the three particles is 10 kg, find the values of m_1 and m_2.

Solution:

$$5 + m_1 + m_2 = 10$$
$$m_1 + m_2 = 5 \quad (1)$$

Use the total mass.

$$(5 \times 2) + (m_1 \times 3) + (m_2 \times -2) = 1 \times 10$$

Use $\Sigma m_i x_i = \bar{x} \Sigma m_i$, and $m_i = 10$.

$$10 + 3m_1 - 2m_2 = 10$$

Simplify.

$$3m_1 - 2m_2 = 0 \quad (2)$$

Adding $(2) + 2 \times (1)$ $2m_1 + 2m_2 = 10$

$$5m_1 = 10$$

Eliminate m_2.

$$m_1 = 2$$

Solve for m_1.

$$m_2 = 3$$

Use (1).

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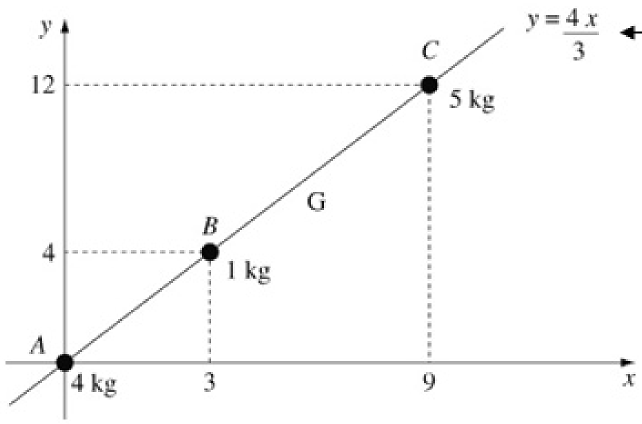
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 10

Question:

Three particles A, B and C have masses 4 kg, 1 kg and 5 kg respectively. The particles are placed on the line with equation $3y - 4x = 0$. Particle A is at the origin, particle B is at the point (3, 4) and particle C is at the point (9, 12). Find the coordinates of the centre of mass of the three particles.

Solution:



Draw a diagram showing the positions of the three particles.

$AB = \sqrt{3^2 + 4^2} = 5$

$AC = \sqrt{9^2 + 12^2} = 15$

We need the distances along the line.

$(4 \times 0) + (1 \times 5) + (5 \times 15) = \bar{x}(4 + 1 + 5)$

$0 + 5 + 75 = 10\bar{x}$

$8 = \bar{x}$

Taking our axis as the line $3y - 4x = 0$ and A as the origin.

Use $\Sigma m_i x_i = \bar{x} \Sigma m_i$.

Solve for $\bar{x}$.

i.e. $AG = 8$

Since $AB = 5$,

$\overrightarrow{AG} = \frac{8}{5} \times \overrightarrow{AB} = \frac{8}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 4.8 \\ 6.4 \end{pmatrix}$

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AG}$ are parallel.

Multiply out.

The coordinates of G are (4.8, 6.4).

Section 2.2 will show you a neater and easier way to solve this problem.

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Exercise B, Question 1

Question:

Two particles of equal mass are placed at the points $(1,-3)$ and $(5, 7)$. Find the centre of mass of the particles.

Solution:

$$m\begin{pmatrix}1\\-3\end{pmatrix}+m\begin{pmatrix}5\\7\end{pmatrix}=(m+m)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\Sigma m_i\mathbf{r}_i=\bar{\mathbf{r}}\,\Sigma m_i$

$$\begin{pmatrix}m\\-3m\end{pmatrix}+\begin{pmatrix}5m\\7m\end{pmatrix}=2m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify by dividing both sides by m.

$$\begin{pmatrix}6m\\4m\end{pmatrix}=2m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Centre of mass is $(3, 2)$.

This is the mid-point of the line joining the two points.

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Edexcel AS and A Level Modular Mathematics

Exercise B, Question 2

Question:

Four particles of equal mass are situated at the points (2, 0), $(-1,3)$, $(2,-4)$ and $(-1,-2)$. Find the coordinates of the centre of mass of the particles.

Solution:

$$m\begin{pmatrix}2\\0\end{pmatrix}+m\begin{pmatrix}-1\\3\end{pmatrix}+m\begin{pmatrix}2\\-4\end{pmatrix}+m\begin{pmatrix}-1\\-2\end{pmatrix}=4m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\Sigma m_i\mathbf{r}_i = \bar{\mathbf{r}}\,\Sigma m_i$ as before.

$$\begin{pmatrix}2\\-3\end{pmatrix}=4\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Divide both sides by m.

$$\begin{pmatrix}\frac{1}{2}\\-\frac{3}{4}\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Solve.

Centre of mass is $\left(\frac{1}{2},-\frac{3}{4}\right)$.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 3

Question:

A system of three particles consists of 10 kg placed at (2, 3), 15 kg placed at (4, 2) and 25 kg placed at (6, 6). Find the coordinates of the centre of mass of the system.

Solution:

$$10\begin{pmatrix}2\\3\end{pmatrix}+15\begin{pmatrix}4\\2\end{pmatrix}+5\begin{pmatrix}6\\6\end{pmatrix}=50\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}4\\6\end{pmatrix}+\begin{pmatrix}12\\6\end{pmatrix}+\begin{pmatrix}30\\30\end{pmatrix}=10\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Divide both sides by 5.

$$\begin{pmatrix}46\\42\end{pmatrix}=10\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}4.6\\4.2\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Solve.

Centre of mass is (4.6, 4.2).

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 4

Question:

Find the position vector of the centre of mass of three particles of masses 0.5 kg, 1.5 kg and 2 kg which are situated at the points with position vectors $(6\mathbf{i}-3\mathbf{j}), (2\mathbf{i}+5\mathbf{j})$ and $(3\mathbf{i}+2\mathbf{j})$ respectively.

Solution:

$$0.5\begin{pmatrix}6\\-3\end{pmatrix}+1.5\begin{pmatrix}2\\5\end{pmatrix}+2\begin{pmatrix}3\\2\end{pmatrix}=4\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

4 is the total mass.

$$\begin{pmatrix}3\\-1.5\end{pmatrix}+\begin{pmatrix}3\\7.5\end{pmatrix}+\begin{pmatrix}6\\4\end{pmatrix}=4\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

It's easier to use column vectors.

$$\begin{pmatrix}12\\10\end{pmatrix}=4\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}3\\2.5\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

You could leave your answer as a column vector.

The position vector is $(3\mathbf{i}+2.5\mathbf{j})$.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 5

Question:

Particles of masses m, $2m$, $5m$ and $2m$ are situated at $(-1,-1),(3,2),(4,-2)$ and $(-2,5)$ respectively. Find the coordinates of the centre of mass of the particles.

Solution:

$$m\begin{pmatrix}-1\\-1\end{pmatrix}+2m\begin{pmatrix}3\\2\end{pmatrix}+5m\begin{pmatrix}4\\-2\end{pmatrix}+2m\begin{pmatrix}-2\\5\end{pmatrix}=10m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}-1\\-1\end{pmatrix}+\begin{pmatrix}6\\4\end{pmatrix}+\begin{pmatrix}20\\-10\end{pmatrix}+\begin{pmatrix}-4\\10\end{pmatrix}=10\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Divide both sides by m.

$$\begin{pmatrix}21\\3\end{pmatrix}=10\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}2.1\\0.3\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Solve.

Centre of mass is at $(2.1, 0.3)$.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 6

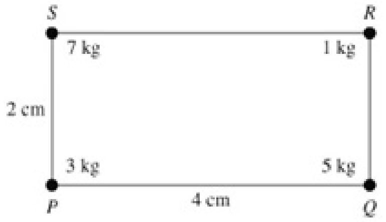
Question:

A light rectangular metal plate $PQRS$ has $PQ = 4$ cm and $PS = 2$ cm. Particles of masses 3 kg, 5 kg, 1 kg and 7 kg are attached respectively to the corners P, Q, R and S of the plate.
Find the distance of the centre of mass of the loaded plate from

a the side PQ,

b the side PS.

Solution:



Draw a diagram. (Note that the plate is light.)

Draw the rectangle with the 2 'axes' (PQ and PS) in the bottom L. H. corner.

Taking P as the origin, and axes, PQ and PS, P is (0, 0); Q is $(4, 0)$; R is $(4, 2)$; S is $(0, 2)$.

$$3\begin{pmatrix}0\\0\end{pmatrix}+5\begin{pmatrix}4\\0\end{pmatrix}+1\begin{pmatrix}4\\2\end{pmatrix}+7\begin{pmatrix}0\\2\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Total mass is 16 kg.

$$\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}20\\0\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}0\\14\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}24\\16\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}1.5\\1\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Solve.

a Distance from PQ is 1 $(\bar{y})$.

b Distance from PS is 1.5 $(\bar{x})$.

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Exercise B, Question 7

Question:

Three particles of masses 1 kg, 2 kg and 3 kg are positioned at the points (1, 0), (4, 3) and (p, q) respectively. Given that the centre of mass of the particles is at the point (2, 0), find the values of p and q.

Solution:

$$1\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}+3\begin{pmatrix}p\\q\end{pmatrix}=(1+2+3)\begin{pmatrix}2\\0\end{pmatrix}$$ Use $\Sigma m_i\mathbf{r}_i=\bar{\mathbf{r}}\,\Sigma m_i$.

$$\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}8\\6\end{pmatrix}+\begin{pmatrix}3p\\3q\end{pmatrix}=\begin{pmatrix}12\\0\end{pmatrix}$$ Simplify.

$$\begin{pmatrix}9+3p\\6+3q\end{pmatrix}=\begin{pmatrix}12\\0\end{pmatrix}$$ Collect terms.

$$9+3p=12$$
$$6+3q=0$$ Equate **i** and **j** components.

$$\Rightarrow p=1$$
$$q=-2$$ Solve for p and q.

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Exercise B, Question 8

Question:

A system consists of three particles with masses $3m$, $4m$ and $5m$. The particles are situated at the points with coordinates $(-3,-4)$, $(0.5, 4)$ and $(0,-5)$ respectively. Find the coordinates of the position of a fourth particle of mass $7m$, given that the centre of mass of all four particles is at the origin.

Solution:

$$3m\begin{pmatrix}-3\\-4\end{pmatrix}+4m\begin{pmatrix}0.5\\4\end{pmatrix}+5m\begin{pmatrix}0\\-5\end{pmatrix}+7m\begin{pmatrix}x\\y\end{pmatrix}=19m\begin{pmatrix}0\\0\end{pmatrix}$$

Use $\Sigma m_i\mathbf{r}_i = \bar{\mathbf{r}}\,\Sigma m_i$.

$$\begin{pmatrix}-9\\-12\end{pmatrix}+\begin{pmatrix}2\\16\end{pmatrix}+\begin{pmatrix}0\\-25\end{pmatrix}+\begin{pmatrix}7x\\7y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Divide by m and simplify.

$$\begin{pmatrix}-7+7x\\-21+7y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Collect terms.

$$-7+7x=0$$
$$-21-7y=0$$

Equate **i** and **j** components.

$$x=1$$
$$y=3$$

Solve for x and y.

Coordinates of particle are (1, 3).

State answer.

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Exercise B, Question 9

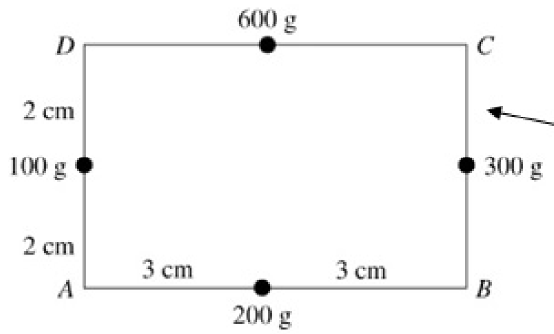
Question:

A light rectangular piece of card $ABCD$ has $AB = 6$ cm and $AD = 4$ m. Four particles of mass 200 g, 300 g, 600 g and 100 g are fixed to the rectangle at the mid-points of the sides AB, BC, CA and AD respectively. Find the distance of the centre of mass of the loaded rectangle from

a the side AB,

b the side AD.

Solution:



The card is light so has no mass.

Draw a diagram showing all the information.

Taking axes through A, the coordinates of the masses are (3, 0), (6, 2), (3, 4) and (0, 2).

Here you have to set up your own axes.

So,

$$200\begin{pmatrix}3\\0\end{pmatrix}+300\begin{pmatrix}6\\2\end{pmatrix}+600\begin{pmatrix}3\\4\end{pmatrix}+100\begin{pmatrix}0\\2\end{pmatrix}=1200\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\Sigma m_i \mathbf{r}_i = \bar{\mathbf{r}}\,\Sigma m_i$.

$$\begin{pmatrix}600\\0\end{pmatrix}+\begin{pmatrix}1800\\600\end{pmatrix}+\begin{pmatrix}1800\\2400\end{pmatrix}+\begin{pmatrix}0\\200\end{pmatrix}=1200\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify and divide by 100.

$$\begin{pmatrix}42\\32\end{pmatrix}=\begin{pmatrix}12\bar{x}\\12\bar{y}\end{pmatrix}$$

Collect terms.

$$42 = 12\bar{x}$$
$$32 = 12\bar{y}$$

Equate **i** and **j** components.

$$3.5 = \bar{x}$$
$$2.\dot{6} = \bar{y}$$

Solve for $\bar{x}$ and $\bar{y}$.

a Distance from AB is $2\frac{2}{3}$ cm ($\bar{y}$)

b Distance from AD is $3\frac{1}{2}$ cm ($\bar{x}$)

State answer.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 10

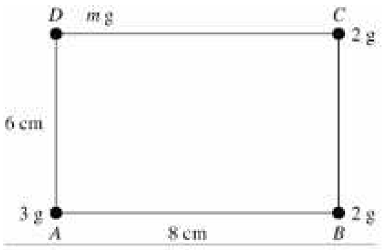
Question:

A light rectangular piece of card $ABCD$ has $AB = 8$ cm and $AD = 6$ cm. Three particles of mass 3 g, 2 g and 2 g are attached to the rectangle at the points A, B and C respectively.

a Find the mass of a particle which must be placed at the point D for the centre of mass of the whole system of four particles to lie 3 cm from the line AB.

b With this fourth particle in place, find the distance of the centre of mass of the system from the side AD.

Solution:



Draw a diagram.

The card has no mass. Here 'g' is grams!

Let mass of particle of D be m g.

Here we have to set up our own axes.

Taking axes through A, the coordinates of the particles are (0, 0), (8, 0), (8, 6) and (0, 6).

$$3\begin{pmatrix}0\\0\end{pmatrix}+2\begin{pmatrix}8\\0\end{pmatrix}+2\begin{pmatrix}8\\6\end{pmatrix}+m\begin{pmatrix}0\\6\end{pmatrix}=(3+2+2+m)\begin{pmatrix}\bar{x}\\3\end{pmatrix}$$

'3 cm from A' means $\bar{y} = 3$.

$$\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}16\\0\end{pmatrix}+\begin{pmatrix}16\\12\end{pmatrix}+\begin{pmatrix}0\\6m\end{pmatrix}=(7+m)\begin{pmatrix}\bar{x}\\3\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}32\\12+6m\end{pmatrix}=\begin{pmatrix}(7+m)\bar{x}\\21+3m\end{pmatrix}$$

Collect terms.

$$12+6m = 21+3m$$

Equate $\mathbf{j}$ components.

$$3m = 9$$

$$m = 3$$

Solve for m.

$$32 = 10\bar{x}$$

Equate $\mathbf{i}$ component and substitute for m.

$$3.2 = \bar{x}$$

Solve for $\bar{x}$.

a $m = 3$

b 3.2 cm

State your answer.

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Exercise C, Question 1

Question:

Find the centre of mass of a uniform triangular lamina whose vertices are

a $(1, 2)$, $(2, 6)$ and $(3, 1)$,

b $(-1, 4)$, $(3, 5)$ and $(7, 3)$

c $(-3, 2)$, $(4, 0)$ and $(0, 1)$

d (a, a), $(3a, 2a)$ and $(4a, 6a)$.

Solution:

a G is $\left(\frac{1+2+3}{3}, \frac{2+6+1}{3}\right)$
i.e. $(2, 3)$

Find the mean of the vertices of the triangle.

b G is $\left(\frac{-1+3+7}{3}, \frac{4+5+3}{3}\right)$
i.e. $(3, 4)$

Find the mean of the vertices of the triangle.

c G is $\left(\frac{-3+4+0}{3}, \frac{2+0+1}{3}\right)$
i.e. $\left(\frac{1}{3}, 1\right)$

Find the mean of the vertices of the triangle.

d G is $\left(\frac{a+3a+4a}{3}, \frac{a+2a+6a}{3}\right)$
i.e. $\left(\frac{8a}{3}, 3a\right)$

Find the mean of the vertices of the triangle.

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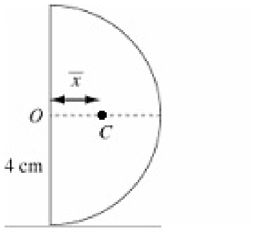
Exercise C, Question 2

Question:

Find the position of the centre of mass of a uniform semi-circular lamina of radius 4 cm and centre O.

Solution:

For a semi-circle,



$$2\alpha = \pi$$
$$\alpha = \frac{\pi}{2}$$
$$\bar{x} = \frac{2r\sin\alpha}{3\alpha} = \frac{8\sin\frac{\pi}{2}}{3\frac{\pi}{2}}$$
$$= \frac{16}{3\pi}$$

This result is in the formula booklet provided by Edexcel.

Centre of mass is on the axis of symmetry at a distance $\frac{16}{3\pi}$ cm from the centre.

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Edexcel AS and A Level Modular Mathematics

Exercise C, Question 3

Question:

The centre of mass of a uniform triangular lamina ABC is at the point $(2, a)$. Given that A is the point $(4, 3)$, B is the point $(b, 1)$ and C is the point $(-1, 5)$, find the values of a and b.

Solution:

$$(2, a) = \left(\frac{4+b-1}{3}, \frac{3+1+5}{3}\right)$$

Use the fact that the centre of mass is at the 'mean point'.

$$\text{i.e. } (2, a) = \left(\frac{3+b}{3}, 3\right)$$

Simplify.

So,

$$2 = \frac{3+b}{3} \quad \text{and} \quad a = 3$$

Compare x and y coordinates.

$$6 = 3 + b$$

$$3 = b$$

Solve for b.

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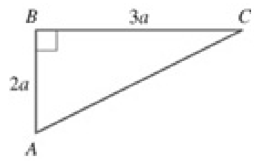
Exercise C, Question 4

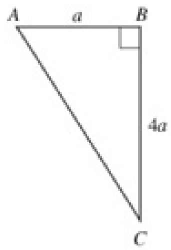
Question:

Find the position of the centre of mass of the following uniform triangular laminas:

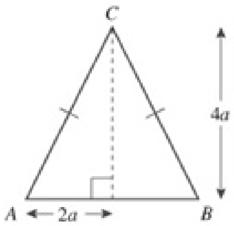
a



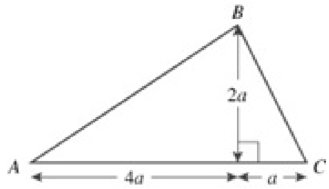
b



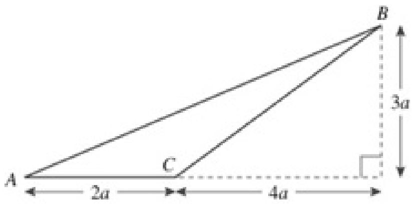
c



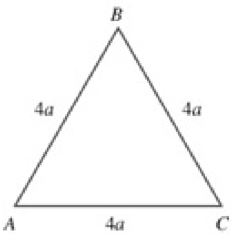
d



e

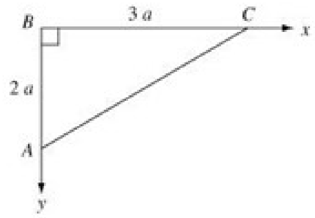


f



Solution:

a



We need to set up our own axes here.

Using the axes shown, B is (0, 0)
C is $(3a,\ 0)$ and A is $(0,\ 2a)$.

Centre of mass G is $\left(\frac{0+0+3a}{3}, \frac{2a+0+0}{3}\right)$

i.e. $\left(a, \frac{2a}{3}\right)$

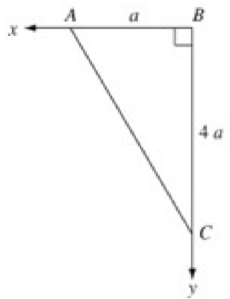
Use the fact that the centre of mass is at the 'mean point'.

Centre of mass is a distance a from AB and a distance $\frac{2a}{3}$ from BC.

State your answer carefully.

B is $(0,0)$
A is $(a,0)$
C is $(0,4a)$

b



Use the axes chosen (see the diagram).

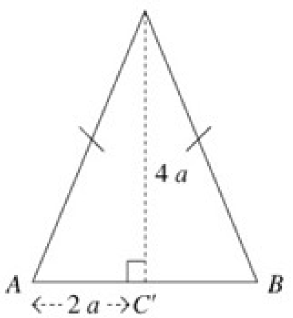
Centre of mass G is $\left(\frac{0+a+0}{3}, \frac{0+0+4a}{3}\right)$

i.e. $\left(\frac{a}{3}, \frac{4a}{3}\right)$

Centre of mass is a distance $\frac{a}{3}$ from BC and a distance $\frac{4a}{3}$ from AB.

Note that $\bar{x}$ gives the distance from the y-axis and $\bar{y}$ gives the distance from the x-axis.

c



Since $AC = BC$ i.e. the Δ is isosceles so $AB = 4a$.

Taking A as the origin with AB as the x-axis, the coordinates of A, B and C are (0, 0), ($4a$, 0) and ($2a$, $4a$) respectively.

We need to set up our own axes here.

G is $\left(\frac{0+4a+2a}{3}, \frac{0+0+4a}{3}\right)$

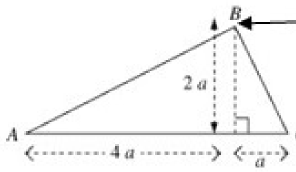
i.e. $\left(2a, \frac{4a}{3}\right)$

Take the mean of the vertices.

Note that we could have found G by using the *symmetry* of the Δ. G must lie on the axis of symmetry and since this line is also a median, G divides CC' in the ratio 2:1 i.e. it is $\frac{2}{3}$ of the way down the median from C.

This type of argument is perfectly acceptable when answering examination questions.

d



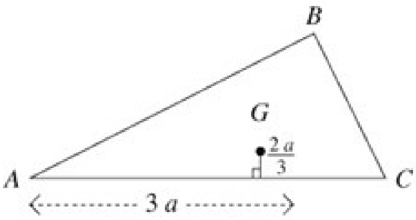
Again we need to set up some axes.

A is (0, 0); B is ($4a$, $2a$); C is ($5a$, 0).

Then G is $\left(\frac{0+4a+5a}{3}, \frac{0+2a+0}{3}\right)$

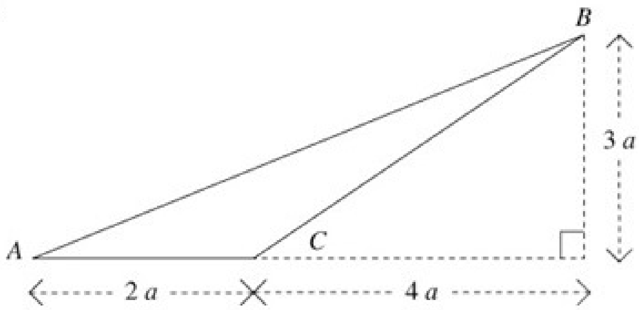
i.e. $\left(3a, \frac{2a}{3}\right)$

Take the mean of the vertices.



The diagram shows the position of G.

e



Take A as the origin with AC as the x-axis.

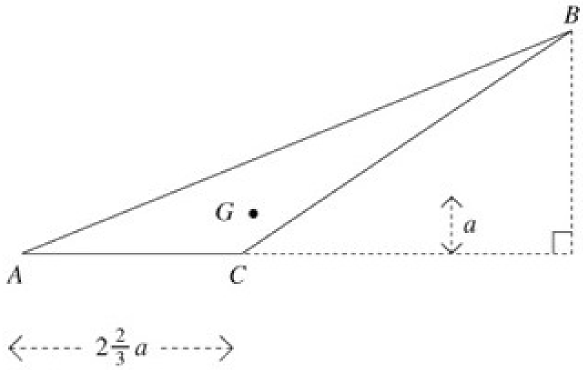
We need to set up axes.

The coordinates of A, B and C are $(0, 0)$, $(6a, 3a)$ and $(2a, 0)$ respectively.

Then G is $\left(\frac{0+6a+2a}{3}, \frac{0+3a+0}{3}\right)$

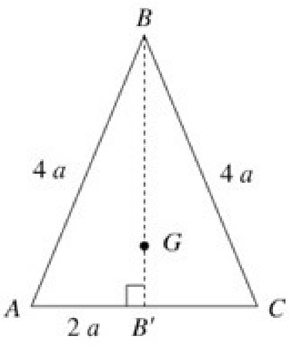
Take the mean point.

i.e. $\left(\frac{8a}{3}, a\right)$



The diagram shows the position of G, the centre of mass.

f



The ΔABC is equilateral, so G is on a line of symmetry, and divides it (BB') in ratio 2:1,

This type of argument is acceptable in an examination.

So,

$$BG = \frac{2}{3} \times 4a \sin 60°$$
$$= \frac{2}{3} \times 4a \times \frac{\sqrt{3}}{2}$$
$$= \frac{4a\sqrt{3}}{3}$$

$\left(GB' = \frac{2a\sqrt{3}}{3}\right)$

OR

If we take A as our origin, and the line AC as the x-axis, then A is $(0, 0)$, C is $(4a, 0)$ and B is $(2a, 4a \sin 60°)$

Set up axes.

so, G is $\left(\frac{0+4a+2a}{3}, \frac{0+0+4a\sin 60°}{3}\right)$

Take the mean point.

i.e. $\left(2a, \frac{2a\sqrt{3}}{3}\right)$

Simplify.

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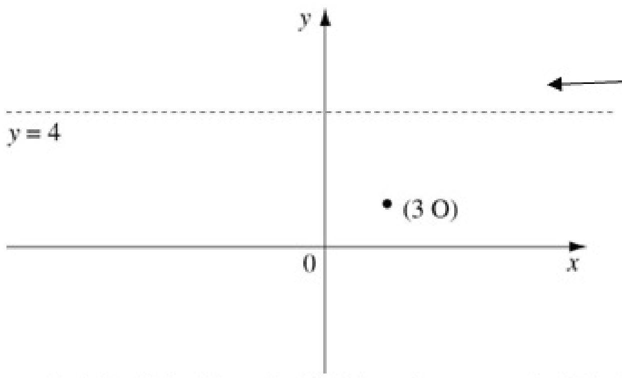
Exercise C, Question 5

Question:

A uniform triangular lamina is isosceles and has the line $y = 4$ as its axis of symmetry. One of the vertices of the triangle is the point (2, 1). Given that the x-coordinate of the centre of mass of the lamina is -3, find the coordinates of the other two vertices.

Solution:



Axis of symmetry $y = 4$

Draw a diagram showing all the information.

Let (2, 1) be the point A. Then since $y = 4$ is the line of symmetry, the point (2, 7) must be another vertex.

A is 3 units below the axis of symmetry, so the other after vertex must be 3 units above the axis of symmetry.

The third vertex must be on the line of symmetry, $(x, 4)$ say.
Then,

$$-3 = \frac{2+2+x}{3}$$

$$-13 = x$$

The mean of the x-coordinates must be -3.

The other two vertices are (2, 7) and $(-13, 4)$.

State the answer.

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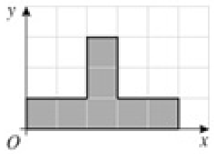
Exercise D, Question 1

Question:

<JN note: I've amended the question stem, as I'm assuming that each of these questions will appear separately in Solution bank.>

The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

1



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

$$5\begin{pmatrix}2\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}+2\begin{pmatrix}2\frac{1}{2}\\ 2\end{pmatrix}=(5+2)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

Use $\Sigma m_i \mathbf{r}_i = \bar{\mathbf{r}}\,\Sigma m_i$.

Clearly, $\bar{x}=2\frac{1}{2}$

This is acceptable in an examination.

OR By symmetry, $\bar{x}=2\frac{1}{2}$

$\frac{5}{2}+4=7\bar{y}$

$\frac{13}{14}=\bar{y}$

Equate the y-components.

Centre of mass is $\left(2\frac{1}{2},\frac{13}{14}\right)$.

State your answer, using both coordinaters.

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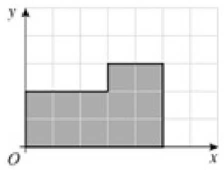
Exercise D, Question 2

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

$$6\begin{pmatrix}1\frac{1}{2}\\1\end{pmatrix}+6\begin{pmatrix}4\\1\frac{1}{2}\end{pmatrix}=(6+6)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

There are other ways of splitting the lamina up (*see below*).

$$\begin{pmatrix}9\\6\end{pmatrix}+\begin{pmatrix}24\\9\end{pmatrix}=12\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}33\\15\end{pmatrix}=12\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\frac{33}{12}=\frac{11}{4}=\bar{x}$$

$$\frac{15}{12}=\frac{5}{4}=\bar{y}$$

Equate **i** and **j** components.

Centre of mass is $\left(\frac{11}{4},\frac{5}{4}\right)$.

OR

$$10\begin{pmatrix}2\frac{1}{2}\\1\end{pmatrix}+2\begin{pmatrix}4\\2\frac{1}{2}\end{pmatrix}=(10+2)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

This splits the lamina differently.

$$\begin{pmatrix}25\\10\end{pmatrix}+\begin{pmatrix}8\\5\end{pmatrix}=12\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

As above, etc.

$$\begin{pmatrix}33\\15\end{pmatrix}=12\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

OR

$$15\begin{pmatrix}2\frac{1}{2}\\1\frac{1}{2}\end{pmatrix}-3\begin{pmatrix}1\frac{1}{2}\\2\frac{1}{2}\end{pmatrix}=(15-3)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

This uses ‘rectangle – rectangle’.

$$\begin{pmatrix}37\frac{1}{2}\\22\frac{1}{2}\end{pmatrix}-\begin{pmatrix}4\frac{1}{2}\\7\frac{1}{2}\end{pmatrix}=12\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}33\\5\end{pmatrix}=12\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

As above.

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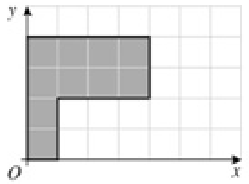
Exercise D, Question 3

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

$$16\begin{pmatrix}2\\2\end{pmatrix}-6\begin{pmatrix}2\frac{1}{2}\\1\end{pmatrix}=(16-6)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use 'square − rectangle'.

$$\begin{pmatrix}32\\32\end{pmatrix}-\begin{pmatrix}15\\6\end{pmatrix}=10\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}17\\26\end{pmatrix}=10\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Equate **i** and **j** components.

$$\begin{pmatrix}1.7\\2.6\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Centre of mass is (1.7, 2.6).

State your answer.

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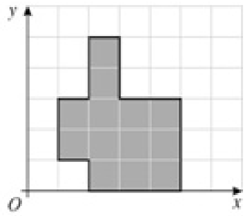
Exercise D, Question 4

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

$$2\begin{pmatrix}1\frac{1}{2}\\2\end{pmatrix}+2\begin{pmatrix}2\frac{1}{2}\\4\end{pmatrix}+9\begin{pmatrix}3\frac{1}{2}\\1\frac{1}{2}\end{pmatrix}=(2+2+9)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\Sigma m_i\mathbf{r}_i=\bar{\mathbf{r}}\Sigma m_i$.

$$\begin{pmatrix}3\\4\end{pmatrix}+\begin{pmatrix}5\\8\end{pmatrix}+\begin{pmatrix}31\frac{1}{2}\\13\frac{1}{2}\end{pmatrix}=13\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}39\frac{1}{2}\\25\frac{1}{2}\end{pmatrix}=13$$

Collect terms.

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}\frac{79}{26}=\bar{x};\frac{51}{26}=\bar{y}$$

Centre of mass is $\left(\frac{79}{26},\frac{51}{26}\right)$.

You could use decimals.

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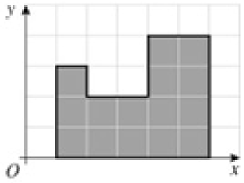
Exercise D, Question 5

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

$$1\begin{pmatrix}1\frac{1}{2}\\2\frac{1}{2}\end{pmatrix}+10\begin{pmatrix}3\frac{1}{2}\\1\end{pmatrix}+4\begin{pmatrix}5\\3\end{pmatrix}=(1+10+4)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\Sigma m_i \mathbf{r}_i = \bar{\mathbf{r}}\,\Sigma m_i$.

$$\begin{pmatrix}1\frac{1}{2}\\2\frac{1}{2}\end{pmatrix}+\begin{pmatrix}35\\10\end{pmatrix}+\begin{pmatrix}20\\12\end{pmatrix}=15\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Multiply out.

$$\begin{pmatrix}56\frac{1}{2}\\24\frac{1}{2}\end{pmatrix}=15\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Collect terms.

$$\frac{113}{30}=\bar{x};\ \frac{49}{30}=\bar{y}$$

Centre of mass is $\left(\frac{113}{30},\frac{49}{30}\right)$.

Check that your answer looks reasonable.

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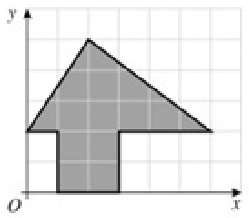
Exercise D, Question 6

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

$$\frac{1}{3}\left\{\begin{pmatrix}0\\2\end{pmatrix}+\begin{pmatrix}6\\2\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}\right\}=\frac{1}{3}\begin{pmatrix}8\\9\end{pmatrix}=\begin{pmatrix}\frac{8}{3}\\3\end{pmatrix}$$

The centre of mass of the triangle can be found by taking the average (mean) of its vertices.

Area of the triangle $=\frac{1}{2}\times 6\times 3=9$

$$4\begin{pmatrix}2\\1\end{pmatrix}+9\begin{pmatrix}\frac{8}{3}\\3\end{pmatrix}=13\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\Sigma m_i\mathbf{r}_i=\bar{\mathbf{r}}\Sigma m_i$.

$$\begin{pmatrix}8\\4\end{pmatrix}+\begin{pmatrix}24\\27\end{pmatrix}=13\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Multiply out.

$$\begin{pmatrix}32\\31\end{pmatrix}=13\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Centre of mass is $\left(\frac{32}{13},\frac{31}{13}\right)$.

Improper fractions are acceptable.

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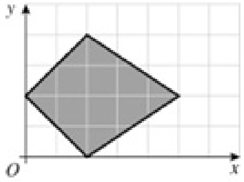
Exercise D, Question 7

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

By symmetry, $\bar{y} = 2$

Area of L.H. triangle $= \frac{1}{2} \times 4 \times 2 = 4$

Area of R.H. triangle $= \frac{1}{2} \times 4 \times 3 = 6$

$\frac{1}{2} \times \text{base} \times \text{height}$

x-coordinate of centre of mass of L.H. triangle $= \frac{1}{3}(0 + 2 + 2) = \frac{4}{3}$

Take the mean of the x-coordinates of its vertices.

x-coordinate of centre of mass of R.H. triangle $= \frac{1}{3}(2 + 2 + 5) = 3$

As before.

So,

$$\left(4 \times \frac{4}{3}\right) + (6 \times 3) = 10\bar{x}$$

Use the x-coordinates only.

$$\frac{16}{3} + 18 = 10\bar{x}$$

$$\frac{70}{3} = 10\bar{x}$$

$$\frac{7}{3} = \bar{x}$$

Centre of mass is $\left(\frac{7}{3}, 2\right)$.

Check that your answer looks sensible.

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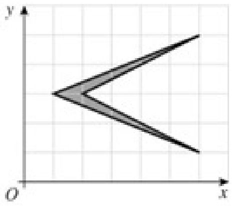
Exercise D, Question 8

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

By symmetry, $\bar{y} = 3$

Area of each triangle $= \frac{1}{2} \times 1 \times 2 = 1$

Split the shape in to two triangles.

x-coordinate of the centre of mass of each triangle is $\frac{1}{3}(1+2+6) = \frac{9}{3} = 3$

Find the mean of the x-coordinates of the vertices.

So,

$$(1 \times 3) + (1 \times 3) = (1+1) \times \bar{x}$$
$$3+3 = 2\bar{x}$$
$$3 = \bar{x}$$

Use $\Sigma m_i x_i = \bar{x} \Sigma m_i$.

Centre of mass is (3, 3).

OR Area of large triangle $= \frac{1}{2} \times 4 \times 5 = 10$

Treat the lamina as a large triangle — a small triangle.

Area of small triangle $= \frac{1}{2} \times 4 \times 4 = 8$

x-coordinate of centre of mass of large triangle $= \frac{1}{3}(1+6+6) = \frac{13}{3}$

Take the mean of the corners.

x-coordinate of centre of mass of small triangle $= \frac{1}{3}(2+6+6) = \frac{14}{3}$

So,

$$\left(10 \times \frac{13}{3}\right) - \left(8 \times \frac{14}{3}\right) = (10-8) \quad \bar{x}$$
$$\frac{130}{3} - \frac{112}{3} = 2\bar{x}$$
$$\frac{18}{3} = 2\bar{x}$$
$$3 = \bar{x}$$

As before.

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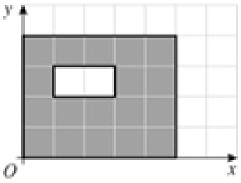
Exercise D, Question 9

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

$$20\begin{pmatrix}2\frac{1}{2}\\2\end{pmatrix}-2\begin{pmatrix}2\\2\frac{1}{2}\end{pmatrix}=(20-2)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

It's much easier to treat this lamina as a rectangle with a rectangle removed.

$$\begin{pmatrix}50\\40\end{pmatrix}-\begin{pmatrix}4\\5\end{pmatrix}=18\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}46\\35\end{pmatrix}=18\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Centre of mass is $\left(\frac{23}{9},\frac{35}{18}\right)$.

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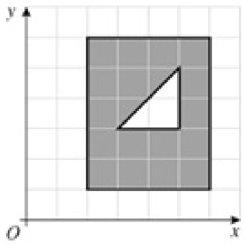
Exercise D, Question 10

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

Centre of mass of triangle $= \frac{1}{3}\left\{\begin{pmatrix}3\\3\end{pmatrix}+\begin{pmatrix}5\\3\end{pmatrix}+\begin{pmatrix}5\\5\end{pmatrix}\right\}$

$= \frac{1}{3}\begin{pmatrix}13\\11\end{pmatrix}$

$= \begin{pmatrix}\frac{13}{3}\\ \frac{11}{3}\end{pmatrix}$

Take the mean of the vertices of the triangle.

$$20\begin{pmatrix}4\\3\frac{1}{2}\end{pmatrix}-2\begin{pmatrix}\frac{13}{3}\\ \frac{11}{3}\end{pmatrix}=(20-2)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

It's much easier to treat the lamina as (a rectangle – a triangle).

$$\begin{pmatrix}80\\70\end{pmatrix}-\begin{pmatrix}\frac{26}{3}\\ \frac{22}{3}\end{pmatrix}=18\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\begin{pmatrix}\frac{214}{3}\\ \frac{188}{3}\end{pmatrix}=18\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

Simplify and collect terms.

So, centre of mass is $\left(\frac{107}{27},\frac{94}{27}\right)$.

State the answer.

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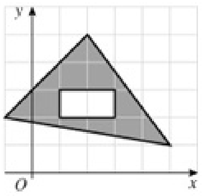
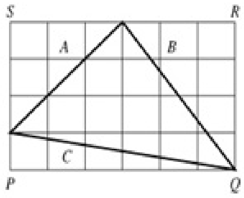
Exercise D, Question 11

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:



$$\begin{aligned}\text{Area of triangle} &= \text{Area of } PQRS - \text{Area of } (A+B+C)\\ &= (6\times 4)-(\tfrac{1}{2}\times 3\times 3+\tfrac{1}{2}\times 3\times 4+\tfrac{1}{2}\times 1\times 6)\\ &= 24-\tfrac{9}{2}-6-3\\ &= \tfrac{21}{2}\end{aligned}$$

This is the easiest way of finding the area of the triangle.

$$\begin{aligned}\text{Centre of mass of triangle} &= \frac{1}{3}\left\{\begin{pmatrix}-1\\3\end{pmatrix}+\begin{pmatrix}5\\2\end{pmatrix}+\begin{pmatrix}2\\6\end{pmatrix}\right\}\\ &= \frac{1}{3}\begin{pmatrix}6\\11\end{pmatrix}\\ &= \begin{pmatrix}2\\ \frac{11}{3}\end{pmatrix}\end{aligned}$$

Take the mean of the vertices of the triangle.

So,

$$\frac{21}{2}\begin{pmatrix}2\\ \frac{11}{3}\end{pmatrix}-2\begin{pmatrix}2\\ 3\frac{1}{2}\end{pmatrix}=\left(\frac{21}{2}-2\right)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

This is the only viable method here.

$$\begin{pmatrix}21\\ \frac{77}{2}\end{pmatrix}-\begin{pmatrix}4\\7\end{pmatrix}=\frac{17}{2}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\begin{pmatrix}17\\ \frac{63}{2}\end{pmatrix}=\frac{17}{2}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}\Rightarrow\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}2\\ \frac{63}{17}\end{pmatrix}$$

Centre of mass is $(2, \frac{63}{17})$.

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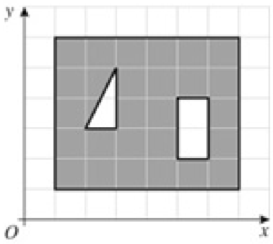
Exercise D, Question 12

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

Centre of mass of triangle $= \frac{1}{3}\left\{\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}3\\3\end{pmatrix}+\begin{pmatrix}3\\5\end{pmatrix}\right\}$

$= \begin{pmatrix}\frac{8}{3}\\ \frac{11}{3}\end{pmatrix}$

Find the 'average' of the vertices.

$$30\begin{pmatrix}4\\3\frac{1}{2}\end{pmatrix}-1\begin{pmatrix}\frac{8}{3}\\ \frac{11}{3}\end{pmatrix}-2\begin{pmatrix}5\frac{1}{2}\\3\end{pmatrix}=(30-1-2)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

This is the only sensible method here.

$$\begin{pmatrix}120\\105\end{pmatrix}-\begin{pmatrix}\frac{8}{3}\\ \frac{11}{3}\end{pmatrix}-\begin{pmatrix}11\\6\end{pmatrix}=27\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\begin{pmatrix}\frac{319}{3}\\ \frac{286}{3}\end{pmatrix}=27\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

Centre of mass is $\left(\frac{319}{81}, \frac{286}{81}\right)$.

Check that your answer looks reasonable for the lamina in question.

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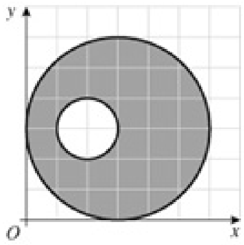
Exercise D, Question 13

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

$$\pi \times 3^2 \begin{pmatrix} 3 \\ 3 \end{pmatrix} - \pi \times 1^2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = (\pi \times 3^2 - \pi \times 1^2) \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Cancel the π s.

$$\begin{pmatrix} 27 \\ 27 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 8 \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

This is the only method possible for this lamina.

$$\begin{pmatrix} 25 \\ 24 \end{pmatrix} = 8 \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{25}{8} \\ 3 \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Note that we could have said $\bar{y} = 3$, by symmetry.

This can be used in the examination.

Centre of mass is $\left(\frac{25}{8}, 3 \right)$.

State your answer.

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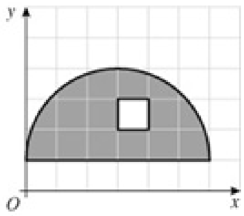
Exercise D, Question 14

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

$$\text{Centre of mass of semi-circle} = \begin{pmatrix} 3 \\ 1+\frac{4\times 3}{3\pi} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ \frac{\pi+4}{\pi} \end{pmatrix}$$

The position of the centre of mass of a semi-circular lamina is given in the formulae booklet.

$$\frac{\pi\times 3^2}{2}\begin{pmatrix} 3 \\ \frac{\pi+4}{\pi} \end{pmatrix} - 1\begin{pmatrix} 3\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} = \left(\frac{\pi\times 3^2}{2} - 1\right)\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Simplify.

$$\begin{pmatrix} \frac{27\pi}{2} \\ \frac{9}{2}(\pi+4) \end{pmatrix} - \begin{pmatrix} 3\frac{1}{2} \\ 2\frac{1}{2} \end{pmatrix} = \left(\frac{9\pi}{2} - 1\right)\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

$$\left(\begin{pmatrix} \frac{27\pi-7}{2} \\ \frac{9\pi+31}{2} \end{pmatrix}\right) = \frac{9\pi-2}{2}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

So, $\bar{x} = \frac{27\pi-7}{9\pi-2}$; $\bar{y} = \frac{9\pi+31}{9\pi-2}$

Decimal answers would, of course, be acceptable.

Centre of mass is $\left(\frac{27\pi-7}{9\pi-2}, \frac{9\pi+31}{9\pi-2}\right)$.

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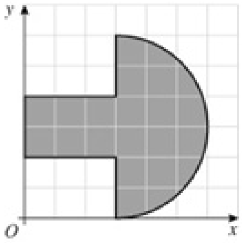
Exercise D, Question 15

Question:



The diagram shows a uniform plane figure drawn on a grid of unit squares. Find the coordinates of the centre of mass.

Solution:

By symmetry, $\bar{y} = 3$

x-coordinate of centre of mass of semi-circle is $3 + \frac{4 \times 3}{3\pi} = \frac{3\pi + 4}{\pi}$

$$\left(6 \times 1\tfrac{1}{2}\right) + \frac{\pi \times 3^2}{2}\left(\frac{3\pi + 4}{\pi}\right) = \left(6 + \frac{\pi \times 3^2}{2}\right)\bar{x}$$

Use $\Sigma m_i x_i = \bar{x}\Sigma m_i$.

$$9 + \frac{9}{2}(3\pi + 4) = \left(6 + \frac{9\pi}{2}\right)\bar{x}$$

$$18 + 27\pi + 36 = (12 + 9\pi)\bar{x}$$

Multiply through by 2 to clear the fractions.

$$\frac{18 + 9\pi}{4 + 3\pi} = \bar{x}$$

Divide by 3.

Centre of mass is $\left(\frac{18 + 9\pi}{4 + 3\pi}, 3\right)$.

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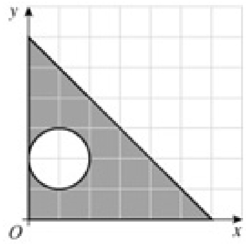
Exercise D, Question 16

Question:



Solution:

Centre of mass of triangle

$$= \frac{1}{3}\left\{\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}6\\0\end{pmatrix}+\begin{pmatrix}0\\6\end{pmatrix}\right\} = \begin{pmatrix}2\\2\end{pmatrix}$$

This is two thirds of the way along the median through O.

$$\left(\frac{1}{2}\times 6\times 6\right)\begin{pmatrix}2\\2\end{pmatrix} - \pi\times 1^2\begin{pmatrix}1\\2\end{pmatrix} = (18-\pi)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\Sigma m_i \mathbf{r}_i = \bar{\mathbf{r}}\,\Sigma m_i$.

$$\begin{pmatrix}36\\36\end{pmatrix} - \begin{pmatrix}\pi\\2\pi\end{pmatrix} = (18-\pi)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\bar{x} = \frac{36-\pi}{18-\pi};\ \bar{y} = \frac{36-2\pi}{18-\pi}$$

Centre of mass is $\left(\dfrac{36-\pi}{18-\pi}, \dfrac{36-2\pi}{18-\pi}\right)$.

Check that your answer is reasonable for the lamina in question.

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Exercise E, Question 1

Question:

Find the coordinates of the centres of mass of the shapes shown in Exercise 2D questions **1** to **5**, regarding them as uniform plane wire frameworks.

Solution:

By symmetry, $\bar{x} = 2\frac{1}{2}$

Always use the symmetry if possible.

$(5\times 0)+(1\times \frac{1}{2})+(2\times 1)+(2\times 2)+(1\times 3)+(2\times 2)+(2\times 1)+(1\times \frac{1}{2})$
$= (5+1+2+2+1+2+2+1)\bar{y}$

Use $\Sigma m_i y_i = \bar{y}\Sigma m_i$.

$$\frac{1}{2}+2+4+3+4+2+\frac{1}{2} = 16\bar{y}$$

The framework has 8 sides.

$$16 = 16\bar{y}$$
$$1 = \bar{y}$$

Simplify.

$\therefore$ Centre of mass is $\left(2\frac{1}{2}, 1\right)$.

Note that it was $(2\frac{1}{2}, \frac{13}{14})$ when regarded as a lamina.

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Exercise E, Question 2

Question:

Find the coordinates of the centres of mass of the shapes shown in Exercise 2D questions **1** to **5**, regarding them as uniform plane wire frameworks.

Solution:

$$5\begin{pmatrix}2\frac{1}{2}\\0\end{pmatrix}+3\begin{pmatrix}5\\1\frac{1}{2}\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}+1\begin{pmatrix}3\\2\frac{1}{2}\end{pmatrix}+3\begin{pmatrix}1\frac{1}{2}\\2\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}$$

Use $\Sigma m_i \mathbf{r}_i = \bar{\mathbf{r}}\,\Sigma m_i$.

$$=(5+3+2+1+3+2)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

The framework has 6 sides.

$$\begin{pmatrix}12\frac{1}{2}\\0\end{pmatrix}+\begin{pmatrix}15\\4\frac{1}{2}\end{pmatrix}+\begin{pmatrix}8\\6\end{pmatrix}+\begin{pmatrix}3\\2\frac{1}{2}\end{pmatrix}+\begin{pmatrix}4\frac{1}{2}\\6\end{pmatrix}+\begin{pmatrix}0\\2\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}43\\21\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify and collect the terms.

$\therefore$ Centre of mass is $\left(\frac{43}{16},\frac{21}{16}\right)$.

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Exercise E, Question 3

Question:

Find the coordinates of the centres of mass of the shapes shown in Exercise 2D questions **1** to **5**, regarding them as uniform plane wire frameworks.

Solution:

$$1\begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}+2\begin{pmatrix}1\\1\end{pmatrix}+3\begin{pmatrix}2\frac{1}{2}\\2\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}+4\begin{pmatrix}2\\4\end{pmatrix}+4\begin{pmatrix}0\\2\end{pmatrix}$$

$$=(1+2+3+2+4+4)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\Sigma m_i \mathbf{r}_i = \bar{\mathbf{r}}\,\Sigma m_i$.

The framework has 6 sides.

$$\begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}+\begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}7\frac{1}{2}\\6\end{pmatrix}+\begin{pmatrix}8\\6\end{pmatrix}+\begin{pmatrix}8\\16\end{pmatrix}+\begin{pmatrix}0\\8\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}26\\38\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$\therefore$ Centre of mass is $\left(\frac{13}{8},\frac{19}{8}\right)$.

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Exercise E, Question 4

Question:

Find the coordinates of the centres of mass of the shapes shown in Exercise 2D questions **1** to **5**, regarding them as uniform plane wire frameworks.

Solution:

$$3\begin{pmatrix}3\frac{1}{2}\\0\end{pmatrix}+3\begin{pmatrix}5\\1\frac{1}{2}\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}+2\begin{pmatrix}3\\4\end{pmatrix}+1\begin{pmatrix}2\frac{1}{2}\\5\end{pmatrix}+2\begin{pmatrix}2\\4\end{pmatrix}+1\begin{pmatrix}1\frac{1}{2}\\3\end{pmatrix}$$

The shape has 10 sides.

$$+2\begin{pmatrix}1\\2\end{pmatrix}+1\begin{pmatrix}1\frac{1}{2}\\1\end{pmatrix}+1\begin{pmatrix}2\\\frac{1}{2}\end{pmatrix}=(3+3+2+2+1+2+1+2+1+1)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}10\frac{1}{2}\\0\end{pmatrix}+\begin{pmatrix}15\\4\frac{1}{2}\end{pmatrix}+\begin{pmatrix}8\\6\end{pmatrix}+\begin{pmatrix}6\\8\end{pmatrix}+\begin{pmatrix}2\frac{1}{2}\\5\end{pmatrix}+\begin{pmatrix}4\\8\end{pmatrix}+\begin{pmatrix}1\frac{1}{2}\\3\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}1\frac{1}{2}\\1\end{pmatrix}+\begin{pmatrix}2\\\frac{1}{2}\end{pmatrix}$$

Simplify.

$$=18\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}53\\40\end{pmatrix}=18\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Centre of mass is $\left(\frac{53}{18},\frac{20}{9}\right)$.

Check the sense of your answer.

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Exercise E, Question 5

Question:

Find the coordinates of the centres of mass of the shapes shown in Exercise 2D questions **1** to **5**, regarding them as uniform plane wire frameworks.

Solution:

$$5\begin{pmatrix}3\frac{1}{2}\\0\end{pmatrix}+4\begin{pmatrix}6\\2\end{pmatrix}+2\begin{pmatrix}5\\4\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}+2\begin{pmatrix}3\\2\end{pmatrix}+1\begin{pmatrix}2\\2\frac{1}{2}\end{pmatrix}+1\begin{pmatrix}1\frac{1}{2}\\3\end{pmatrix}+3\begin{pmatrix}1\\1\frac{1}{2}\end{pmatrix}$$

The framework has 8 sides.

$$=(5+4+2+2+2+1+1+3)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\Sigma m_i \mathbf{r}_i = \bar{\mathbf{r}}\Sigma m_i$.

$$\begin{pmatrix}17\frac{1}{2}\\0\end{pmatrix}+\begin{pmatrix}24\\8\end{pmatrix}+\begin{pmatrix}10\\8\end{pmatrix}+\begin{pmatrix}8\\6\end{pmatrix}+\begin{pmatrix}6\\4\end{pmatrix}+\begin{pmatrix}2\\2\frac{1}{2}\end{pmatrix}+\begin{pmatrix}1\frac{1}{2}\\3\end{pmatrix}+\begin{pmatrix}3\\4\frac{1}{2}\end{pmatrix}=20\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}72\\36\end{pmatrix}=20\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Centre of mass is $\left(\frac{18}{5},\frac{9}{5}\right)$.

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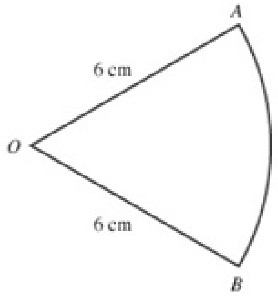
Exercise E, Question 6

Question:

Find the position of the centre of mass of the framework shown in the diagram which is formed by bending a uniform piece of wire of total length $(12+2\pi)$ cm to form a sector of a circle, centre O, radius 6 cm.



Solution:

$AB=(12+2\pi)-12=2\pi$

Let $A\hat{O}B=\theta$ (radians) $(=2\alpha)$

Then

$6\theta=2\pi$

$\theta=\frac{2\pi}{6}=\frac{\pi}{3}=60^\circ$

Use $S=r\theta$.

$\frac{r\sin\alpha}{\alpha}$; α in RADIANS.

from the formula booklet.

Distance of G from O

$=\bar{x}$ say.

Then,

$$\left(6\times 3\cos 30^\circ\right)\times 2+2\pi\times\frac{6\sin\frac{\pi}{6}}{\frac{\pi}{6}}=\bar{x}(12+2\pi)$$

Use $\Sigma m_i x_i=\bar{x}\Sigma m_i$.

$$18\sqrt{3}+36=\bar{x}(12+2\pi)$$

$$\cos 30^\circ=\frac{\sqrt{3}}{2}$$

$$\sin\frac{\pi}{6}=\frac{1}{2}$$

$$\therefore\ \bar{x}=\frac{18\sqrt{3}+36}{12+2\pi}$$

$$=\frac{9(\sqrt{3}+2)}{6+\pi}$$

Simplify by dividing top and bottom by 2.

Centre of mass is on line of symmetry through O,
and a distance of $\frac{9(\sqrt{3}+2)}{6+\pi}$ from O.

State your answer.

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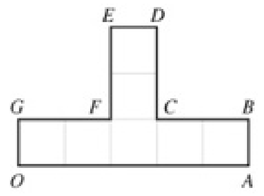
Exercise F, Question 1

Question:

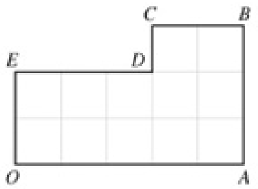
a The lamina from question 1 in Exercise 2D is shown.

The lamina is freely suspended from the point O and hangs in equilibrium.

Find the angle between OA and the downward vertical.

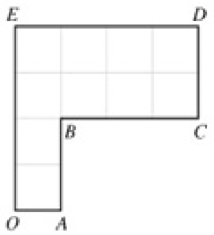


b The lamina from question 2 in Exercise 2D is shown below.



The lamina is freely suspended from the point O and hangs in equilibrium. Find the angle between OA and the downward vertical.

c The lamina from question 3 in Exercise 2D is shown below.



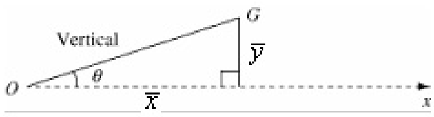
The lamina is freely suspended from the point O and hangs in equilibrium. Find the angle between OA and the downward vertical.

Solution:

a From question 1**a** in Exercise 2D,

$\bar{x} = 2\frac{1}{2}; \bar{y} = \frac{13}{14}$

Vertical



In equilibrium, G will be vertically below O i.e. OG is the vertical.

$$\tan\theta = \frac{\bar{y}}{\bar{x}} = \frac{\frac{13}{14}}{\frac{5}{2}}$$

$$= \frac{13}{14} \times \frac{2}{5} = \frac{13}{35}$$

$$\theta = \tan^{-1}\left(\frac{13}{35}\right) = 20.4° \text{ (3 s.f.)}$$

b From question 1**b** in Exercise 2D,

$\bar{x} = \frac{11}{4}; \bar{y} = \frac{5}{4}$

As above, $\tan\theta = \frac{\bar{y}}{\bar{x}} = \frac{\frac{5}{4}}{\frac{11}{4}}$

i.e. $\tan\theta = \frac{5}{11}$

$$\theta = \tan^{-1}\left(\frac{5}{11}\right) = 24.4° \text{ (3 s.f.)}$$

c From question 1**c** in Exercise 2D,

$\bar{x} = 1.7; \bar{y} = 2.6$

$$\tan\theta = \frac{2.6}{1.7} = \frac{26}{17}$$

$$\theta = \tan^{-1}\left(\frac{26}{17}\right) = 56.8° \text{ (3 s.f.)}$$

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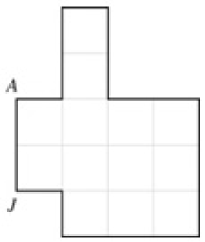
Exercise F, Question 2

Question:

The lamina from question 4 in Exercise 2D is shown below.



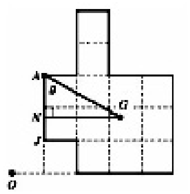
The lamina is freely suspended from the point A and hangs in equilibrium. Find the angle between AJ and the downward vertical.

Solution:

From question 1**a** in Exercise 2D,

$\bar{x} = \frac{79}{26}; \bar{y} = \frac{51}{26}$

These are the coordinates of the centre of mass, G, referred to O as origin.



A is the point of suspension.

G is $\left(\frac{79}{26}, \frac{51}{26}\right)$.

When the lamina hangs in equilibrium from A,
AG will be the downward vertical.
Let N be the point on AJ such that GN is perpendicular to AJ.

See diagram.

Then $N\hat{A}G = \theta$ is the required angle.

$$\tan\theta = \frac{GN}{AN} = \frac{\bar{x}-1}{3-\bar{y}}$$

Since A is the point $(1, 3)$.

$$= \frac{\frac{79}{26}-1}{3-\frac{51}{26}} = \frac{79-26}{78-51}$$

Multiply top and bottom by 26.

$$= \frac{53}{27} \Rightarrow \theta = 63.0^\circ \text{ (3 s.f.)}$$

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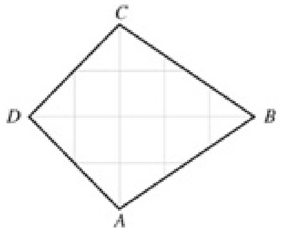
Exercise F, Question 3

Question:

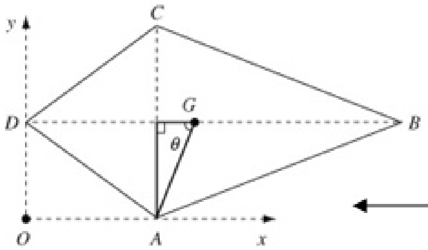
The lamina from question 7 in Exercise 2D is shown below.



The lamina is free to rotate about a fixed smooth horizontal axis, perpendicular to the plane of the lamina, passing through the point A.

Find the angle between AC and the horizontal.

Solution:



G, the centre of mass has coordinates $\left(\frac{7}{3}, 2\right)$ taking O as origin.

Since AG will be *vertical* in equilibrium, the angle between AC and the *horizontal* will be θ.

θ is the required angle.

$$\tan\theta = \frac{2}{\frac{7}{3}-2}$$

Multiply top and bottom by 3 to clear fractions.

$$= \frac{6}{7-6}$$

$$= 6$$

$$\theta = 80.5^\circ \text{ (3 s.f.)}$$

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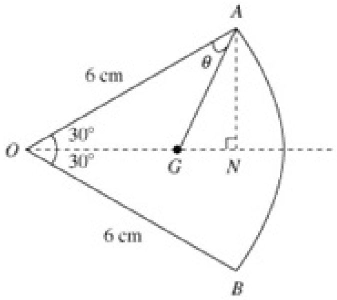
Exercise F, Question 4

Question:

The framework in question 6, Exercise 2E is freely suspended from the point A and allowed to hang in equilibrium. Find the angle between OA and the downward vertical.

Solution:



G is the centre of mass if the framework

$$OG = \bar{x} = \frac{9(\sqrt{3}+2)}{(6+\pi)}$$

G is on the line of symmetry.

(see question 2 in Exercise 2E)

AG will be vertical, when the framework hangs in equilibrium.

θ (see diagram) is the required angle.

$\theta = 60° - G\hat{A}N$

$$\tan G\hat{A}N = \frac{GN}{AN} = \frac{6\cos 30° - \bar{x}}{6\sin 30°}$$

$$= \frac{3\sqrt{3} - \bar{x}}{3}$$

$$= \sqrt{3} - \frac{3(\sqrt{3}+2)}{6+\pi}$$

So, $G\hat{A}N = 26.898°\ldots$

So, $\theta = 33.1°$ (3 s.f.)

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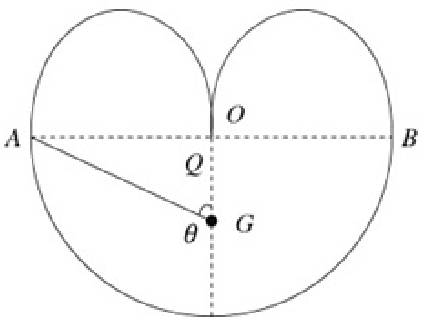
Exercise F, Question 5

Question:

The shape in question 7, Exercise 2E is freely suspended from the point A and allowed to hang in equilibrium. Find the angle between OA and the horizontal.

Solution:



$OG = \frac{3}{2\pi}$, where G is the centre of mass so, AG will be vertical in equilibrium.

See question 3 in Exercise 2E.

Required angle is $A\hat{G}O = \theta$

Since the angle with the horizontal will be $90° -$ angle with the vertical.

$$\tan\theta = \frac{AO}{OG} = \frac{3}{\frac{3}{2\pi}} = 2\pi$$

$$\theta = \tan^{-1}(2\pi) = 81.0° \text{ (3 s.f.)}$$

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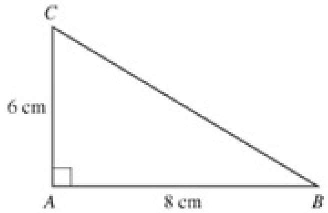
Exercise F, Question 6

Question:

The uniform triangular lamina ABC shown below is placed on a rough plane inclined at an angle α to the horizontal.



The edge AB is in contact with the plane, with A below B.

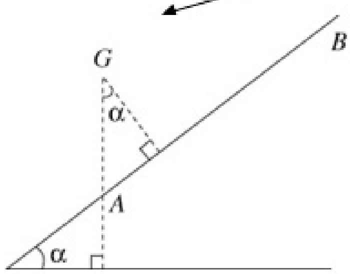
Given that the lamina is on the point of toppling about A, find the value of α.

Solution:

G, the centre of mass of the lamina, has position vector $\frac{1}{3}\left\{\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}8\\0\end{pmatrix}+\begin{pmatrix}0\\6\end{pmatrix}\right\}$ referred to axes, AB

and AC respectively. i.e.: $\begin{pmatrix}\frac{8}{3}\\2\end{pmatrix}$

G will be vertically above A when the lamina is about to topple.



$$\tan\alpha=\frac{\frac{8}{3}}{2}=\frac{4}{3}$$
$$\alpha=53.1°$$

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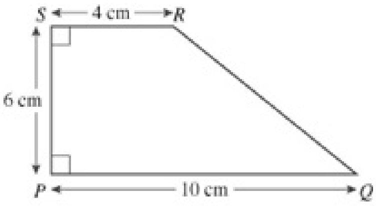
Exercise F, Question 7

Question:

$PQRS$ is a uniform lamina.

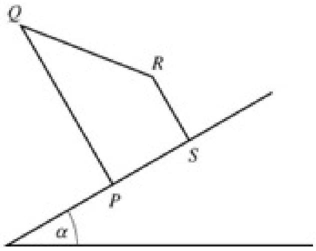


a Find the distance of the centre of mass of the lamina from

i PS

ii PQ.

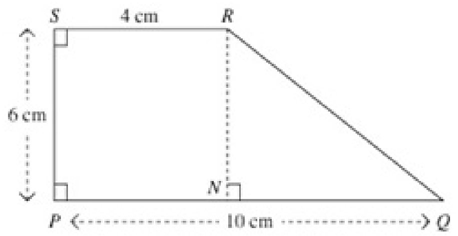
b The diagram shows the lamina on a rough inclined plane of angle α.



Given that the lamina is about to topple about the point P, find the value of α, giving your answer to 3 s.f.

Solution:

a



Centre of mass of ΔRNQ has position vector

Taking PQ and PS as axes.

$$\frac{1}{3}\left\{\begin{pmatrix}4\\0\end{pmatrix}+\begin{pmatrix}10\\0\end{pmatrix}+\begin{pmatrix}4\\6\end{pmatrix}\right\}=\begin{pmatrix}6\\2\end{pmatrix}$$

Use $\Sigma m_i \mathbf{r}_i = \bar{\mathbf{r}}\Sigma m_i$.

$$24\begin{pmatrix}2\\3\end{pmatrix}+18\begin{pmatrix}6\\2\end{pmatrix}=42\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

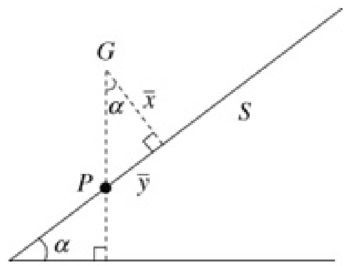
$$\begin{pmatrix}48\\72\end{pmatrix}+\begin{pmatrix}108\\36\end{pmatrix}=42\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}156\\108\end{pmatrix}=42\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}\frac{26}{7}\\\frac{18}{7}\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

b



G will be above P, on the point of toppling.

$$\tan\alpha = \frac{\bar{y}}{\bar{x}}$$

$$= \frac{\frac{18}{7}}{\frac{26}{7}}$$

$$= \frac{9}{13} = 0.6924$$

$$\alpha = 34.7°$$

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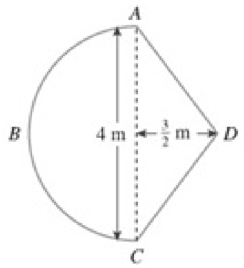
Exercise G, Question 1

Question:

The diagram shows a uniform lamina consisting of a semi-circle joined to a triangle ADC.



The sides AD and DC are equal.

a Find the distance of the centre of mass of the lamina from AC.

The lamina is freely suspended from A and hangs at rest.

b Find, to the nearest degree, the angle between AC and the vertical.

The mass of the lamina is M. A particle P of mass kM is attached to the lamina at D. When suspended from A, the lamina now hangs with its axis of symmetry, BD, horizontal.

c Find, to 3 s.f., the value of k.

Solution:

a

$$\frac{\pi\times 2^2}{2}\times\left(\frac{4\times 2}{3\pi}\right)+2\times\frac{3}{2}\times\left(-\frac{1}{2}\right)=\left(\frac{\pi\times 2^2}{2}+2\times\frac{3}{2}\right)\bar{x}$$

$$\frac{16}{3}-\frac{3}{2}=(2\pi+3)\bar{x}$$

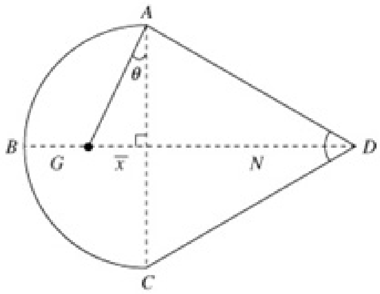
$$\frac{23}{6(2\pi+3)}=\bar{x}$$

Use $\Sigma m_i x_i = \bar{x}\Sigma m_i$ taking AC as the y-axis.

0.413m (3 s.f.)

A decimal answer is acceptable.

b



G is the centre of mass

G will be on the line of symmetry.

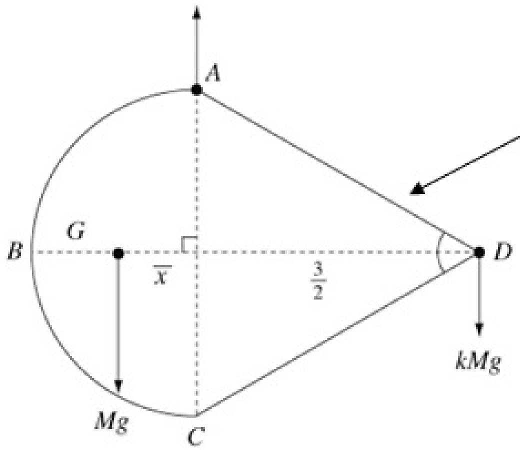
θ is the required angle

In equilibrium, AG will be vertical.

$$\tan\theta = \frac{\bar{x}}{2}=\frac{23}{12(2\pi+3)}$$

$$\theta = 13^\circ \text{ (nearest degree)}$$

c



Draw a diagram showing all the forces.

M(A),

$$Mg\bar{x} = kMg\times\frac{3}{2}$$

$$\Rightarrow k=\frac{2}{3}\times\frac{23}{6(2\pi+3)}$$

$$=\frac{23}{9(2\pi+3)}=0.275 \text{ (3 s.f.)}$$

Taking moments about A means we don't need to know the force A.

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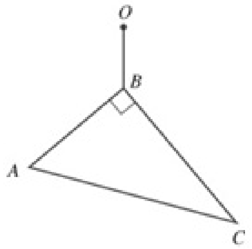
Exercise G, Question 2

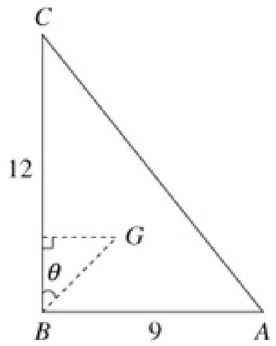
Question:

A uniform triangular lamina ABC is in equilibrium, suspended from a fixed point O by a light inextensible string attached to the point B of the lamina, as shown in the diagram.



Given that $AB = 9$ cm, $BC = 12$ cm and $A\hat{B}C = 90°$, find the angle between BC and the downward vertical.

Solution:



A is (9, 0)
B is (0, 0)
C is (0, 12)
then G is (3, 4)

Take BA and BC as axes.

Take the mean of the 3 points.

G will be vertically below B.

In equilibrium, BG will be vertical.

Hence required angle is $G\hat{B}C = \theta$.

$\tan\theta = \frac{3}{4} \Rightarrow \theta = 36.9°$.

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Exercise G, Question 3

Question:

Four particles P, Q, R and S of masses 3 kg, 5 kg, 2 kg and 4 kg are placed at the points (1, 6), $(-1,5)$, $(2,-3)$ and $(-1,-4)$ respectively. Find the coordinates of the centre of mass of the particles.

Solution:

$$3\begin{pmatrix}1\\6\end{pmatrix}+5\begin{pmatrix}-1\\5\end{pmatrix}+2\begin{pmatrix}2\\-3\end{pmatrix}+4\begin{pmatrix}-1\\-4\end{pmatrix}=(3+5+2+4)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\Sigma m_i\mathbf{r}_i=\bar{\mathbf{r}}\Sigma m_i$.

$$\begin{pmatrix}3\\18\end{pmatrix}+\begin{pmatrix}-5\\25\end{pmatrix}+\begin{pmatrix}4\\-6\end{pmatrix}+\begin{pmatrix}-4\\-16\end{pmatrix}=14\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}-2\\21\end{pmatrix}=14\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Hence, coordinates of the centre of mass are $\left(-\frac{1}{7},\frac{3}{2}\right)$.

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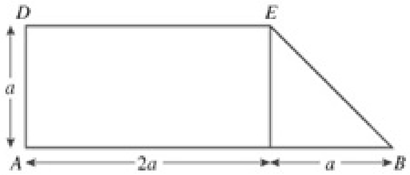
Exercise G, Question 4

Question:

A uniform rectangular piece of card $ABCD$ has $AB = 3a$ and $BC = a$. One corner of the rectangle is folded over to form a trapezium $ABED$ as shown in the diagram:



Find the distance of the centre of mass of the trapezium from

a AD,

b AB.

The lamina $ABED$ is freely suspended from E and hangs at rest.

c Find the angle between DE and the horizontal.

The mass of the lamina is M. A particle of mass m is attached to the lamina at the point B. The lamina is freely suspended from E and it hangs at rest with AB horizontal.

d Find m in terms of M.

Solution:

Taking AB and AD as axes:

$$2a^2\begin{pmatrix} a \\ \frac{1}{2}a \end{pmatrix} + 2\times\frac{1}{2}a^2\begin{pmatrix} \frac{7a}{3} \\ \frac{a}{3} \end{pmatrix} = 3a^2\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Centre of mass of the *two* triangles.

$$\begin{pmatrix} 2a \\ a \end{pmatrix} + \begin{pmatrix} \frac{7a}{3} \\ \frac{a}{3} \end{pmatrix} = 3\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

$$\frac{1}{3}\left\{\begin{pmatrix} 2a \\ 0 \end{pmatrix} + \begin{pmatrix} 3a \\ 0 \end{pmatrix} + \begin{pmatrix} 2a \\ a \end{pmatrix}\right\} = \begin{pmatrix} \frac{7a}{3} \\ \frac{a}{3} \end{pmatrix}$$

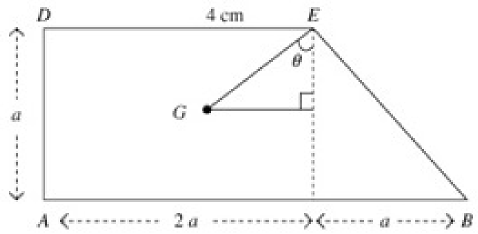
$$\begin{pmatrix} \frac{13a}{9} \\ \frac{4a}{9} \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

.

a Distance from AD is $\frac{13a}{9}$.

b Distance from AB is $\frac{4a}{9}$.

c



EG will be vertical in equilibrium.

θ is the required angle

$D\hat{E}G$ is the angle between DE and the vertical so $(90^\circ - D\hat{E}G)$ will be the angle between DE and the horizontal.

$$\tan\theta = \frac{2a-\bar{x}}{a-\bar{y}}$$

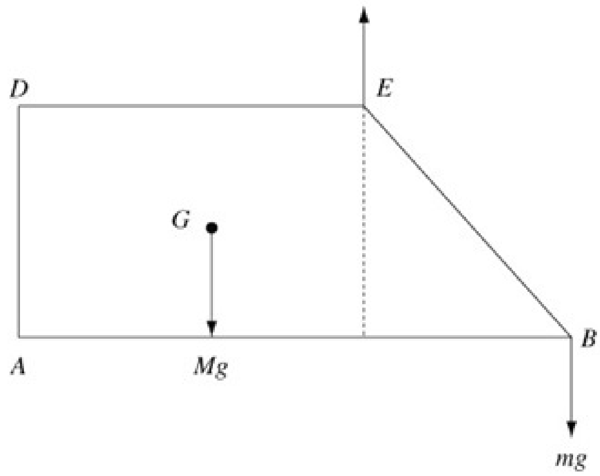
$$= \frac{2a - \frac{13a}{9}}{a - \frac{4a}{9}}$$

$$= \frac{18-13}{9-4}$$

$$= 1$$

So, θ is 45°

d



$$M(\mathrm{E}),\ Mg(2a-\bar{x}) = mga$$
$$M\frac{5a}{9} = ma$$
$$\text{i.e. } m = \frac{5M}{9}$$

Take moments about E to give an equation relating M and m.

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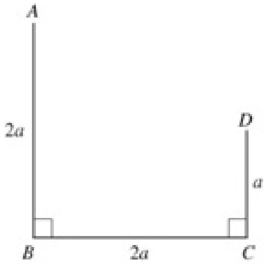
Exercise G, Question 5

Question:

A thin uniform wire of length $5a$ is bent to form the shape $ABCD$, where $AB = 2a$, $BC = 2a$, $CD = a$ and BC is perpendicular to both AB and CD, as shown in the diagram:



a Find the distance of the centre of mass of the wire from

i AB,

ii BC.

The wire is freely suspended from B and hangs at rest.

b Find, to the nearest degree, the angle between AB and the vertical.

Solution:

a Taking axes BC and BA:

$$2a\begin{pmatrix}0\\a\end{pmatrix}+2a\begin{pmatrix}a\\0\end{pmatrix}+a\begin{pmatrix}2a\\\frac{1}{2}a\end{pmatrix}=5a\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}0\\2a\end{pmatrix}+\begin{pmatrix}2a\\0\end{pmatrix}+\begin{pmatrix}2a\\\frac{1}{2}a\end{pmatrix}=5\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

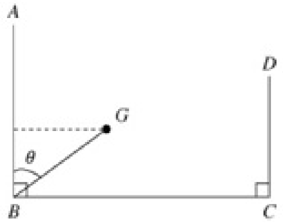
$$\begin{pmatrix}4a\\\frac{5a}{2}\end{pmatrix}=5\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}\frac{4a}{5}\\\frac{a}{2}\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Take axes through the point B, and use $\Sigma m_i\mathbf{r}_i = \bar{\mathbf{r}}\Sigma m_i$.

i $\frac{4a}{5}$ **ii** $\frac{a}{2}$

b



BG will be vertical when the wire hangs in equilibrium.

θ is the required angle.

$$\tan\theta = \frac{\bar{x}}{\bar{y}}$$

$$= \frac{4a}{5}\times\frac{2}{a}=\frac{8}{5}$$

$$\Rightarrow \theta = 58^\circ \text{ (nearest degree)}$$

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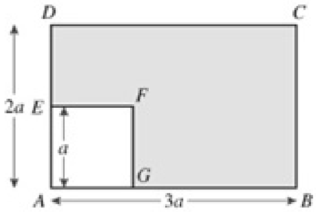
Exercise G, Question 6

Question:

A uniform lamina consists of a rectangle $ABCD$, where $AB = 3a$ and $AD = 2a$, with a square hole $EFGA$, where $EF = a$, as shown in the diagram:

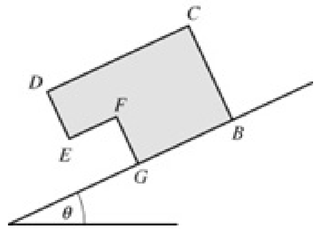


a Find the distance of the centre of mass of the lamina from

i AD,

ii AB.

The lamina is balanced on a rough plane inclined to the horizontal at an angle θ. The plane of the lamina is vertical and the inclined plane is sufficiently rough to prevent the lamina from slipping. The side GB is in contact with the plane with G lower than B, as shown in the diagram:



b Find, in degrees to 1 decimal place, the greatest value of θ for which the lamina can rest in equilibrium without toppling.

Solution:

a Taking AB and AD as axes:

$$6a^2\begin{pmatrix}\frac{3a}{2}\\ a\end{pmatrix} - a^2\begin{pmatrix}\frac{1}{2}a\\ \frac{1}{2}a\end{pmatrix} = 5a^2\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

Treat the lamina as a rectangle with a square removed.

$$\begin{pmatrix}9a\\ 6a\end{pmatrix} - \begin{pmatrix}\frac{1}{2}a\\ \frac{1}{2}a\end{pmatrix} = 5\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\begin{pmatrix}1.7a\\ 1.1a\end{pmatrix} = \begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

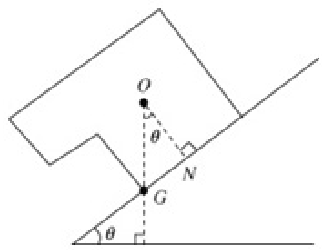
Simplify.

i $1.7a$

ii $1.1a$

State your answers.

b



At critical point, the centre of mass, O, will be vertically above the point G.

$$\tan\theta = \frac{GN}{ON}$$

$$= \frac{\bar{x}-a}{\bar{y}}$$

Since $\bar{x}$ was measured from the point A.

$$= \frac{1.7a-a}{1.1a}$$

Substitute for $\bar{x}$.

$$= \frac{0.7}{1.1}$$

$$= \frac{7}{11}$$

Simplify.

So, $\theta = 32.5°$ (1d.p.)

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Exercise A, Question 1

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

Calculate the work done by a horizontal force of magnitude 0.6 N which pulls a particle a distance of 4.2 m across a horizontal floor.

Solution:

$$\begin{aligned}\text{Work done} &= F \times s\\ &= 0.6 \times 4.2\\ &= 2.52\end{aligned}$$

The work done is 2.52 J.

Solutionbank M2

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Exercise A, Question 2

Question:

A box is pulled 12 m across a smooth horizontal floor by a constant horizontal force. The work done by the force is 102 J. Calculate the magnitude of the force.

Solution:

$$\text{Work done} = F \times s$$

$$102 = F \times 12$$

$$F = \frac{102}{12} = 8.5$$

The magnitude of the force is 8.5 N.

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Exercise A, Question 3

Question:

Calculate the work done against gravity when a particle of mass 0.35 kg is raised a vertical distance of 7 m.

Solution:

$$\begin{aligned}\text{Work done against gravity} &= mgh \\ &= 0.35\times9.8\times7 \\ &= 24.01\end{aligned}$$

The work done against gravity is 24.0 J.

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Exercise A, Question 4

Question:

A crate of mass 15 kg is raised through a vertical distance of 4 m. Calculate the work done against gravity.

Solution:

$$\begin{aligned}\text{Work done against gravity} &= mgh\\ &= 15\times9.8\times4\\ &= 588\end{aligned}$$

The work done against gravity is 588 J.

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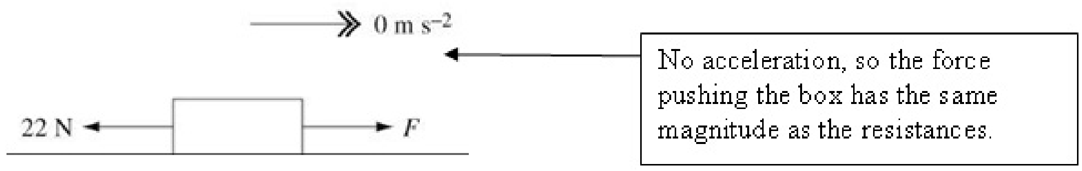
Exercise A, Question 5

Question:

A box is pushed 15 m across a horizontal surface. The box moves at a constant speed and the resistances to motion total 22 N. Calculate the work done by the force pushing the box.

Solution:



$F = 22$

$$\begin{aligned}\text{Work done} &= F \times s \\ &= 22 \times 15 \\ &= 330\end{aligned}$$

The work done by the force pushing the box is 330 N.

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Exercise A, Question 6

Question:

A ball of mass 0.5 kg falls vertically 15 m from rest. Calculate the work done by gravity.

Solution:

$$\begin{aligned}\text{Work done by gravity} &= mgh \\ &= 0.5 \times 9.8 \times 15 \\ &= 73.5\end{aligned}$$

The work done by gravity is 73.5 J.

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Exercise A, Question 7

Question:

A cable is attached to a crate of mass 80 kg. The crate is raised vertically at a constant speed from the ground to the top of a building. The work done in raising the crate is 30 kJ. Calculate the height of the building.

Solution:

$$\text{Work done} = mgh$$
$$30 \times 1000 = 80 \times 9.8\,h$$
$$h = \frac{30\,000}{80 \times 9.8}$$
$$h = 38.26$$

$\boxed{1\,\text{kJ} = 1000\,\text{J}}$

The building is 38.3 m high.

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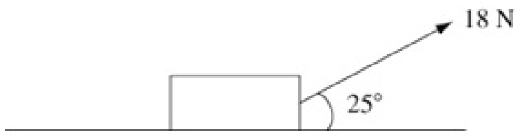
Exercise A, Question 8

Question:

A sledge is pulled 14 m across a horizontal sheet of ice by a rope inclined at $25°$ to the horizontal. The tension in the rope is 18 N and the ice can be assumed to be a smooth surface. Calculate the work done.

Solution:



$$\text{Work done} = 18\cos 25° \times 14$$
$$= 228.3$$

Work done = *horizontal* component of tension × *horizontal* distance moved.

The work done is 228 J.

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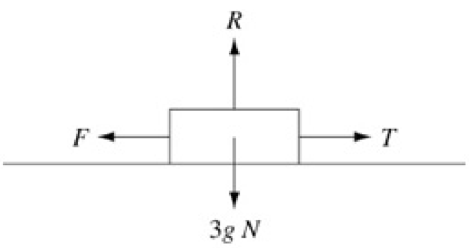
Exercise A, Question 9

Question:

A parcel of mass 3 kg is pulled at a distance of 4 cm across a rough horizontal floor. The parcel moves at a constant speed. The work done against friction is 30 J. Calculate the coefficient of friction between the parcel and the surface.

Solution:



$$\text{Work done} = F \times s$$
$$30 = T \times 4$$
$$T = 7.5$$
$$F = ma$$
$$7.5 - F = 0$$
$$F = 7.5$$

The parcel moves at a constant speed so the acceleration is 0 m s^{-2}.

$$\text{R}(\uparrow) \quad R = mg$$
$$R = 3 \times 9.8$$

Friction is limiting $\quad F = \mu R$

$$7.5 = \mu \times 3 \times 9.8$$
$$\mu = \frac{7.5}{3 \times 9.8} = 0.2551$$

The coefficient of friction is 0.255.

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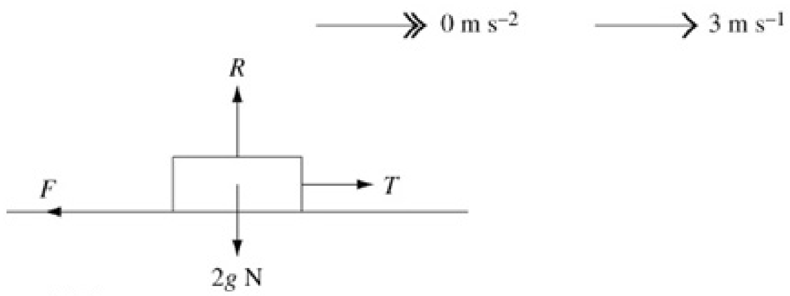
Exercise A, Question 10

Question:

A block of wood of mass 2 kg is pushed across a rough horizontal floor. The block moves at 3 m s^{-1} and the coefficient of friction between the block and the floor is 0.55. Calculate the work done in 2 seconds.

Solution:



$\mu = 0.55$

$R(\uparrow) \quad R = 2g$

Friction is limiting

$$
\begin{aligned}
F &= \mu R \\
F &= 0.55 \times 2g \\
R(\rightarrow) \quad T - F &= 0 \\
T &= 0.55 \times 2g \\
\text{Work done} &= F \times s \\
&= 0.55 \times 2g \times (3 \times 2) \\
&= 0.55 \times 2 \times 9.8 \times 6 \\
&= 64.68
\end{aligned}
$$

Distance moved = speed × time

The work done is 64.7 J.

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Exercise A, Question 11

Question:

A girl of mass 52 kg climbs a vertical cliff which is 46 m high. Calculate the work she does against gravity.

Solution:

$$\begin{aligned}\text{Work done against gravity} &= mgh \\ &= 52 \times 9.8 \times 46 \\ &= 23\,441\end{aligned}$$

The work done against gravity is 23 400 J.

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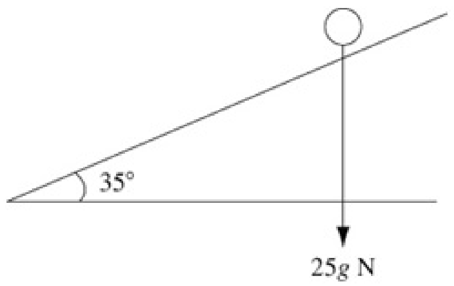
Exercise A, Question 12

Question:

A child of mass 25 kg slides 2 m down a smooth slope inclined at 35° to the horizontal. Calculate the work done by gravity.

Solution:



$$\begin{aligned}\text{Work done by gravity} &= mgh\\ &= 25\times 9.8\times(2\sin 35^\circ)\\ &= 281.0\end{aligned}$$

Vertical distance moved $= 2\sin 35^\circ$

The work done by gravity is 281 J.

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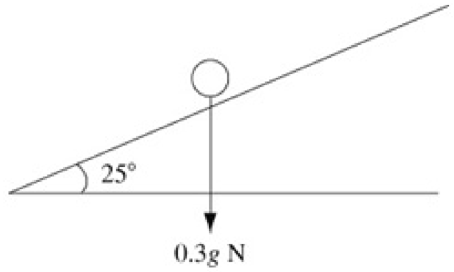
Exercise A, Question 13

Question:

A particle of mass 0.3 kg is pulled 2 m up a line of greatest slope of a plane which is inclined at $25°$ to the horizontal. Assuming that the particle moves along a line of greatest slope of the plane, calculate the work done against gravity.

Solution:



$$\begin{aligned}\text{Work done against gravity} &= mgh\\ &= 0.3\times 9.8\times(2\sin 25°)\\ &= 2.484\end{aligned}$$

The work done against gravity is 2.48 J.

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Exercise A, Question 14

Question:

A rough plane surface is inclined at an angle arcsin $\frac{5}{13}$ to the horizontal. A packet of mass 8 kg is pulled at a constant speed up a line of greatest slope of the plane. The coefficient of friction between the packet and the plane is 0.3.

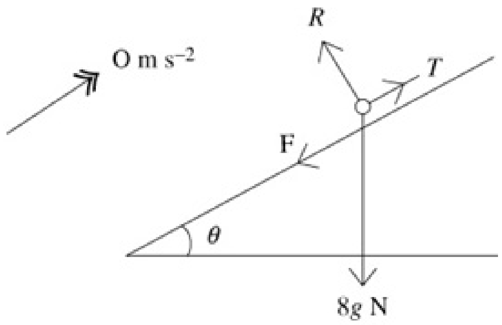
a Calculate the magnitude of the frictional force acting on the packet.

The packet moves a distance of 15 m up the plane. Calculate

b the work done against friction,

c the work done against gravity.

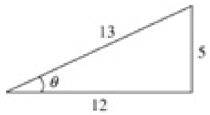
Solution:



a

R(↖) $R = 8g\cos\theta$

$= 8g \times \frac{12}{13}$



$\mu = 0.3$

Draw a small right-angled triangle to show information about θ. Use exact values for $\sin\theta$ and $\cos\theta$.

Friction is limiting:

$F = \mu R$

$F = 0.3 \times 8 \times 9.8 \times \frac{12}{13}$

$= 21.71$

The frictional force has magnitude 21.7 N.

b

$$\begin{aligned}\text{Work done against friction} &= F \times s \\ &= 21.71 \times 15 \\ &= 325.6\end{aligned}$$

The work done against friction is 326 J.

c

$$\begin{aligned}\text{Work done against gravity} &= mgh \\ &= 8 \times 9.8 \times (15\sin\theta) \\ &= 8 \times 9.8 \times \left(15 \times \frac{5}{13}\right) \\ &= 452.3\end{aligned}$$

The work done against gravity is 452 J.

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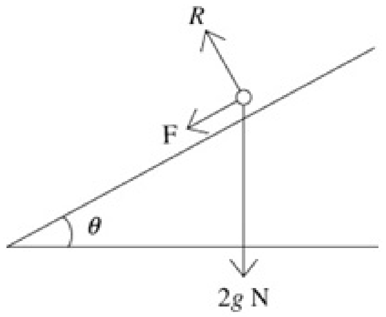
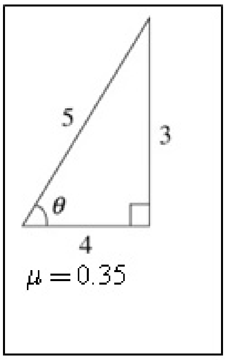
Exercise A, Question 15

Question:

A particle P of mass 2 kg is projected up a line of greatest slope of a rough plane which is inclined at an angle $\arcsin \frac{3}{5}$ to the horizontal. The coefficient of friction between P and the plane is 0.35. The particle travels 3 m up the plane. Calculate the work done by friction.

Solution:





R(↖) $R = 2g\cos\theta$

$R = 2\times 9.8\times\frac{4}{5}$

Friction is limiting

$F = \mu R$

$F = 0.35\times 2\times 9.8\times\frac{4}{5}$

$$\text{Work done} = F\times s$$
$$= 0.35\times 2\times 9.8\times\frac{4}{5}\times 3$$
$$= 16.46$$

The work done by friction is 16.5 J.

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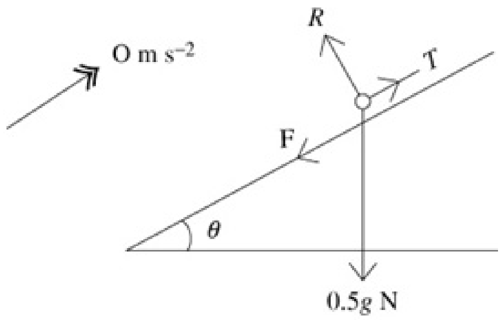
Exercise A, Question 16

Question:

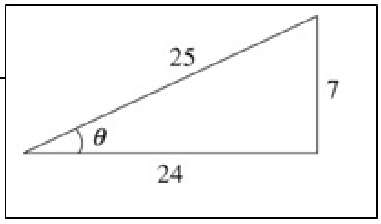
A rough surface is inclined at an angle arcsin $\frac{7}{25}$ to the horizontal. A particle of mass 0.5 kg is pulled 3 m at a constant speed up the surface by a force acting along a line of greatest slope. The only resistances to the motion are those due to friction and gravity. The work done by the force is 12 J. Calculate the coefficient of friction between the particle and the surface.

Solution:



R(↖) $R = 0.5g\cos\theta$

$= 0.5g \times \frac{24}{25}$



R(↗)

$F + 0.5g\sin\theta = T$

Friction is limiting: $F = \mu R$

$F = \mu \times 0.5g \times \frac{24}{25}$

$T = \mu \times 0.5g \times \frac{24}{25} + 0.5g \times \frac{7}{25}$

Work done by force $= F \times s$

$12 = T \times 3$

$T = 4$

$\therefore 4 = \mu \times 0.5g \times \frac{24}{25} + 0.5g \times \frac{7}{25}$

$$\mu = \frac{4 - 0.5 \times 9.8 \times \frac{7}{25}}{0.5 \times 9.8 \times \frac{24}{25}}$$

$\mu = 0.5586$

The coefficient of friction is 0.559.

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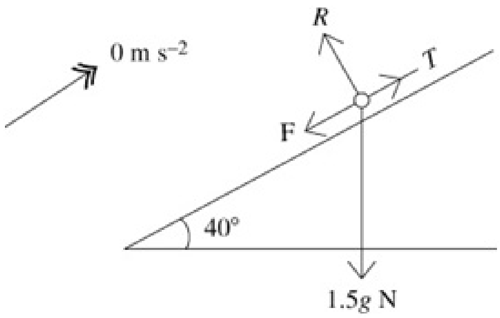
Exercise A, Question 17

Question:

A rough surface is inclined at $40°$ to the horizontal. A particle of mass 1.5 kg is pulled at a constant speed up the surface by a force T acting along a line of greatest slope. The coefficient of friction between the particle and the surface is 0.4. Calculate the work done by T when the particle travels 8 m. You may assume that the only resistances to motion are due to gravity and friction.

Solution:



$\mu = 0.4$

R(↖) $R = 1.5g\cos 40°$

Friction is limiting: $F = \mu R$

$F = 0.4 \times 1.5g\cos 40°$

R(↗)

$$T = F + 1.5g\sin 40°$$
$$T = 0.4 \times 1.5g\cos 40° + 1.5g\sin 40°$$

Work done by $T = T \times s$

$= (0.4 \times 1.5g\cos 40° + 1.5g\sin 40°) \times 8$

$= 111.6$

The work done by T is 112 J.

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Exercise B, Question 1

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.

Calculate the kinetic energy of:

a a particle of mass 0.3 kg moving at 15 m s^{-1}

b a particle of mass 3 kg moving at 2 m s^{-1}

c a box of mass 5 kg moving at 7.5 m s^{-1}

d an arrow of mass 0.5 kg moving at 200 m s^{-1}

e a boy of mass 25 kg running at 4 m s^{-1}

f a ball of mass 0.4 kg moving at 15 m s^{-1}

g a car of mass 800 kg moving at 20 m s^{-1}

Solution:

a $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.3 \times 15^2 = 33.75 = 33.8\text{ J}$

b $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 3 \times 2^2 = 6\text{ J}$

c $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 5 \times 7.5^2 = 140.625 = 141\text{ J}$

d $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.5 \times 200^2 = 10\,000\text{ J}$

e $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 25 \times 4^2 = 200\text{ J}$

f $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.4 \times 15^2 = 45\text{ J}$

g $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 800 \times 20^2 = 160\,000\text{ J}$

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Exercise B, Question 2

Question:

Find the change in potential energy of each of the following, stating in each case whether it is a loss or a gain:

a a particle or mass 1.5 kg raised through a vertical distance of 3 m

b a woman of mass 55 kg ascending a vertical distance of 15 m

c a man of mass 75 kg descending a vertical distance of 30 m

d a lift of mass 580 kg descending a vertical distance of 6 m

e a man of mass 70 kg ascending a vertical distance of 36 m

f a ball of mass 0.6 kg falling a vertical distance of 12 m

g a lift of mass 800 kg ascending a vertical distance of 16 m

Solution:

a gain of P.E. $= mgh = 1.5 \times 9.8 \times 3 = 44.1 \text{ J}$

b gain of P.E. $= mgh = 55 \times 9.8 \times 15 = 8085 \text{ J}$

c loss of P.E. $= mgh = 75 \times 9.8 \times 30 = 22\,050 \text{ J}$

d loss of P.E. $= mgh = 580 \times 9.8 \times 6 = 34\,104 \text{ J}$

e gain of P.E. $= mgh = 70 \times 9.8 \times 36 = 24\,696 \text{ J}$

f loss of P.E. $= mgh = 0.6 \times 9.8 \times 12 = 70.56 = 70.6 \text{ J}$

g gain of P.E. $= mgh = 800 \times 9.8 \times 16 = 125\,440 \text{ J}$

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Exercise B, Question 3

Question:

A particle of mass 1.2 kg decreases its speed from 12 m s^{-1} to 4 m s^{-1}. Calculate the decrease in the particle's kinetic energy.

Solution:

$$\begin{aligned}\text{Decrease in K.E.} &= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 \\ &= \frac{1}{2}\times 1.2\times 12^2 - \frac{1}{2}\times 1.2\times 4^2 \\ &= 76.8\end{aligned}$$

The decrease in the particle's K.E. is 76.8 J.

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Exercise B, Question 4

Question:

A van of mass 900 kg increases its speed from 5 m s^{-1} to 20 m s^{-1}. Calculate the increase in the van's kinetic energy.

Solution:

$$\begin{aligned}\text{Increase in K.E.} &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \\ &= \frac{1}{2}\times 900\times 20^2 - \frac{1}{2}\times 900\times 5^2 \\ &= 168\,750\end{aligned}$$

The increase in the van's K.E. is 168 750 J.

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Exercise B, Question 5

Question:

A particle of mass 0.2 kg increases its speed from 2 m s^{-1} to $v\text{ m s}^{-1}$. The particle's kinetic energy increases by 6 J. Calculate the value of v.

Solution:

$$\begin{aligned}
\text{Increase in K.E.} &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \\
6 &= \frac{1}{2}\times 0.2\times v^2 - \frac{1}{2}\times 0.2\times 2^2 \\
6 &= 0.1v^2 - 0.4 \\
v^2 &= \frac{6.4}{0.1} = 64 \\
v &= 8 \quad (v > 0)
\end{aligned}$$

The value of v is 8.

Solutionbank M2

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Exercise B, Question 6

Question:

An ice-skater of mass 45 kg is initially moving at 5 m s^{-1}. She decreases her kinetic energy by 100 J. Calculate her final speed.

Solution:

$$\text{Decrease in K.E.} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

$$100 = \frac{1}{2}\times 45 \times 5^2 - \frac{1}{2}\times 45 v^2$$

$$100 = 562.5 - 22.5v^2$$

$$v^2 = \frac{462.5}{22.5}$$

$$v = \pm 4.533$$

$$v = 4.533 \quad (v > 0)$$

Speed is positive.

The skater's final speed is 4.53 m s^{-1}.

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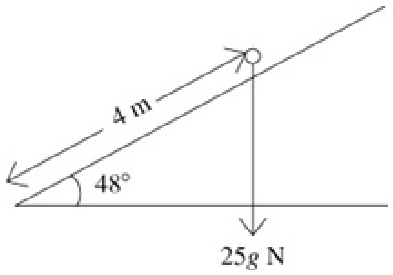
Exercise B, Question 7

Question:

A playground side is a plane inclined at $48°$ to the horizontal. A child of mass 25 kg slides down the slide for 4 m. Calculate the potential energy lost by the child.

Solution:



$$\begin{aligned}\text{P.E. lost} &= mgh\\ &= 25\times 9.8\times(4\sin 48°)\\ &= 728.2\end{aligned}$$

Vertical distance moved is $4\sin 48°$.

The P.E. lost by the child is 728 J.

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Exercise B, Question 8

Question:

A ball of mass 0.6 kg is dropped from a height of 2 m into a pond.

a Calculate the kinetic energy of the ball as its hits the surface of the water.

The ball begins to sink in the water with a speed of 4.8 m s^{-1}.

b Calculate the kinetic energy lost when the ball strikes the water.

Solution:

a

$s = 2 \text{ m}$

$a = 9.8 \text{ m s}^{-2}$

$u = 0$

$v = ?$

$$v^2 = u^2 + 2as$$
$$v^2 = 0 + 2 \times 9.8 \times 2$$
$$v^2 = 39.2$$

Use $v^2 = u^2 + 2as$ to find the speed of the ball as it hits the water.

$$\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.6 \times 39.2$$
$$= 11.76$$

The K.E. of the ball as it hits the surface of the water is 11.8 J.

b

$$\text{K.E. lost} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$
$$= 11.76 - \frac{1}{2} \times 0.6 \times 4.8^2$$
$$= 4.848$$

The K.E. lost by the ball is 4.85 J.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 9

Question:

A lorry of mass 2000 kg is initially travelling at 35 m s^{-1}. The brakes are applied, causing the lorry to decelerate at 1.2 m s^{-2} for 5 s. Calculate the loss of kinetic energy of the lorry.

Solution:

$$u = 35 \text{ m s}^{-1}$$
$$a = -1.2 \text{ m s}^{-2}$$
$$t = 5 \text{ s}$$
$$v = ?$$

$$v = u + at$$
$$v = 35 - 1.2 \times 5$$
$$v = 29$$

Use $v = u + at$ to find the final speed of the lorry.

$$\text{Loss of K.E.} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$
$$= \frac{1}{2} \times 2000 \times 35^2 - \frac{1}{2} \times 2000 \times 29^2$$
$$= 384\,000$$

The loss of K.E. of the lorry is 384 000 J.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 10

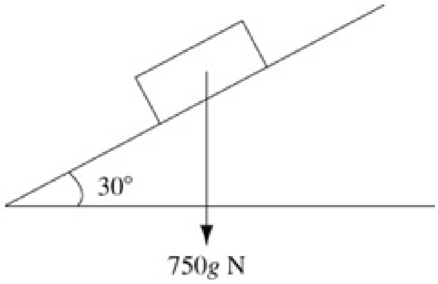
Question:

A car of mass 750 kg moves along a stretch of road which can be modelled as a line of greatest slope of a plane inclined to the horizontal at $30°$. As the car moves up the road for 500 m its speed reduces from 20 m s^{-1} to 15 m s^{-1}. Calculate

a the loss of kinetic energy of the car,

b the gain of potential energy of the car.

Solution:



a

$$\text{Loss of K.E.} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$
$$= \frac{1}{2}\times 750\times 20^2 - \frac{1}{2}\times 750\times 15^2$$
$$= 65\,625$$

The loss of K.E. of the car is 65 625 J.

b

$$\text{Gain of P.E.} = mgh$$
$$= 750\times 9.8\times(500 \sin 30°)$$
$$= 1837\,500$$

The gain of P.E. of the car is 1 837 500 J.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 11

Question:

A man of mass 80 kg climbs a vertical cliff face of height h m. His potential energy increases by 15.7 kJ. Calculate the height of the cliff.

Solution:

$$\text{Increase of P.E.} = mgh$$

$$15.7 \times 1000 = 80 \times 9.8\, h \qquad \boxed{1\,\text{kJ} = 1000\ \text{J}}$$

$$h = \frac{15.7 \times 1000}{80 \times 9.8}$$

$$h = 20.02$$

The cliff is 20.0 m high.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 1

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A particle of mass 0.4 kg falls a vertical distance of 7 m from rest.

a Calculate the potential energy lost.

b By assuming that air resistance can be neglected, calculate the final speed of the particle.

Solution:

a

$$\text{P.E. lost} = mgh = 0.4 \times 9.8 \times 7$$
$$= 27.44$$

The P.E. lost is 27.4 J.

b

$$\text{K.E. gained} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$
$$= \frac{1}{2} \times 0.4 \times v^2 - 0$$
$$\text{P.E. lost} = \text{K.E. gained}$$
$$27.44 = \frac{1}{2} \times 0.4 \times v^2$$
$$v^2 = \frac{27.44}{0.2}$$
$$v = 11.71$$

The final speed of the particle is 11.7 m s^{-1}.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 2

Question:

A stone of mass 0.5 kg is dropped from the top of a tower and falls vertically to the ground. It hits the ground with a speed of 12 m s^{-1}. Find

a the kinetic energy gained by the stone,

b the potential energy lost by the stone,

c the height of the tower.

Solution:

a

$$\begin{aligned}\text{K.E. gained} &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \\ &= \frac{1}{2}\times 0.5\times 12^2 - 0 \\ &= 36\end{aligned}$$

The K.E. gained by the stone is 36 J.

b

$$\begin{aligned}\text{P.E. lost} &= \text{K.E. gained} \\ &= 36\text{ J}\end{aligned}$$

P.E. lost = K.E. gained

The P.E. lost by the stone is 36 J.

c

$$\begin{aligned}\text{P.E. lost} &= mgh \\ 36 &= 0.5\times 9.8\times h \\ h &= \frac{36}{0.5\times 9.8} \\ h &= 7.346\end{aligned}$$

The height of the tower is 7.35 m.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 3

Question:

A box of mass 6 kg is pulled in a straight line across a smooth horizontal floor by a constant horizontal force of magnitude 10 N. The box has speed 2.5 m s^{-1} when it passes through point P and speed 5 m s^{-1} when it passes through point Q.

a Find the increase in kinetic energy of the box.

b Write down the work done by the force.

c Find the distance PQ.

Solution:

a

$$\begin{aligned}\text{Increase in K.E.} &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2\\ &= \frac{1}{2}\times 6\times 5^2 - \frac{1}{2}\times 6\times 2.5^2\\ &= 56.25\end{aligned}$$

The increase in K.E. of the box is 56.3 J.

b The work done by the force is 56.3 J.

c

$$\begin{aligned}\text{Work done} &= F\times s\\ 56.25 &= 10\times s\\ s &= \frac{56.25}{10} = 5.625\end{aligned}$$

Work done = change in energy

The distance PQ is 5.63 m.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 4

Question:

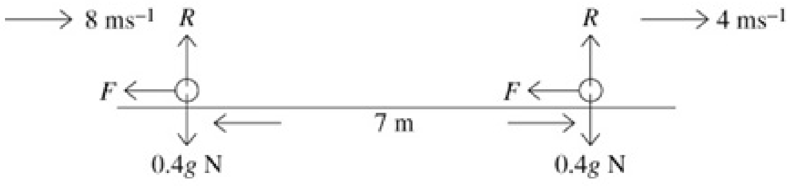
A particle of mass 0.4 kg moves in a straight line across a rough horizontal surface. The speed of the particle decreases from 8 m s^{-1} to 4 m s^{-1} as it travels 7 m.

a Calculate the kinetic energy lost by the particle.

b Write down the work done against friction.

c Calculate the coefficient of friction between the particle and the surface.

Solution:



a

$$\begin{aligned}\text{K.E. lost} &= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 \\ &= \frac{1}{2}\times 0.4\times 8^2 - \frac{1}{2}\times 0.4\times 4^2 \\ &= 9.6\end{aligned}$$

The K.E. lost by the particle is 9.6 J.

b The work done against friction is 9.6 J.

Work done = change in energy

c

$$\begin{aligned}\text{R}(\uparrow) \quad R &= 0.4g \\ \text{Friction is limiting} \quad F &= \mu R \\ F &= 0.4g\times\mu \\ \text{Work done} &= F\times s \\ 9.6 &= 0.4g\times\mu\times 7 \\ \mu &= \frac{9.6}{0.4\times 9.8\times 7} = 0.3498\end{aligned}$$

The coefficient of friction is 0.350.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 5

Question:

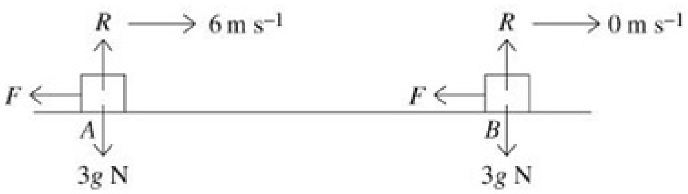
A box of mass 3 kg is projected from point A of a rough horizontal floor with speed 6 m s^{-1}. The box moves in a straight line across the floor and comes to rest at point B. The coefficient of friction between the box and the floor is 0.4.

a Calculate the kinetic energy lost by the box.

b Write down the work done against friction.

c Calculate the distance AB.

Solution:



a

$\mu = 0.4$

$$\begin{aligned} \text{K.E. lost} &= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 \\ &= \frac{1}{2} \times 3 \times 6^2 - 0 \\ &= 54 \end{aligned}$$

The kinetic energy lost by the box is 54 J.

b The work done against friction is 54 J.

c

$$\begin{aligned} \text{R}(\uparrow) \quad R &= 3g \\ \text{Friction is limiting}: \quad F &= \mu R \\ F &= 0.4 \times 3g \\ \text{Work done} &= F \times s \\ 54 &= 0.4 \times 3g \times s \\ s &= \frac{54}{0.4 \times 3g} = 4.591 \end{aligned}$$

The distance AB is 4.59 m.

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Exercise C, Question 6

Question:

A particle of mass 0.8 kg falls a vertical distance of 5 m from rest. By considering energy, find the speed of the particle as it hits the ground. (You may assume that air resistance can be neglected.)

Solution:

$$\begin{aligned}
\text{P.E. lost} &= mgh \\
&= 0.8 \times 9.8 \times 5 \\
&= 39.2 \\
\text{K.E. gained} &= \text{P.E. lost} \\
&= 39.2 \\
\text{K.E. gained} &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \\
39.2 &= \frac{1}{2} \times 0.8v^2 - 0 \\
v^2 &= \frac{39.2 \times 2}{0.8} \\
v &= 9.899
\end{aligned}$$

The particle hits the ground at 9.90 m s^{-1}.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 7

Question:

A stone of mass 0.3 kg is dropped from the top of a vertical cliff and falls freely under gravity. It hits the ground below with a speed of 20 m s^{-1}. Use energy considerations to calculate the height of the cliff.

Solution:

$$\begin{aligned}
\text{K.E. gained} &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \\
&= \frac{1}{2} \times 0.3 \times 20^2 - 0 \\
&= 60 \\
\text{P.E. lost} &= \text{K.E. gained} \\
&= 60 \\
\text{P.E. lost} &= mgh \\
60 &= 0.3 \times 9.8 \times h \\
h &= \frac{60}{0.3 \times 9.8} \\
h &= 20.40
\end{aligned}$$

The cliff is 20.4 m high.

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Edexcel AS and A Level Modular Mathematics

Exercise C, Question 8

Question:

A particle of mass 0.3 kg is projected vertically upwards and moves freely under gravity. The initial speed of the particle is u m s^{-1}. When the particle is 5 m above the point of projection its kinetic energy is 2.1 J. Calculate the value of u.

Solution:

$$\begin{aligned}
\text{P.E. gained} &= mgh \\
&= 0.3\times 9.8\times 5 \\
\text{K.E. lost} &= \text{initial K.E.} - \text{final K.E.} \\
&= \frac{1}{2}mu^2 - 2.1 \\
&= \frac{1}{2}\times 0.3u^2 - 2.1 \\
\text{K.E. lost} &= \text{P.E. gained} \\
\frac{1}{2}\times 0.3u^2 - 2.1 &= 0.3\times 9.8\times 5 \\
u^2 &= \frac{0.3\times 9.8\times 5 + 2.1}{\frac{1}{2}\times 0.3} \\
u &= 10.58 \\
u &= 10.6
\end{aligned}$$

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Exercise C, Question 9

Question:

A bullet of mass 0.1 kg travelling at 500 m s^{-1} horizontally hits a vertical wall. The bullet penetrates the wall to a depth of 50 mm. The resistive force exerted on the bullet by the wall is constant. Calculate the magnitude of the resistive force.

Solution:

$$\begin{aligned}\text{Loss of K.E.} &= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 \\ &= \frac{1}{2}\times 0.1\times 500^2 - 0\end{aligned}$$

$$\begin{aligned}\text{Work done by resistance} &= F\times s \\ &= F\times 0.05 \quad \boxed{1\text{ mm} = 0.001\text{ m}}\end{aligned}$$

$$\begin{aligned}\text{Work done by resistance} &= \text{Loss of K.E.} \\ F\times 0.05 &= \frac{1}{2}\times 0.1\times 500^2 \\ F &= \frac{\frac{1}{2}\times 0.1\times 500^2}{0.05} \\ &= 250\,000\end{aligned}$$

The magnitude of the resistive force is 250 000 N (or 250 kN).

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Edexcel AS and A Level Modular Mathematics

Exercise C, Question 10

Question:

A bullet of mass 150 g travelling at 500 m s^{-1} horizontally hits a vertical wall. The wall exerts a constant resistance of magnitude 250 000 N on the bullet. Calculate the distance the bullet penetrates the wall.

Solution:

$$\begin{aligned}
\text{Loss of K.E.} &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \\
&= \frac{1}{2} \times 0.15 \times 500^2 - 0 \\
\text{Work done by resistance} &= F \times s \\
&= 250\,000\, s \\
\text{Work done by resistance} &= \text{Loss of K.E.} \\
250\,000\, s &= \frac{1}{2} \times 0.15 \times 500^2 \\
s &= \frac{\frac{1}{2} \times 0.15 \times 500^2}{250\,000} \\
&= 0.075
\end{aligned}$$

$\boxed{15 \text{ g} = 0.15 \text{ kg}}$

The distance the bullet penetrates the wall is 0.075 m (or 75 mm).

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 11

Question:

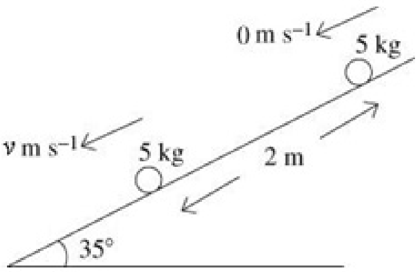
A package of mass 5 kg is released from rest and slides 2 m down a line of greatest slope of a smooth plane inclined at 35° to the horizontal.

a Calculate the potential energy lost by the package.

b Write down the kinetic energy gained by the package.

c Calculate the final speed of the package.

Solution:



a

$$\begin{aligned}\text{P.E. lost} &= mgh \\ &= 5 \times 9.8 \times (2\sin 35°) \\ &= 56.21\end{aligned}$$

The P.E. lost is 56.2 J.

b

The K.E. gained is 56.2 J.

c

$$\begin{aligned}\text{K.E. gained} &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \\ 56.2 &= \frac{1}{2} \times 5 \times v^2 - 0 \\ v^2 &= \frac{56.2 \times 2}{5} \\ v &= 4.741\end{aligned}$$

The final speed of the package is 4.74 m s^{-1}.

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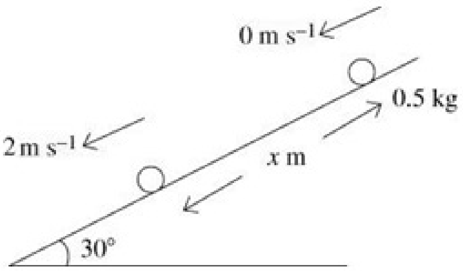
Exercise C, Question 12

Question:

A particle of mass 0.5 kg is released from rest and slides down a line of greatest slope of a smooth plane inclined at $30°$ to the horizontal. When the particle has moved a distance x m, its speed is 2 m s^{-1}. Find the value of x.

Solution:



$$\begin{aligned}\text{K.E. gained} &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2\\ &= \frac{1}{2}\times 0.5\times 2^2 - 0\\ &= 1\end{aligned}$$

$$\begin{aligned}\text{P.E. lost} &= mgh = 0.5\times 9.8\times(x\sin 30^\circ)\\ \text{P.E. lost} &= \text{K.E. gained}\\ 0.5\times 9.8\times(x\sin 30^\circ) &= 1\\ x &= \frac{1}{0.5\times 9.8\times \sin 30^\circ}\\ &= 0.4081\\ x &= 0.408\end{aligned}$$

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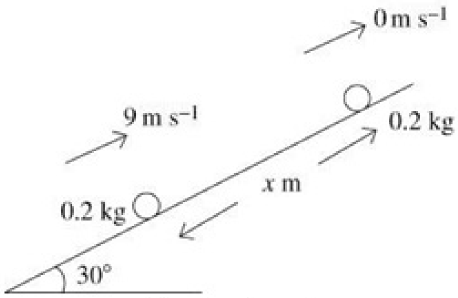
Exercise C, Question 13

Question:

A particle of mass 0.2 kg is projected with speed 9 m s^{-1} up a line of greatest slope of a smooth plane inclined at $30°$ to the horizontal. The particle travels a distance x m before first coming to rest. By considering energy, calculate the value of x.

Solution:



$$\text{K.E. lost} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$
$$= \frac{1}{2}\times 0.2\times 9^2 - 0$$
$$\text{P.E. gained} = mgh$$
$$= 0.2\times 9.8(x\sin 30°)$$
$$\text{P.E. gained} = \text{K.E. lost}$$
$$0.2\times 9.8(x\sin 30°) = \frac{1}{2}\times 0.2\times 9^2$$
$$x = \frac{\frac{1}{2}\times 0.2\times 9^2}{0.2\times 9.8\sin 30°}$$
$$= 8.265$$
$$= 8.27$$

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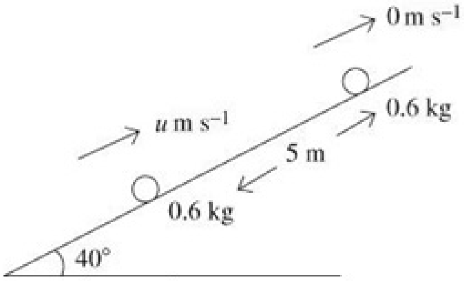
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 14

Question:

A particle of mass 0.6 kg is projected up a line of greatest slope of a smooth plane inclined at 40° to the horizontal. The particle travels 5 m before first coming to rest. Use energy considerations to calculate the speed of projection.

Solution:



$$\text{K.E. lost} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$
$$= \frac{1}{2}\times 0.6u^2 - 0$$
$$\text{P.E. gained} = mgh$$
$$= 0.6\times 9.8\times 5\sin 40^\circ$$
$$\text{K.E. lost} = \text{P.E. gained}$$
$$\frac{1}{2}\times 0.6u^2 = 0.6\times 9.8\times 5\sin 40^\circ$$
$$u^2 = \frac{0.6\times 9.8\times 5\sin 40^\circ}{\frac{1}{2}\times 0.6}$$
$$u = 7.936$$

The speed of projection is 7.94 m s^{-1}.

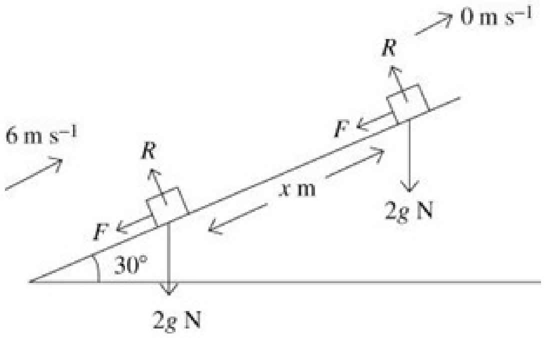
Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 15

Question:

A box of mass 2 kg is projected with speed 6 m s^{-1} up a line of greatest slope of a rough plane inclined at $30°$ to the horizontal. The coefficient of friction between the box and the plane is $\frac{1}{3}$. Use the work–energy principle to calculate the distance the box travels up the plane before first coming to rest.

Solution:



$$\text{K.E. lost} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$
$$= \frac{1}{2}\times 2\times 6^2 - 0$$
$$= 36$$
$$\text{P.E. gained} = mgh$$
$$= 2\times 9.8(x\sin 30°)$$
$$= 9.8x$$
$$\text{R}(\nwarrow) R = 2g\sin 30°$$
$$\text{Friction is limiting:} \quad F = \mu R$$
$$F = \frac{1}{3}\times 2g\sin 30° = \frac{1}{3}g$$
$$\text{Work done against friction} = \text{F}x$$
$$= \frac{1}{3}gx$$
$$\text{K.E. lost} = \text{P.E. gained} + \text{work done against } F$$
$$36 = 9.8\times x + \frac{1}{3}\times 9.8\times x$$
$$36 = \frac{4}{3}\times 9.8\times x$$
$$x = \frac{36\times 3}{4\times 9.8}$$
$$x = 2.755$$

The particle moves 2.76 m up the plane.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 16

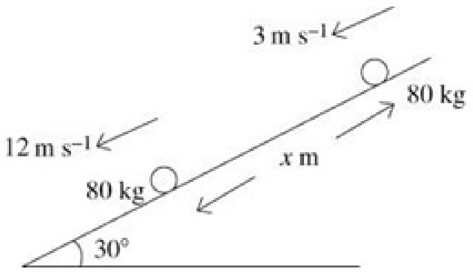
Question:

A cyclist freewheels down a hill inclined at $30°$ to the horizontal. The cyclist and his cycle have a combined mass of 80 kg. His speed increases from 3 m s^{-1} to 12 m s^{-1}. Assuming that resistances can be ignored, calculate

a the potential energy lost by the cyclist,

b the distance travelled by the cyclist.

Solution:



a

$$\text{K.E. gained} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$
$$= \frac{1}{2}\times 80\times 12^2 - \frac{1}{2}\times 80\times 3^2$$
$$= 5400$$

P.E. lost = K.E. gained

$\therefore$ The P.E. lost is 5400 J.

P.E. lost = K.E. gained

b

$$\text{P.E. lost} = mgh$$
$$5400 = 80\times 9.8\times(x\sin 30°)$$
$$x = \frac{5400}{80\times 9.8\times\sin 30°}$$
$$= 13.77$$

The cyclist travels 13.8 m.

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Exercise C, Question 17

Question:

A cyclist starts from rest and freewheels down a hill inclined at $20°$ to the horizontal. After travelling 60 m the road becomes horizontal and the cyclist travels a further 50 m before coming to rest. The cyclist and her cycle have a combined mass of 70 kg and the resistance to motion remains constant throughout. Calculate the magnitude of the resistance.

Solution:

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Exercise D, Question 1

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A force of 1500 N pulls a van up a slope at a constant speed of 12 m s^{-1}. Calculate, in kW, the power developed.

Solution:

$$\begin{aligned}\text{Power} &= F \times v \\ &= 1500 \times 12 \\ &= 18\,000\end{aligned}$$

The power is 18 kW.

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Exercise D, Question 2

Question:

A car is travelling at 15 m s^{-1} and its engine is producing a driving force of 1000 N.
Calculate the power developed.

Solution:

$$\begin{aligned}\text{Power} &= F \times v \\ &= 1000 \times 15 \\ &= 15\,000\end{aligned}$$

The power is 15 000 W (or 15 kW).

Solutionbank M2

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Exercise D, Question 3

Question:

The engine of a van is working at 5 kW and the van is travelling at 18 m s^{-1}. Find the magnitude of the driving force produced by the van's engine.

Solution:

$$\begin{aligned} \text{Power} &= F \times v \\ 5000 &= F \times 18 \\ F &= \frac{5000}{18} \\ &= 277.7 \end{aligned}$$

The driving force has magnitude 278 N.

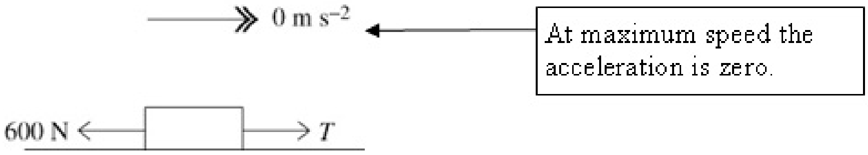
Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 4

Question:

A car's engine is working at 15 kW. The car is travelling along a horizontal road. The total resistance to motion has a magnitude of 600 N. Calculate the maximum speed of the car.

Solution:



$R(\rightarrow)\ T = 600$

$$\text{Power} = F \times v$$
$$15000 = 600v$$
$$v = \frac{15\,000}{600}$$
$$= 25$$

The maximum speed is 25 m s^{-1}.

Solutionbank M2
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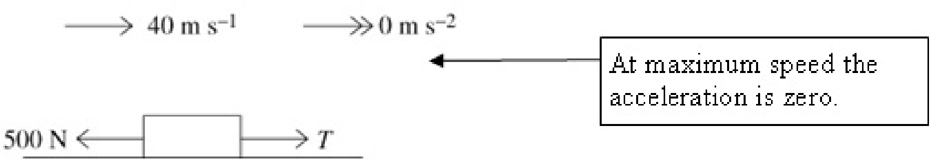
Exercise D, Question 5

Question:

A car has a maximum speed of 40 m s^{-1} when travelling along a horizontal road against a constant resistance of 500 N. Calculate the power the car's engine must develop to maintain this speed.

Solution:



$R(\rightarrow)\ T = 500$

$$\begin{aligned}\text{Power} &= F \times v\\ &= 500 \times 40\\ &= 20\,000\end{aligned}$$

The power is 20 000 W (or 20 kW).

Solutionbank M2

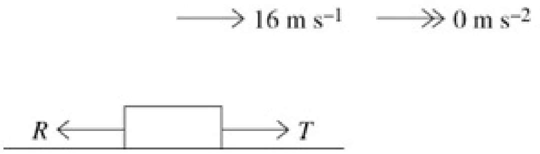
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 6

Question:

A van is travelling along a horizontal road at a constant speed of 16 m s^{-1}. The van's engine is working at 8.8 kW. Calculate the magnitude of the resistance to motion.

Solution:



$$\text{Power} = F \times v$$
$$8.8 \times 10^3 = T \times 16$$
$$T = \frac{8800}{16}$$
$$T = 550$$

$$\text{R}(\rightarrow)\ R = T$$
$$R = 550$$

The magnitude of the resistance is 550 N.

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Exercise D, Question 7

Question:

A car of mass 850 kg is travelling along a straight horizontal road against resistances totalling 350 N. The car's engine is working at 9 kW. Calculate

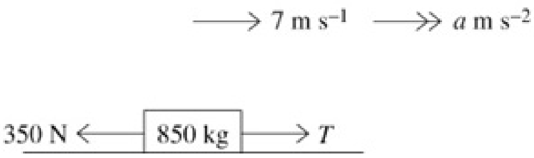
a the acceleration when the car is travelling at 7 m s^{-1},

b the acceleration when the car is travelling at 15 m s^{-1},

c the maximum speed of the car.

Solution:

a



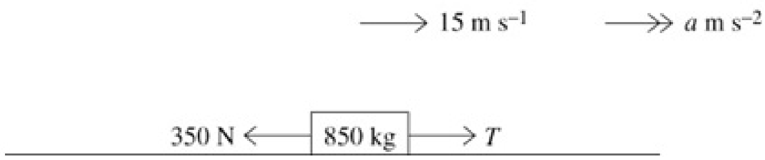
$$\text{Power} = F \times v$$
$$9000 = T \times 7$$
$$T = \frac{9000}{7}$$
$$F = ma$$

First find the fractive force produced by the engine and then use $F = ma$ to find the acceleration.

$$\frac{9000}{7} - 350 = 850a$$
$$a = \frac{\frac{9000}{7} - 350}{850}$$
$$a = 1.100$$

The acceleration is 1.10 m s^{-2}.

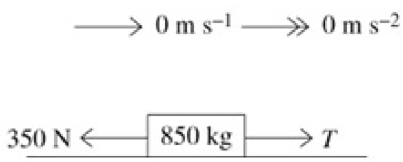
b



$$\text{Power} = F \times v$$
$$9000 = T \times 15$$
$$T = \frac{9000}{15} = 600$$
$$F = ma$$
$$600 - 350 = 850a$$
$$a = \frac{250}{850}$$
$$a = 0.2941$$

The acceleration is 0.294 m s^{-2}.

c



$$\text{R}(\rightarrow) T = 350$$
$$\text{Power} = F \times v$$
$$9000 = 350v$$
$$v = \frac{9000}{350}$$
$$v = 25.71$$

The maximum speed is 25.7 m s^{-1}.

Solutionbank M2

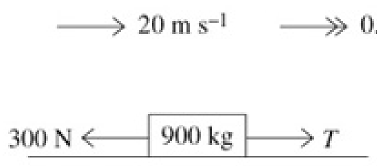
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 8

Question:

A car of mass 900 kg is travelling along a straight horizontal road at a speed of 20 m s^{-1}. The constant resistances to motion total 300 N. The car is accelerating at 0.3 m s^{-2}. Calculate the power developed by the engine.

Solution:



$$
\begin{aligned}
F &= ma \\
T - 300 &= 900 \times 0.3 \\
T &= 900 \times 0.3 + 300 \\
&= 570 \\
\text{Power} &= F \times v \\
&= 570 \times 20 \\
&= 1140
\end{aligned}
$$

The power developed by the engine is 1140 W (or 1.14 kW).

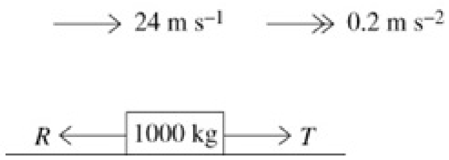
Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 9

Question:

A car of mass 1000 kg is travelling along a straight horizontal road. The car's engine is working at 12 kW. When its speed is 24 m s^{-1} its acceleration is 0.2 m s^{-2}. The resistances to motion have a total magnitude of R newtons. Calculate the value of R.

Solution:



$$\begin{aligned}
\text{Power} &= F \times v \\
12\,000 &= T \times 24 \\
T &= \frac{12\,000}{24} = 500 \\
F &= ma \\
T - R &= 1000 \times 0.2 \\
500 - R &= 200 \\
R &= 500 - 200 \\
R &= 300
\end{aligned}$$

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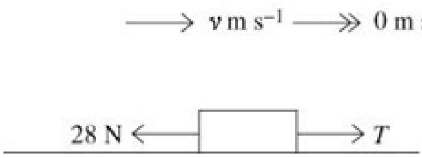
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 10

Question:

A cyclist is travelling along a straight horizontal road. The resistance to his motion is constant and has magnitude 28 N. The maximum rate at which he can work is 280 W. Calculate his maximum speed.

Solution:



$$
\begin{aligned}
R(\rightarrow)\ T &= 28\\
\text{Power} &= F \times v\\
280 &= 28v\\
v &= 10
\end{aligned}
$$

His maximum speed is 10 m s^{-1}.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 11

Question:

A van of mass 1200 kg is travelling up a straight road inclined at $5°$ to the horizontal. The van moves at a constant speed of 20 m s^{-1} and its engine is working at 24 kW. The resistance to motion from non-gravitational forces has magnitude R newtons.

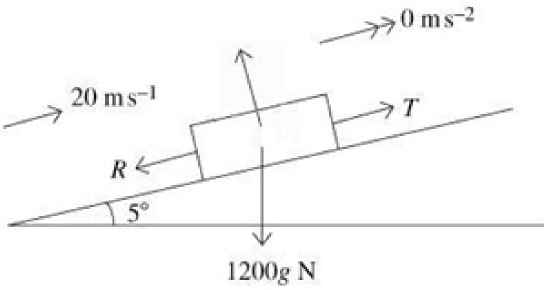
a Calculate the value of R.

The road now becomes horizontal. The resistance to motion from non-gravitational forces is unchanged.

b Calculate the initial acceleration of the car.

Solution:

a

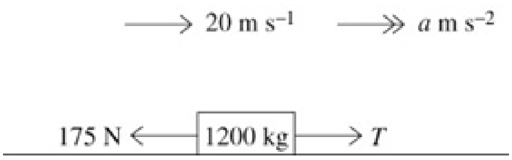


$$\text{Power} = F \times v$$
$$24000 = T \times 20$$
$$T = \frac{24\,000}{20} = 1200$$

$$\text{R}(\nearrow) \quad T = R + 1200g \sin 5°$$
$$1200 = R + 1200g \sin 5°$$
$$R = 1200 - 1200g \sin 5°$$
$$R = 175.0$$

The magnitude of the resistance is 175 N.

b



From above, $T = 1200$.

$$F = ma$$
$$1200 - 175 = 1200a$$
$$a = \frac{1200 - 175}{1200}$$
$$a = 0.8541$$

The initial acceleration is 0.854 m s^{-2}.

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Exercise D, Question 12

Question:

A car of mass 800 kg is travelling at 18 m s^{-1} along a straight horizontal road. The car's engine is working at a constant rate of 26 kW against a constant resistance of magnitude 750 N.

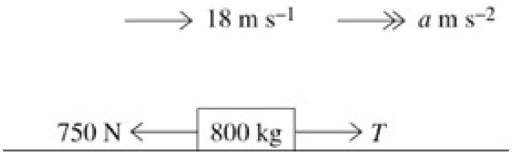
a Find the acceleration of the car.

The car now ascends a straight road, inclined at $9°$ to the horizontal. The resistance to motion from non-gravitational forces is unchanged and the car's engine works at the same rate.

b Find the maximum speed at which the car can travel up the road.

Solution:

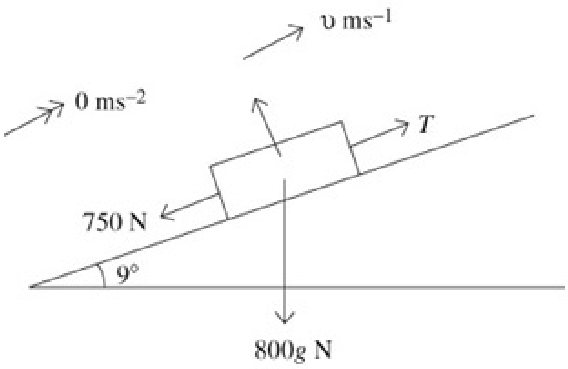
a



$$\text{Power} = F \times s$$
$$26000 = T \times 18$$
$$T = \frac{26000}{18}$$
$$F = ma$$
$$T - 750 = 800a$$
$$800a = \frac{26\,000}{18} - 750$$
$$a = 0.8680$$

The acceleration is 0.868 m s^{-2}.

b



$$R(\nearrow)\ T = 750 + 800g \sin 9^\circ$$
$$\text{Power} = F \times v$$
$$26\,000 = T \times v$$
$$26\,000 = (750 + 800 \times 9.8 \sin 9^\circ)v$$
$$v = \frac{26\,000}{(750 + 800 \times 9.8 \sin 9^\circ)}$$
$$v = 13.15$$

The maximum speed is 13.2 m s^{-1}.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 13

Question:

A van of mass 1500 kg is travelling at its maximum speed of 30 m s^{-1} along a straight horizontal road against a constant resistance of magnitude 600 N.

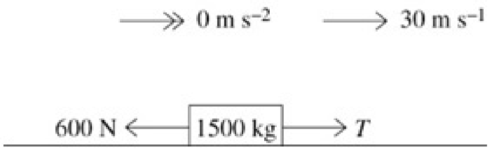
a Find the power developed by the van's engine.

The van now travels up a straight road inclined at $8°$ to the horizontal. The van's engine works at the same rate and the resistance to motion from non-gravitational forces is unchanged.

b Find the maximum speed at which the van can ascend the road.

Solution:

a

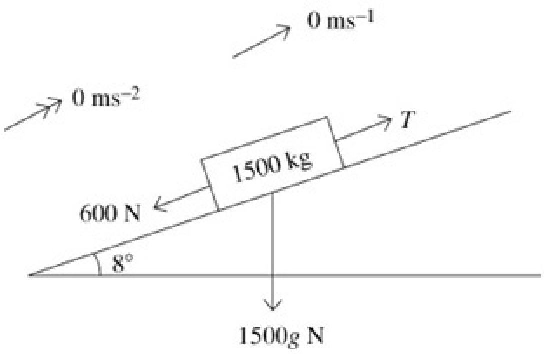


$R(\rightarrow)\ T = 600$

$$\begin{aligned} \text{Power } &= F \times v \\ &= 600 \times 30 \\ &= 18\,000 \end{aligned}$$

The power is 18 000 W (or 18 kW).

b



$R(\rightarrow)\ T = 600 + 1500g \sin 8°$

$$\begin{aligned} \text{Power } &= F \times v \\ 18\,000 &= (600 + 1500g \sin 8°)v \\ v &= \frac{18000}{(600 + 1500g \sin 8°)} = 6.803 \end{aligned}$$

The maximum speed is 6.80 m s^{-1}.

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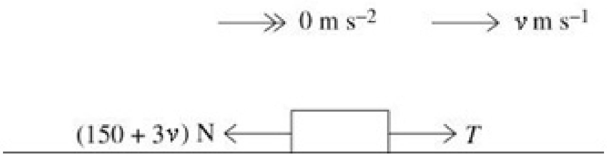
Exercise D, Question 14

Question:

A car is moving along a straight horizontal road with speed v m s^{-1}. The magnitude of the resistance to motion of the car is given by the formula $(150+3v)$ N. The car's engine is working at 10 kW. Calculate the maximum value of v.

Solution:



$$R(\rightarrow) \quad T = 150+3v$$
$$\text{Power} = F \times v$$
$$10\,000 = (150+3v)v$$
$$3v^2 + 150v - 10\,000 = 0$$
$$v = \frac{-150 \pm \sqrt{\left(150^2 - 4\times 3\times(-10\,000)\right)}}{2\times 3}$$
$$= 37.91 \quad (v > 0)$$

The maximum value of v is 37.9

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Exercise D, Question 15

Question:

A train of mass 150 tonnes is moving up a straight track which is inclined at $2°$ to the horizontal. The resistance to the motion of the train from non-gravitational forces has magnitude 6 kN and the train's engine is working at a constant rate of 350 kW.

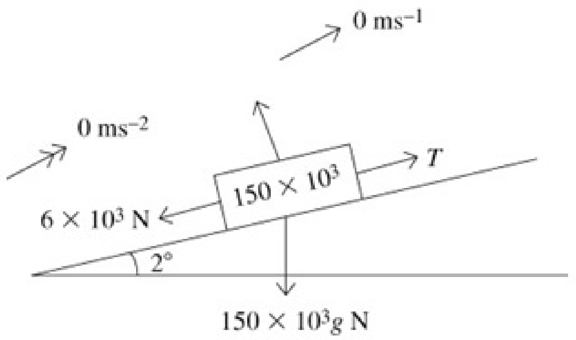
a Calculate the maximum speed of the train.

The track now becomes horizontal. The engine continues to work at 350 kW and the resistance to motion remains 6 kN.

b Find the initial acceleration of the train.

Solution:

a



$$R(\nearrow)\ T = 6\times10^3 + 150\times10^3 g\sin 2^\circ$$

$$\text{Power} = F\times v$$

$$350\times10^3 = (6\times10^3 + 150\times10^3 g\sin 2^\circ)\times v$$

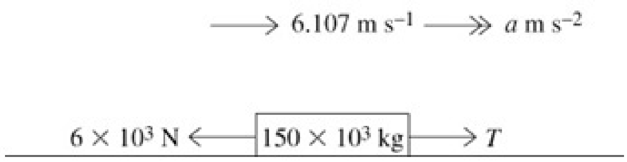
$$v = \frac{350}{(6+150\times9.8\sin 2^\circ)}$$

$$= 6.107$$

The maximum speed is $6.11\ \text{m s}^{-1}$.

1 tonne $= 10^3$ kg
When tonnes, kilonewtons and kilowatts are used the 10^3 will cancel, leaving easier numbers.

b



$$\text{Power} = F\times v$$

$$350\times10^3 = T\times6.107$$

$$T = \frac{350\times10^3}{6.107}$$

$$F = ma$$

$$T - 6\times10^3 = 150\times10^3 a$$

$$\frac{350\times10^3}{6.107} - 6\times10^3 = 150\times10^3 a$$

$$150a = \frac{350}{6.107} - 6$$

$$a = 0.3420$$

The initial acceleration is $0.342\ \text{m s}^{-2}$.

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Edexcel AS and A Level Modular Mathematics

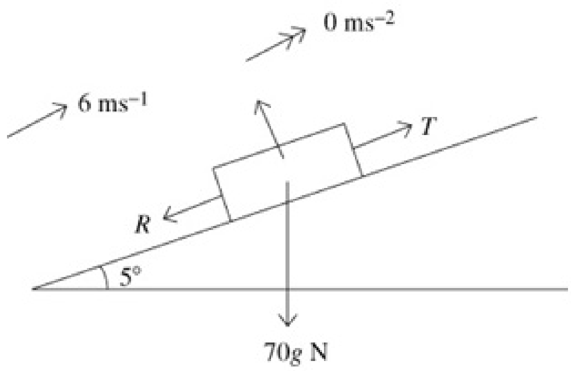
Exercise E, Question 1

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A cyclist and her bicycle have a combined mass of 70 kg. She is cycling at a constant speed of 6 m s^{-1} on a straight road up a hill inclined at $5°$ to the horizontal. She is working at a constant rate of 480 W. Calculate the magnitude of the resistance to motion from non-gravitational forces.

Solution:



$$\text{Power} = F \times v$$
$$480 = T \times 6$$
$$T = \frac{480}{6} = 80$$
$$\text{R}(\rightarrow) T = R + 70g \sin 5°$$
$$80 = R + 70 \times 9.8 \sin 5°$$
$$R = 80 - 70 \times 9.8 \sin 5°$$
$$R = 20.21$$

The magnitude of the resistance is 20.2N.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 2

Question:

A boy hauls a bucket of water through a vertical distance of 25 m. The combined mass of the bucket and water is 12 kg. The bucket starts from rest and finishes at rest.

a Calculate the work done by the boy.

The boy takes 30 s to raise the bucket.

b Calculate the average rate of working of the boy.

Solution:

a

$$\begin{aligned}\text{P.E. gained by water and bucket} &= mgh\\ &= 12\times 9.8\times 25\\ &= 2940\end{aligned}$$

Initial K.E. = final K.E. = 0

$$\begin{aligned}\text{Work done by the boy} &= \text{P.E. gained by bucket}\\ &= 2940\text{ J}\end{aligned}$$

b

$$\begin{aligned}\text{Average rate of working} &= \frac{\text{work done}}{\text{time taken}} = \frac{2940}{30}\\ &= 98\end{aligned}$$

$$\boxed{\text{Rate of doing work} = \frac{\text{work done}}{\text{time taken}}}$$

The average rate of working of the boy is 98 J s^{-1} (or 98 W).

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 3

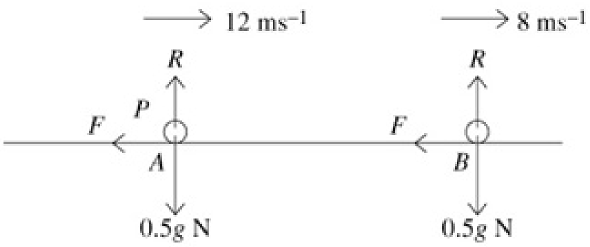
Question:

A particle P of mass 0.5 kg is moving in a straight line from A to B on a rough horizontal plane. At A the speed of P is 12 m s^{-1}, and at B its speed is 8 m s^{-1}. The distance from A to B is 25 m. The only resistance to motion is the friction between the particle and the plane. Find

a the work done by friction as P moves from A to B,

b the coefficient of friction between the particle and the plane.

Solution:



a

$$\text{K.E. lost by particle} = \frac{1}{2}\times 0.5\times 12^2 - \frac{1}{2}\times 0.5 \times 8^2$$
$$= 20$$

Work done by friction = K.E. lost by particle

$\therefore$ Work done by friction = 20 J

Work done by friction = K.E. lost by particle.

b

$$R(\uparrow) R = 0.5g$$

Friction is limiting $F = \mu R = \mu \times 0.5g$

Work done by friction $= F \times s$

$$20 = \mu \times 0.5g \times 25$$

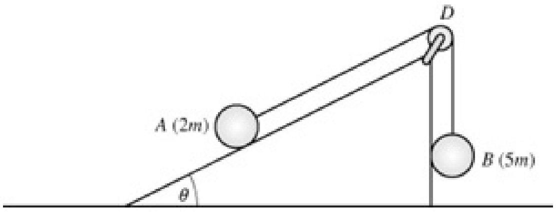
$$\mu = \frac{20}{0.5g \times 25} = 0.1632$$

The coefficient of friction is 0.163.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 4

Question:



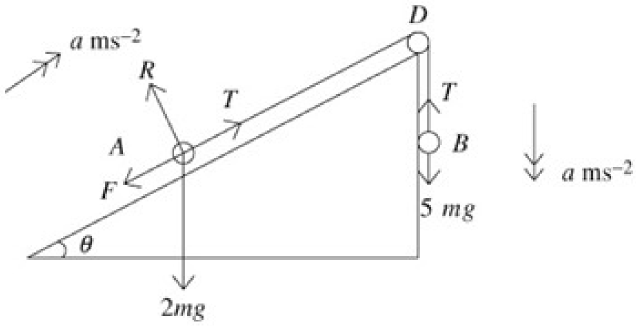
The diagram shows a particle A of mass $2m$ which can move on the rough surface of a plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{3}{5}$. A second particle B of mass $5m$ hangs freely attached to a light inextensible string which passes over a smooth light pulley fixed at D. The other end of the string is attached to A. The coefficient of friction between A and the plane is $\frac{3}{8}$. Particle B is initially hanging 2 m above the ground and A is 4 m from D. When the system is released from rest with the string taut A moves up a line of greatest slope of the plane.

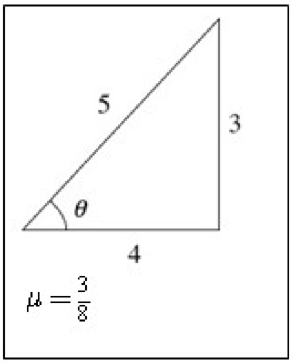
a Find the initial acceleration of A.

When B has descended 1 m the string breaks.

b By using the principle of conservation of energy calculate the total distance moved by A before it first comes to rest.

Solution:





a R(↖) for A:

$R = 2mg\cos\theta$

Friction is limiting:

$F = \mu R$

$F = \frac{3}{8} \times 2mg\cos\theta$

$= \frac{3}{8} \times 2mg \times \frac{4}{5}$

$= \frac{3}{5}mg$

$F = ma$ for A: $T - (F + 2mg\sin\theta) = 2ma$

$T - \left(\frac{3}{5}mg + 2mg \times \frac{3}{5}\right) = 2ma$

$T - \frac{9mg}{5} = 2ma \quad (1)$

$F = ma$ for B: $5mg - T = 5ma \quad (2)$

$(1)+(2)$: $5mg - \frac{9mg}{5} = 7ma$

$\frac{16mg}{5} = 7ma$

$a = \frac{16g}{35} = \frac{16 \times 9.8}{35}$

$a = 4.48$

The acceleration of A is $4.48\ \text{m s}^{-2}$.

b For the first 1 m A travels

The motion must be considered in two parts, before and after the string breaks.
The friction force acting on A is the same throughout the motion.

$u = 0$

$a = 4.48\ \text{m s}^{-2}$

$s = 1\ \text{m}$

$v = ?$

$v^2 = u^2 + 2as$

$v^2 = 2 \times 4.48 \times 1$

$v^2 = 8.96$

After string breaks:

$$\text{Loss of K.E. (of } A) = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$
$$= \frac{1}{2}\times 2m \times 8.96 - 0$$
$$= 8.96\,m$$

$$\text{Gain of P.E. (of } A) = mgh$$
$$= 2mg \times (x \sin\theta)$$
$$= 2mg \times x \times \frac{3}{5}$$
$$= \frac{6mgx}{5}$$

where x is the distance moved up the plane.

$$\text{Work done by friction} = \frac{3mg}{5}\times x$$

Work–energy principle:

$$\frac{3mgx}{5} + \frac{6mgx}{5} = 8.96m$$
$$\frac{9}{5}gx = 8.96$$
$$x = \frac{8.96\times 5}{9\times 9.8}$$
$$x = 0.5079$$

$$\text{Total distance moved} = 1 + 0.5079$$
$$= 1.51\text{ m}$$

Solutionbank M2

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Exercise E, Question 5

Question:

A car of mass 800 kg is travelling along a straight horizontal road. The resistance to motion from non-gravitational forces has a constant magnitude of 500 N. The engine of the car is working at a rate of 16 kW.

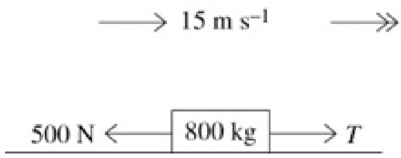
a Calculate the acceleration of the car when its speed is 15 m s^{-1}.

The car comes to a hill at the moment when it is travelling at 15 m s^{-1}. The road is still straight but is now inclined at $5°$ to the horizontal. The resistance to motion from non-gravitational forces is unchanged. The rate of working of the engine is increased to 24 kW.

b Calculate the new acceleration of the car.

Solution:

a



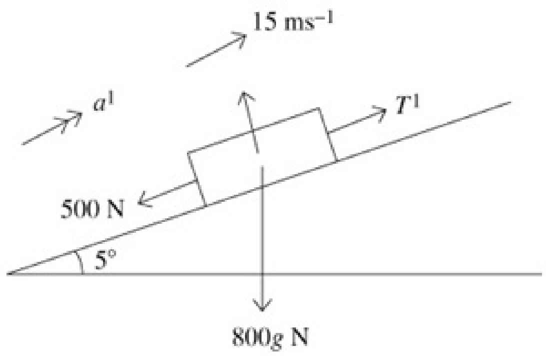
$$\text{Power} = F \times v$$
$$16\,000 = T \times 15$$
$$T = \frac{16\,000}{15}$$

Ensure units are consistent.

$$F = ma$$
$$T - 500 = 800a$$
$$\frac{16\,000}{15} - 500 = 800a$$
$$a = \frac{\frac{16\,000}{15} - 500}{800}$$
$$a = 0.7083$$

The acceleration is 0.708 m s^{-2}.

b



$$\text{Power} = F \times v$$
$$24\,000 = T' \times 15$$
$$T' = \frac{24\,000}{15}$$

$$\text{R}(\nearrow) F = ma$$
$$T' - 500 - 800g \sin 5^\circ = 800a$$
$$\frac{24\,000}{15} - 500 - 800 \times 9.8 \sin 5^\circ = 800a$$
$$800a = 416.698$$
$$a = 0.5208$$

The new acceleration is 0.521 m s^{-2}.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 6

Question:

A car of mass 750 kg is moving at a constant speed of 18 m s^{-1} down a straight road inclined at an angle θ to the horizontal, where $\tan\theta = \frac{1}{20}$. The resistance to motion from non-gravitational forces has a constant magnitude of 1000 N.

a Find, in kW, the rate of working of the car's engine.

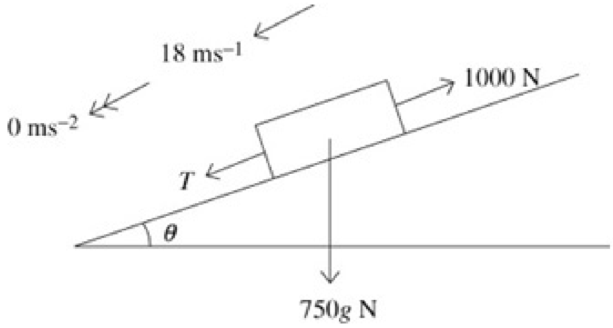
The engine of the car is now switched off and the car comes to rest T seconds later.

The resistance to motion from non-gravitational forces is unchanged.

b Find the value of T.

Solution:

a

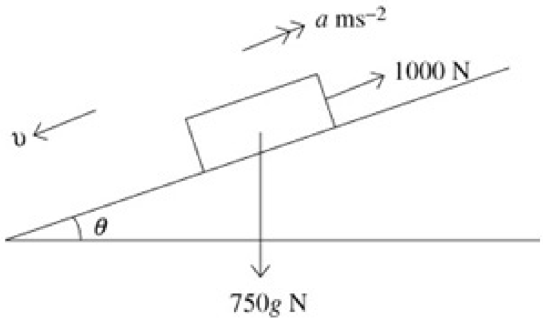


$\tan\theta = \frac{1}{20} \quad \theta = 2.8624^\circ$

$$
\begin{aligned}
\mathrm{R}(\nearrow)\ T + 750g\sin\theta &= 1000 \\
T &= 1000 - 750 \times 9.8 \sin 2.8624^\circ \\
T &= 632.95 \\
\text{Power} &= F \times v \\
&= 632.95 \times 18 \\
&= 11\,393 \text{ W}
\end{aligned}
$$

The rate of working of the car's engine is 11.4 kW.

b



The tractive force is zero.

$$
\begin{aligned}
\mathrm{R}(\nearrow)\ F &= ma \\
1000 - 750 \times 9.8 \times \sin\theta &= 750a \\
a &= \frac{1000 - 750 \times 9.8 \sin 2.8624^\circ}{750} \\
a &= 0.8439
\end{aligned}
$$

$u = 18 \text{ m s}^{-1}$
$v = 0 \text{ m s}^{-1}$
$a = -0.8439 \text{ m s}^{-2}$
$t = T$

$v = u + at$
$0 = 18 - 0.8439 \times T$
$T = \frac{18}{0.8439}$
$T = 21.32$

The value of T is 21.3.

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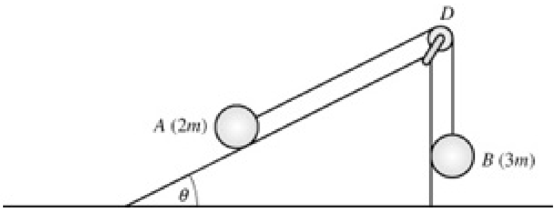
Exercise E, Question 7

Question:



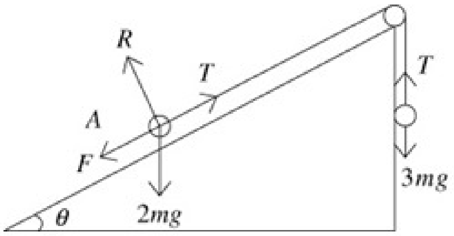
The diagram shows a particle A of mass $2m$ which can move on the rough surface of a plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{3}{5}$. A second particle B of mass $3m$ hangs freely attached to a light inextensible string which passes over a smooth pulley fixed at D. The other end of the string is attached to A. The coefficient of friction between A and the plane is $\frac{1}{4}$. The system is released from rest with the string taut and A moves up a line of greatest slope of the plane. When each particle has moved a distance s, A has not reached the pulley and B has not reached the ground.

a Find an expression for the potential energy lost by the system when each particle has moved a distance s.

When each particle has moved a distance s they are moving with speed v.

b Find an expression for v^2, in terms of s.

Solution:



a

P.E. gained by $A = mgh$

$= 2mg \times (s \times \sin\theta)$

$= 2mg \times s \times \frac{3}{5}$

$= \frac{6mgs}{5}$

P.E. lost by $B = mgh$

$= 3\,mgs$

$\therefore$ P.E. lost by system $= 3mgs - \frac{6mgs}{5} = \frac{9mgs}{5}$

b Consider A:

Find the frictional force and use the work–energy principle.

R(↖) $R = 2mg\cos\theta$

$= 2mg \times \frac{4}{5}$

$= \frac{8mg}{5}$

Friction is limiting:

$F = \mu R$

$= \frac{1}{4} \times \frac{8mg}{5}$

$= \frac{2mg}{5}$

$$\text{Work done against friction} = F \times s$$
$$= \frac{2mgs}{5}$$
$$\text{K.E. gained by } A \text{ and } B = \frac{1}{2}(2m)v^2 + \frac{1}{2}(3m)v^2$$
$$= \frac{5mv^2}{2}$$

Work–energy principle:

K.E. gained + work done against friction = P.E. lost

$$\frac{5mv^2}{2} + \frac{2mgs}{5} = \frac{9mgs}{5}$$
$$\frac{5mv^2}{2} = \frac{7mgs}{5}$$
$$v^2 = \frac{2 \times 7mgs}{5 \times 5m}$$
$$v^2 = \frac{14gs}{25}$$

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Edexcel AS and A Level Modular Mathematics

Exercise E, Question 8

Question:

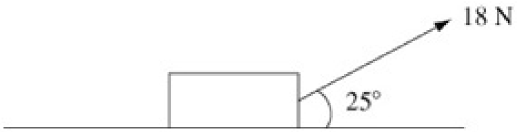
A parcel of mass 5 kg is resting on a platform inclined at $25°$ to the horizontal.

The coefficient of friction between the parcel and the platform is 0.3. The parcel is released from rest and slides down a line of greatest slope of the platform. Calculate

a the speed of the parcel after it has been moving for 2 s,

b the potential energy lost by the parcel during this time.

Solution:



$$\text{Work done} = 18\cos 25^\circ \times 14$$
$$= 228.3$$

Work done = *horizontal* component of tension × *horizontal* distance moved

The work done is 228 J.

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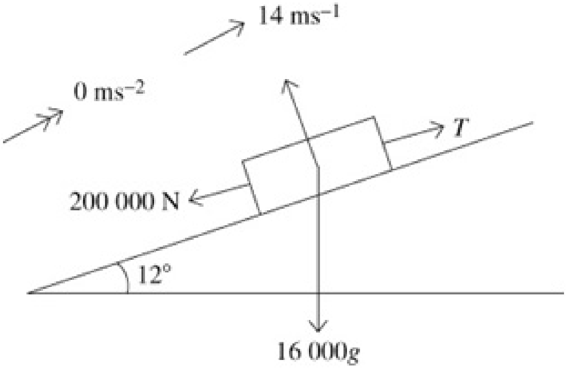
Exercise E, Question 9

Question:

A lorry of mass 16 000 kg is travelling up a straight road inclined at $12°$ to the horizontal.

The lorry is travelling at a constant speed of 14 m s^{-1} and the resistance to motion from non-gravitational forces has a constant magnitude of 200 kN. Find the work done in 10 s by the engine of the lorry.

Solution:



$$R(\nearrow)\ T = 200\,000 + 16\,000g\sin 12°$$
$$T = 232\,600$$
$$\text{Work done in 10 s} = \text{force} \times \text{distance moved}$$
$$= 232\,600 \times (14 \times 10)$$
$$= 32\,564\,000$$

The work done in 10s is 32 600 000 J (or 32 600 kJ).

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Exercise E, Question 10

Question:

A particle P of mass 0.3 kg is moving in a straight line on a smooth horizontal surface under the action of a constant horizontal force. The particle passes point A with speed 6 m s^{-1} and point B with speed 12 m s^{-1}.

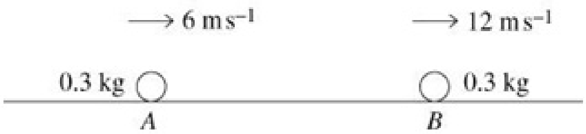
a Find the kinetic energy gained by P while moving from A to B.

b Write down the work done by the constant force.

The distance from A to B is 4 m.

c Calculate the magnitude of the force.

Solution:



a

$$\text{K.E. gained} = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$
$$= \frac{1}{2}\times 0.3\times 12^2 - \frac{1}{2}\times 0.3\times 6^2$$
$$= 16.2$$

The K.E. gained is 16.2 J.

b

The work done by the force is 16.2 J.

c

$$\text{Work done} = F\times s$$
$$16.2 = F\times 4$$
$$F = \frac{16.2}{4}$$
$$F = 4.05$$

The force has magnitude 4.05 N.

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Exercise E, Question 11

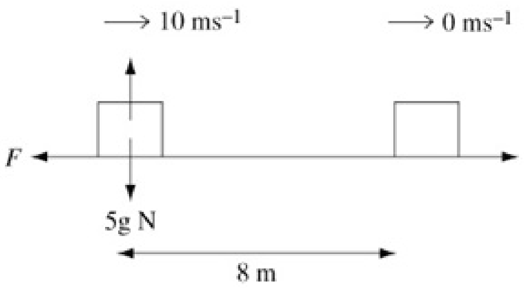
Question:

A box of mass 5 kg slides in a straight line across a rough horizontal floor. The initial speed of the box is 10 m s^{-1}. The only resistance to the motion is the frictional force between the box and the floor. The box comes to rest after moving 8 m. Calculate

a the kinetic energy lost by the box in coming to rest,

b the coefficient of friction between the box and the floor.

Solution:



a

$$\begin{aligned} \text{K.E. lost} &= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 \\ &= \frac{1}{2} \times 5 \times 10^2 - 0 \\ &= 250 \end{aligned}$$

The K.E. lost is 250 J.

b Work done against friction $= 250$ J

$$\begin{aligned} \text{Work done} &= F \times s \\ 250 &= F \times 8 \\ F &= \frac{250}{8} \end{aligned}$$

$$\text{R}(\uparrow) \; R = 5g$$

$$\text{Friction is limiting} \quad F = \mu R$$

$$\frac{250}{8} = \mu \times 5g$$

$$\mu = \frac{250}{8 \times 5 \times 9.8} = 0.6377$$

The coefficient of friction is 0.638.

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Exercise E, Question 12

Question:

A car of mass 900 kg is moving along a straight horizontal road. The resistance to motion has a constant magnitude. The engine of the car is working at a rate of 15 kW. When the car is moving with speed 20 m s^{-1}, the acceleration of the car is 0.3 m s^{-2}.

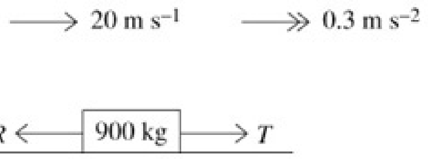
a Find the magnitude of the resistance.

The car now moves downhill on a straight road inclined at $4°$ to the horizontal. The engine of the car is now working at a rate of 8 kW. The resistance to motion from non-gravitational forces remains unchanged.

b Calculate the speed of the car when its acceleration is 0.5 m s^{-2}.

Solution:

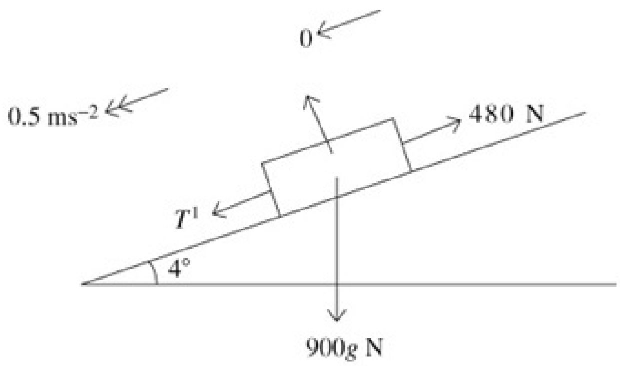
a



$$\begin{aligned}
\text{Power} &= F \times v \\
T &= \frac{15\,000}{20} = 750 \\
F &= ma \\
T - R &= 900 \times 0.3 \\
750 - R &= 270 \\
R &= 750 - 270 \\
R &= 480
\end{aligned}$$

The magnitude of the resistance is 480 N.

b



$$\begin{aligned}
F &= ma \\
T' + 900g \sin 4^\circ - 480 &= 900 \times 0.5 \\
T' &= 450 + 480 - 900g \sin 4^\circ \\
\text{Power} &= F \times v \\
8000 &= (450 + 480 - 900g \sin 4^\circ)v \\
v &= \frac{8000}{(450 + 480 - 900g \sin 4^\circ)} \\
v &= 25.41
\end{aligned}$$

The speed of the car is 25.4 m s^{-1}.

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Exercise E, Question 13

Question:

A block of wood of mass 4 kg is pulled across a rough horizontal floor by a rope inclined at 15° to the horizontal. The tension in the rope is constant and has magnitude 75 N.

The coefficient of friction between the block and the floor is $\frac{3}{8}$.

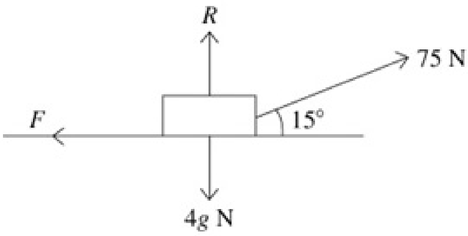
a Find the magnitude of the frictional force opposing the motion.

b Find the work done by the tension when the block moves 6 m.

The block is initially at rest.

c Find the speed of the block when it has moved 6 m.

Solution:



a

$$R(\uparrow)\ R + 75\sin 15^\circ = 4g$$

$$R = 4g - 75\sin 15^\circ$$

Friction is limiting: $F = \mu R$

$$F = \frac{3}{8}\times(4\times 9.8 - 75\sin 15^\circ)$$

$$F = 7.420$$

The magnitude of the frictional force is 7.42 N.

b

$$\text{Work done} = F\times s$$

$$= 75\cos 15^\circ \times 6$$

$$= 434.66$$

The work done is 435 J.

c

K.E. gained = work done by tension − work done against friction

$$\frac{1}{2}\times 4v^2 = 434.66 - 7.42\times 6$$

$$v^2 = \frac{1}{2}(434.66 - 7.42\times 6)$$

$$v = 13.96$$

Use the work–energy principle.

The block is moving at $14.0\ \text{m s}^{-1}$.

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Exercise E, Question 14

Question:

The engine of a lorry works at a constant rate of 20 kW. The lorry has a mass of 1800 kg.

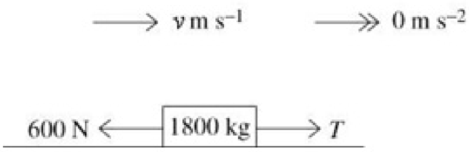
When moving along a straight horizontal road there is a constant resistance to motion of magnitude 600 N. Calculate

a the maximum speed of the lorry,

b the acceleration of the lorry, in $\mathrm{m\ s^{-2}}$, when its speed is $20\ \mathrm{m\ s^{-1}}$.

Solution:

a



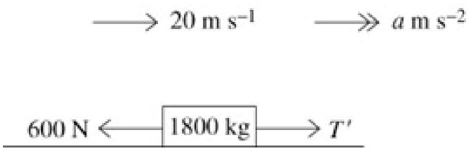
$\mathrm{R}(\rightarrow)\ T = 600$

Ensure units are consistent.

$$\text{Power} = F \times v$$
$$20\,000 = 600v$$
$$v = \frac{20\,000}{600}$$
$$v = 33.33$$

The lorry's maximum speed is $33.3\ \mathrm{m\ s^{-1}}$.

b



$$\text{Power} = F \times v$$
$$20000 = T' \times 20$$
$$T' = 1000$$
$$F = ma$$
$$T' - 600 = 1800a$$
$$1000 - 600 = 1800a$$
$$a = \frac{400}{1800}$$
$$a = 0.2222$$

The acceleration is $0.222\ \mathrm{m\ s^{-2}}$.

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Exercise E, Question 15

Question:

A car of mass 1200 kg is travelling at a constant speed of 20 m s^{-1} along a straight horizontal road. The constant resistance to motion has magnitude 600 N.

a Calculate the power, in kW, developed by the engine of the car.

The rate of working of the engine of the car is suddenly increased and the initial acceleration of the car is 0.5 m s^{-2}. The resistance to motion is unchanged.

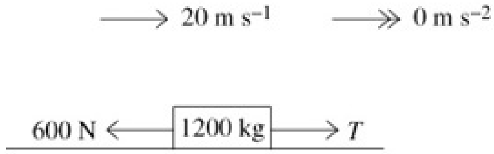
b Find the new rate of working of the engine of the car.

The car now comes to a hill. The road is still straight but is now inclined at $20°$ to the horizontal. The rate of working of the engine of the car is increased further to 50 kW.

The resistance to motion from non-gravitational forces still has magnitude 600 N. The car climbs the hill at a constant speed $V \text{ m s}^{-1}$.

c Find the value of V.

Solution:

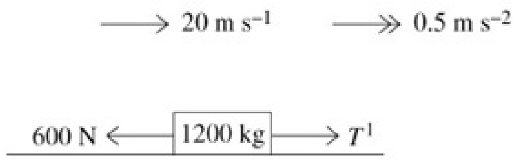


a $R(\rightarrow)\ T = 600$

$$\begin{aligned}\text{Power} &= F \times v\\ &= 600 \times 20\\ &= 12\,000\ \text{W}\\ &= 12\ \text{kW}\end{aligned}$$

The power is 12 kW.

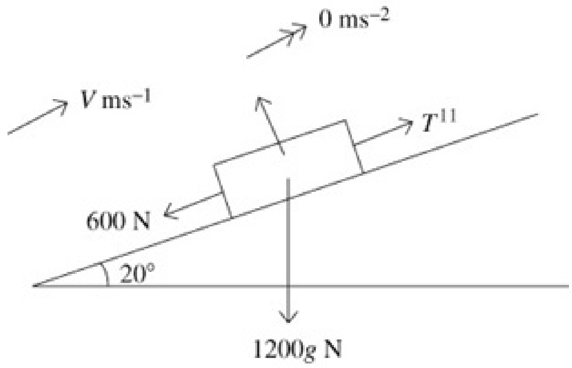
b



$$\begin{aligned}F &= ma\\ T' - 600 &= 1200 \times 0.5\\ T' &= 600 + 600\\ T' &= 1200\\ \text{Power} &= F \times v\\ &= 1200 \times 20\\ &= 24\,000\end{aligned}$$

The new rate of working is 24 kW.

c



$R(\nearrow)\ T'' = 600 + 1200g \sin 20^\circ$

$$\begin{aligned}\text{Power} &= F \times v\\ 50\,000 &= (600 + 1200g \sin 20^\circ)V\\ V &= \frac{50\,000}{(600 + 1200g \sin 20^\circ)}\\ V &= 10.82\\ &= 10.8\end{aligned}$$

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Exercise A, Question 1

Question:

In this exercise $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors.

A particle of mass 0.25 kg is moving with velocity $(12\mathbf{i}+4\mathbf{j})$ m s^{-1} when it receives an impulse $(8\mathbf{i}-7\mathbf{j})$ Ns. Find the new velocity of the particle.

Solution:

$$8\mathbf{i}-7\mathbf{j}=0.25\mathbf{v}-0.25(12\mathbf{i}+4\mathbf{j})$$
$$\therefore 8\mathbf{i}-7\mathbf{j}=0.25\mathbf{v}-3\mathbf{i}-\mathbf{j}$$
$$\therefore 0.25\mathbf{v}=11\mathbf{i}-6\mathbf{j}$$
$$\therefore \mathbf{v}=44\mathbf{i}-24\mathbf{j}$$

Use impulse $= m\mathbf{v}-m\mathbf{u}$, then make $\mathbf{v}$ the subject of the formula.

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Exercise A, Question 2

Question:

A particle of mass 0.5 kg is moving with velocity $(2\mathbf{i}-2\mathbf{j})$ m s^{-1} when it receives an impulse $(3\mathbf{i}+5\mathbf{j})$ Ns. Find the new velocity of the particle.

Solution:

$$\begin{aligned}3\mathbf{i}+5\mathbf{j} &= 0.5\mathbf{v}-0.5(2\mathbf{i}-2\mathbf{j})\\ &= 0.5\mathbf{v}-\mathbf{i}+\mathbf{j}\\ \therefore 0.5\mathbf{v} &= 4\mathbf{i}+4\mathbf{j}\\ \therefore \mathbf{v} &= 8\mathbf{i}+8\mathbf{j}\end{aligned}$$

Use impulse $= m\mathbf{v}-m\mathbf{u}$ (change in momentum).

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Exercise A, Question 3

Question:

A particle of mass 2 kg moves with velocity $(3\mathbf{i}+2\mathbf{j})$ m s^{-1} immediately after it has received an impulse $(4\mathbf{i}+8\mathbf{j})$ Ns. Find the original velocity of the particle.

Solution:

$$
\begin{aligned}
4\mathbf{i}+8\mathbf{j} &= 2\times(3\mathbf{i}+2\mathbf{j})-2\mathbf{u} \\
&= 6\mathbf{i}+4\mathbf{j}-2\mathbf{u} \\
\therefore 2\mathbf{u} &= 6\mathbf{i}+4\mathbf{j}-4\mathbf{i}-8\mathbf{j} \\
&= 2\mathbf{i}-4\mathbf{j} \\
\therefore \mathbf{u} &= \mathbf{i}-2\mathbf{j}
\end{aligned}
$$

Use impulse = change in momentum.

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Exercise A, Question 4

Question:

A particle of mass 1.5 kg moves with velocity $(5\mathbf{i} - 8\mathbf{j})$ m s^{-1} immediately after it has received an impulse $(3\mathbf{i} - 6\mathbf{j})$ Ns. Find the original velocity of the particle.

Solution:

$$
\begin{aligned}
3\mathbf{i} - 6\mathbf{j} &= 1.5(5\mathbf{i} - 8\mathbf{j}) - 1.5\mathbf{u} \\
\therefore 1.5\mathbf{u} &= 7.5\mathbf{i} - 12\mathbf{j} - 3\mathbf{i} + 6\mathbf{j} \\
&= 4.5\mathbf{i} - 6\mathbf{j} \\
\therefore \mathbf{u} &= 3\mathbf{i} - 4\mathbf{j}
\end{aligned}
$$

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Exercise A, Question 5

Question:

A body of mass 3 kg is initially moving with a constant velocity of $(\mathbf{i}+\mathbf{j})$ m s^{-1} when it is acted on by a force of $(6\mathbf{i}-8\mathbf{j})$ N for 3 seconds. Find the impulse exerted on the body and find its velocity when the force ceases to act.

Solution:

$$\text{impulse} = \text{force} \times \text{time}$$
$$\therefore \text{impulse} = (6\mathbf{i}-8\mathbf{j}) \times 3$$
$$= 18\mathbf{i} - 24\mathbf{j}$$

Use impulse $=$ force $\times$ time

But impulse $=$ change in momentum

$$\therefore 18\mathbf{i} - 24\mathbf{j} = 3\left(\mathbf{v} - (\mathbf{i}+\mathbf{j})\right)$$
$$\therefore 18\mathbf{i} - 24\mathbf{j} + 3\mathbf{i} + 3\mathbf{j} = 3\mathbf{v}$$
$$\therefore 3\mathbf{v} = 21\mathbf{i} - 21\mathbf{j}$$
$$\therefore \mathbf{v} = 7\mathbf{i} - 7\mathbf{j}$$

Then use
impulse $=$ change in momentum
$= m\mathbf{v} - m\mathbf{u}$

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Exercise A, Question 6

Question:

A body of mass 0.5 kg is initially moving with a constant velocity of $(5\mathbf{i}+12\mathbf{j})$ m s^{-1} when it is acted on by a force of $(2\mathbf{i}-\mathbf{j})$ N for 5 seconds. Find the impulse exerted on the body and find its velocity when the force ceases to act.

Solution:

$$\begin{aligned}\text{impulse} &= \text{force} \times \text{time}\\ &= (2\mathbf{i}-\mathbf{j})\times 5\\ &= 10\mathbf{i}-5\mathbf{j}\end{aligned}$$

Use impulse = force × time.

But impulse = change in momentum

$$\begin{aligned}\therefore 10\mathbf{i}-5\mathbf{j} &= 0.5\left(\mathbf{v}-(5\mathbf{i}+12\mathbf{j})\right)\\ \therefore 10\mathbf{i}-5\mathbf{j}+2.5\mathbf{i}+6\mathbf{j} &= 0.5\mathbf{v}\\ \therefore 0.5\mathbf{v} &= 12.5\mathbf{i}-\mathbf{j}\\ \therefore \mathbf{v} &= 25\mathbf{i}-2\mathbf{j}\end{aligned}$$

Then impulse = change in momentum.

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Exercise A, Question 7

Question:

A particle of mass 2 kg is moving with velocity $(5\mathbf{i}+3\mathbf{j})$ m s^{-1} when it hits a wall. It rebounds with velocity $(-\mathbf{i}-3\mathbf{j})$ m s^{-1}. Find the impulse exerted by the wall on the particle.

Solution:

$$\begin{aligned}\text{impulse} &= \text{change in momentum}\\ &= 2(-\mathbf{i}-3\mathbf{j})-2(5\mathbf{i}+3\mathbf{j})\\ &= -12\mathbf{i}-12\mathbf{j}\end{aligned}$$

Use impulse $= m\mathbf{v}-m\mathbf{u}$.

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Exercise A, Question 8

Question:

A particle of mass 0.5 kg is moving with velocity $(11\mathbf{i} - 2\mathbf{j})$ m s^{-1} when it hits a wall. It rebounds with velocity $(-\mathbf{i} + 7\mathbf{j})$ m s^{-1}. Find the impulse exerted by the wall on the particle.

Solution:

$$\begin{aligned}\text{impulse} &= \text{change in momentum}\\ &= 0.5\times(-\mathbf{i}+7\mathbf{j}) - 0.5\times(11\mathbf{i}-2\mathbf{j})\\ &= -6\mathbf{i} + 4\tfrac{1}{2}\mathbf{j}\end{aligned}$$

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Exercise A, Question 9

Question:

A particle P of mass 3 kg receives an impulse Q Ns. Immediately before the impulse the velocity of P is $5\mathbf{i}$ m s^{-1} and immediately afterwards it is $(13\mathbf{i} - 6\mathbf{j})$ m s^{-1}. Find the magnitude of Q and the angle between Q and $\mathbf{i}$.

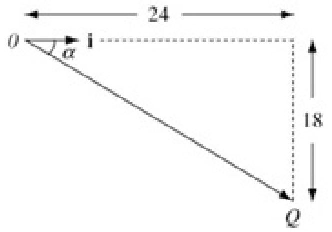
Solution:

$$
\begin{aligned}
\mathrm{Q} &= m\mathbf{v} - m\mathbf{u} \\
&= 3(13\mathbf{i} - 6\mathbf{j}) - 3(5\mathbf{i}) \\
&= 24\mathbf{i} - 18\mathbf{j}
\end{aligned}
$$

Use impulse = change in momentum

$$
\begin{aligned}
\therefore |\mathrm{Q}| &= \sqrt{(24)^2 + (-18)^2} \\
&= 30
\end{aligned}
$$

Find the magnitude of Q by using Pythagoras' Theorem, and find the angle between Q and $\mathbf{i}$ by using trigonometry.



Let α be the acute angle between $\mathbf{i}$ and Q.

Then

$$\tan\alpha = \frac{18}{24}$$

$$\therefore \alpha = 36.9^\circ \text{ (3 s.f.)}$$

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Exercise A, Question 10

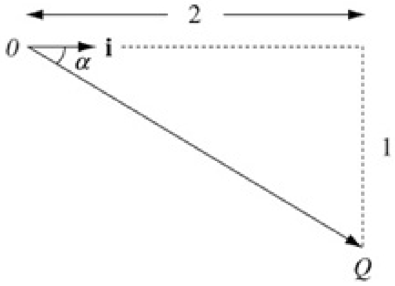
Question:

A particle P of mass 0.5 kg receives an impulse Q Ns. Immediately before the impulse the velocity of P is $(-\mathbf{i}-2\mathbf{j})$ m s^{-1} and immediately afterwards it is $(3\mathbf{i}-4\mathbf{j})$ m s^{-1}. Find the magnitude of Q and the angle between Q and $\mathbf{i}$.

Solution:

Use impulse = change in momentum.

$$
\begin{aligned}
\mathrm{Q} &= 0.5(3\mathbf{i}-4\mathbf{j}) - 0.5(-\mathbf{i}-2\mathbf{j}) \\
&= 2\mathbf{i}-\mathbf{j} \\
\therefore\ |\mathrm{Q}| &= \sqrt{2^2+(-1)^2} \\
&= \sqrt{5} = 2.24\ (3\text{ s.f.})
\end{aligned}
$$



Let α be the acute angle between Q and $\mathbf{i}$.

Use

$\tan\alpha = \frac{1}{2}$

$\therefore \alpha = 26.6°$ (3 s.f.)

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Exercise A, Question 11

Question:

A cricket ball of mass 0.5 kg is hit by a bat. Immediately before being hit the velocity of the ball is $(20\mathbf{i}-4\mathbf{j})$ m s^{-1} and immediately afterwards it is $(-16\mathbf{i}+8\mathbf{j})$ m s^{-1}. Find the magnitude of the impulse exerted on the ball by the bat.

Solution:

$$\begin{aligned}\text{Impulse} &= \text{change in momentum}\\ &= m\mathbf{v}-m\mathbf{u}\\ &= 0.5\times(-16\mathbf{i}+8\mathbf{j})-0.5\times(20\mathbf{i}-4\mathbf{j})\\ &= -8\mathbf{i}+4\mathbf{j}-10\mathbf{i}+2\mathbf{j}\\ &= -18\mathbf{i}+6\mathbf{j}\end{aligned}$$

$$\begin{aligned}\therefore \text{Magnitude of the impulse} &= \sqrt{(-18)^2+6^2}=6\sqrt{10}\\ &= 19.0\text{ Ns (3 s.f.)}\end{aligned}$$

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Exercise A, Question 12

Question:

A ball of mass 0.2 kg is hit by a bat. Immediately before being hit by the bat the velocity of the ball is $-15\mathbf{i}$ m s^{-1} and the bat exerts an impulse of $(2\mathbf{i}+6\mathbf{j})$ Ns on the ball. Find the velocity of the ball after the impact.

Solution:

Use impulse = change in momentum

$$\begin{aligned} 2\mathbf{i}+6\mathbf{j} &= 0.2\mathbf{v}-0.2(-15\mathbf{i}) \\ &= 0.2\mathbf{v}+3\mathbf{i} \\ \therefore 0.2\mathbf{v} &= 2\mathbf{i}+6\mathbf{j}-3\mathbf{i} \\ &= -\mathbf{i}+6\mathbf{j} \\ \therefore \mathbf{v} &= -5\mathbf{i}+30\mathbf{j} \end{aligned}$$

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Exercise A, Question 13

Question:

A particle of mass 0.25 kg has velocity $\mathbf{v}$ m s^{-1} at time t s where $\mathbf{v} = (t^2 - 3)\mathbf{i} + 4t\mathbf{j}$.

When $t = 3$, the particle receives an impulse of $2\mathbf{i} + 2\mathbf{j}$ Ns. Find the velocity of the particle immediately after the impulse.

Solution:

$\mathbf{v} = (t^2 - 3)\mathbf{i} + 4t\mathbf{j}$

When $t = 3$ let $\mathbf{v} = \mathbf{u}$

Substitute $t = 3$ into expression for velocity, to find the velocity before the impact.

$\mathbf{u} = 6\mathbf{i} + 12\mathbf{j}$

Use impulse = change in momentum

Then $2\mathbf{i} + 2\mathbf{j} = 0.25\mathbf{v} - 0.25(6\mathbf{i} + 2\mathbf{j})$

$$\therefore 0.25\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + 0.25(6\mathbf{i} + 2\mathbf{j})$$
$$= 3.5\mathbf{i} + 2.5\mathbf{j}$$
$$\therefore \mathbf{v} = 14\mathbf{i} + 10\mathbf{j}$$

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Exercise A, Question 14

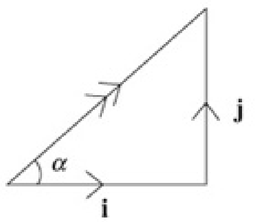
Question:

A ball of mass 2 kg is initially moving with a velocity of $(\mathbf{i}+\mathbf{j})$ m s^{-1}. It receives an impulse of $2\mathbf{j}$ Ns. Find the velocity immediately after the impulse and the angle through which the ball is deflected as a result. Give your answer to the nearest degree.

Solution:

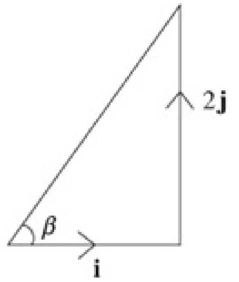
Use impulse = change in momentum

$$\therefore 2\mathbf{j} = 2\mathbf{v} - 2(\mathbf{i}+\mathbf{j})$$
$$\therefore 2\mathbf{v} = 2\mathbf{j} + 2(\mathbf{i}+\mathbf{j})$$
$$= 2\mathbf{i} + 4\mathbf{j}$$
$$\therefore \mathbf{v} = \mathbf{i} + 2\mathbf{j}$$



Before impact the velocity was $\mathbf{i}+\mathbf{j}$ and so the direction of the ball was at an angle α with **i**, where $\tan\alpha = \frac{1}{1}$, i.e. $\alpha = 45°$.

Find the angle between the direction of the velocity and the direction **i**, both before and after the impulse.



After impact the velocity is $\mathbf{i}+2\mathbf{j}$ and so the direction of the ball is at an angle β with **i**, where $\tan\beta = \frac{2}{1}$, i.e. $\beta = 63.4°$.

Then calculate the angle of deflection.

$\therefore$ The ball is deflected through an angle of 18.4°.

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Exercise A, Question 15

Question:

A particle of mass 0.5 kg moving with velocity $3\mathbf{i}$ m s^{-1} collides with a particle of mass 0.25 kg moving with velocity $12\mathbf{i}$ m s^{-1}. The two particles coalesce and move as one particle of mass 0.75 kg. Find the velocity of the combined particle.

Solution:

Let the new velocity be $x\mathbf{i}$.

Let the new velocity be $x\mathbf{i}$ and use conservation of momentum. Equate $\mathbf{i}$ component to find x.

Using conservation of momentum

$$0.5 \times 3\mathbf{i} + 0.25 \times 12\mathbf{i} = 0.75x\mathbf{i}$$
$$\therefore 1.5\mathbf{i} + 3\mathbf{i} = 0.75x\mathbf{i}$$
$$\therefore 0.75x\mathbf{i} = 4.5\mathbf{i}$$
$$\therefore x = \frac{4.5}{0.75}$$
$$= 6$$

So the velocity of the combined particle is $6\mathbf{i}$ m s^{-1}.

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Exercise A, Question 16

Question:

A particle of mass 5 kg moving with velocity $(\mathbf{i}-\mathbf{j})$ m s^{-1} collides with a particle of mass 2 kg moving with velocity $(-\mathbf{i}+\mathbf{j})$ m s^{-1}. The two particles coalesce and move as one particle of mass 7 kg. Find the magnitude of the velocity $\mathbf{v}$ m s^{-1} of the combined particle.

Solution:

Use conservation of momentum to find $\mathbf{v}$, then use Pythagoras' Theorem and trigonometry to find $|\mathbf{v}|$ and the angle between $\mathbf{v}$ and $\mathbf{i}$.

Let the new velocity be $x\mathbf{i}+y\mathbf{j}$.

Use conservation of momentum:

$5(\mathbf{i}-\mathbf{j})+2(-\mathbf{i}+\mathbf{j})=7(x\mathbf{i}+y\mathbf{j})$

Then $5\mathbf{i}-5\mathbf{j}-2\mathbf{i}+2\mathbf{j}=7x\mathbf{i}+7y\mathbf{j}$

$\therefore 3\mathbf{i}-3\mathbf{j}=7x\mathbf{i}+7y\mathbf{j}$

Equate coefficients of $\mathbf{i}$ and $\mathbf{j}$ to give

$7x=3$ and $7y=-3$

$\therefore x=\frac{3}{7}$ and $y=-\frac{3}{7}$

$\therefore$ velocity is $\frac{3}{7}\mathbf{i}-\frac{3}{7}\mathbf{j}$

The magnitude of the velocity is $\sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{3}{7}\right)^2}=\frac{3}{7}\sqrt{2}=0.606$ m s^{-1}(3 s.f.)

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Exercise B, Question 1

Question:

In each part of this question the two diagrams show the speeds of two particles A and B just before and just after a collision. The particles move on a smooth horizontal plane. Find the coefficient of restitution e in each case.

	Before collision		After collision	
a	6 ms^{-1} $\rightarrow$ A	At rest B	At rest A	4 ms^{-1} $\rightarrow$ B
b	4 ms^{-1} $\rightarrow$ A	2 ms^{-1} $\rightarrow$ B	2 ms^{-1} $\rightarrow$ A	3 ms^{-1} $\rightarrow$ B
c	9 ms^{-1} $\rightarrow$ A	6 ms^{-1} $\leftarrow$ B	3 ms^{-1} $\leftarrow$ A	2 ms^{-1} $\rightarrow$ B

Solution:

a $e=\dfrac{4-0}{6-0}=\dfrac{2}{3}$

b $e=\dfrac{3-2}{4-2}=\dfrac{1}{2}$

c

$$e=\frac{2-(-3)}{9-(-6)}$$
$$=\frac{5}{15}$$
$$=\frac{1}{3}$$

Use

$$e=\frac{v_2-v_1}{u_1-u_2}=\frac{\text{speed of separation}}{\text{speed of approach}}.$$

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Exercise B, Question 2

Question:

In each part of this question the two diagrams show the speeds of two particles A and B just before and just after a collision. The particles move on a smooth horizontal plane. The masses of A and B and the coefficients of restitution e are also given. Find the speeds v_1 and v_2 in each case.

		Before collision		After collision	
a	$e = \frac{1}{2}$	6 m s^{-1} → A (0.25kg)	At rest B (0.5kg)	$v_1\text{ m s}^{-1}$ → A (0.25kg)	$v_2\text{ m s}^{-1}$ → B (0.5kg)
b	$e = 0.25$	4 m s^{-1} → A (2kg)	2 m s^{-1} → B (3kg)	$v_1\text{ m s}^{-1}$ → A (2kg)	$v_2\text{ m s}^{-1}$ → B (3kg)
c	$e = \frac{1}{7}$	8 m s^{-1} → A (3kg)	6 m s^{-1} ← B (1kg)	$v_1\text{ m s}^{-1}$ → A (3kg)	$v_2\text{ m s}^{-1}$ → B (1kg)
d	$e = \frac{2}{3}$	6 m s^{-1} → A (400g)	6 m s^{-1} ← B (400g)	$v_1\text{ m s}^{-1}$ → A (400g)	$v_2\text{ m s}^{-1}$ → B (400g)
e	$e = \frac{1}{5}$	3 m s^{-1} → A (5kg)	12 m s^{-1} ← B (4kg)	$v_1\text{ m s}^{-1}$ → A (5kg)	$v_2\text{ m s}^{-1}$ → B (4kg)

Solution:

Use conservation of linear momentum and Newton's Law of Restitution.

a Using conservation of linear momentum:

$0.25\times 6+0.5\times 0=0.25v_1+0.5v_2$

Multiply equation by 4.

$6=v_1+2v_2 \qquad (1)$

Using Newton's Law of Restitution:

$$e=\frac{\text{speed of separation}}{\text{speed of approach}}$$

i.e. $\frac{1}{2}=\frac{v_2-v_1}{6-0}$

$\therefore 3=v_2-v_1 \qquad (2)$

Add equations (1) and (2)

Then

$9=3v_2$

$\therefore v_2=3$

Substitute into equation (1)

$\therefore 6=v_1+2\times 3$

$\therefore v_1=0$

$\therefore$ A is at rest and B moves at 3 m s^{-1} after the collision.

b Using conservation of linear momentum:

$2\times 4+3\times 2=2v_1+3v_2$

$\therefore 14=2v_1+3v_2 \qquad (1)$

Using Newton's Law of Restitution:

$$e=\frac{\text{speed of separation}}{\text{speed of approach}}$$

$\therefore 0.25=\frac{v_2-v_1}{4-2}$

$\therefore 0.5=v_2-v_1 \qquad (2)$

Multiply equation (2) by 2 and add to equation (1).

$\therefore 15=5v_2$

$\therefore v_2=3$

Substitute into equation (1) $v_1=2\frac{1}{2}$

$\therefore$ A and B move with speeds $2\frac{1}{2}\text{ m s}^{-1}$ and 3 m s^{-1} respectively after the collision.

c Using conservation of linear momentum:

$$3\times 8 + 1\times(-6) = 3v_1 + 1v_2$$
$$\therefore 18 = 3v_1 + v_2 \quad (1)$$

Remember that the speed 6 m s^{-1} appears in the equations as -6 because it is directed to the left in the diagram.

Using Newton's Law of Restitution:

$$e = \frac{1}{7} = \frac{v_2 - v_1}{8-(-6)}$$
$$\therefore \frac{1}{7} = \frac{v_2 - v_1}{14}$$
$$\therefore 2 = v_2 - v_1 \quad (2)$$

Subtract to give equation (1) − equation (2)

$$\therefore 16 = 4v_1$$
$$\therefore v_1 = 4$$

Substitute in equation (1) to give $v_2 = 6$.

[This answer may be checked in equation (2).]

$\therefore$ Speed of A is 4 m s^{-1} and speed of B is 6 m s^{-1} after the collision.

d Using conservation of linear momentum:

$$0.4\times 6 - 0.4\times 6 = 0.4v_1 + 0.4v_2$$
$$\therefore 0 = v_1 + v_2 \quad (1)$$

Using Newton's Law of Restitution:

$$e = \frac{2}{3} = \frac{v_2 - v_1}{6-(-6)}$$
$$\text{i.e. } \frac{2}{3} = \frac{v_2 - v_1}{12}$$
$$\therefore v_2 - v_1 = 8 \quad (2)$$

Add equations (1) and (2)

$$\therefore 2v_2 = 8$$
$$\text{i.e. } v_2 = 4$$

Substitute into equation (1)

$$\therefore v_1 = -4.$$

The speeds of A and B are 4 m s^{-1} after the collision, and both change direction after the collision.

e Using conservation of linear momentum:

Remember that a particle moving in the opposite direction (i.e. to the left) has a *negative* velocity in the equations.

$$5\times3-4\times12=5v_1+4v_2$$
$$\therefore -33=5v_1+4v_2 \quad (1)$$

Using Newton's Law of Restitution:

$$e=\frac{1}{5}=\frac{v_2-v_1}{3-(-12)}$$
$$\therefore \frac{1}{5}=\frac{v_2-v_1}{15}$$
$$\therefore 3=v_2-v_1 \qquad (2)$$

Multiply equation (2) by 5 and add to equation (1)

$$\therefore -18=9v_2$$
$$\therefore v_2=-2$$

Substitute into equation (1)

$$\therefore -33=5v_1-8$$
$$\therefore -25=5v_1$$

[Check your answers in equation (2).]

$$\therefore v_1=-5$$

The speeds of A and B are 5 m s^{-1} and 2 m s^{-1} after the collision and both change direction.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 3

Question:

A small smooth sphere A of mass 1 kg is travelling along a straight line on a smooth horizontal plane with speed 4 m s^{-1} when it collides with a second smooth sphere B of the same radius, with mass 2 kg and travelling in the same direction as A with speed 2.5 m s^{-1}.

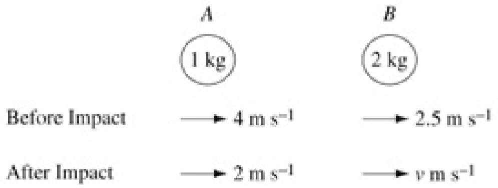
After the collision, A continues in the same direction with speed 2 m s^{-1}.

a Find the speed of B after the collision.

b Find the coefficient of restitution for the spheres.

Solution:

a



Draw a clearly labelled diagram.

Let the speed of B, after the collision, be $v \text{ m s}^{-1}$.

Use conservation of linear momentum:

$$1 \times 4 + 2 \times 2.5 = 1 \times 2 + 2v$$
$$\therefore 9 = 2 + 2v$$
$$\therefore 2v = 7$$
$$v = 3.5$$

Speed of B after the collision is 3.5 m s^{-1}.

b Use Newton's Law of Restitution:

$$e = \frac{v-2}{4-2.5} = \frac{3.5-2}{4-2.5} = \frac{1.5}{1.5} = 1$$

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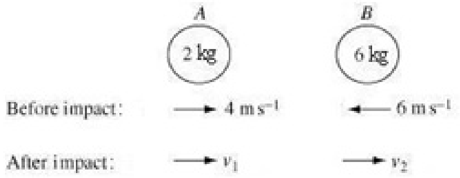
Exercise B, Question 4

Question:

Two spheres A and B are of equal radius and have masses 2 kg and 6 kg respectively. A and B move towards each other along the same straight line on a smooth horizontal surface with velocities 4 m s^{-1} and 6 m s^{-1} respectively. If the coefficient of friction is $\frac{1}{5}$, find the velocities of the spheres after the collision and the magnitude of the impulse given to each sphere.

Solution:



Using conservation of linear momentum:

Use conservation of linear momentum and Newton's Law of Restitution.

$$2\times4-6\times6=2v_1+6v_2$$
$$\therefore -28=2v_1+6v_2$$

$\div$ by 2

Before impact:
$\therefore -14=v_1+3v_2 \quad (1)$

Using Newton's Law of Restitution:

$$e=\frac{1}{5}=\frac{v_2-v_1}{4-(-6)}$$
$$\therefore \frac{1}{5}=\frac{v_2-v_1}{10}$$

i.e. $2=v_2-v_1 \quad (2)$

Add equations (1) and (2)

$$-12=4v_2$$
$$\therefore v_2=-3$$

Substitute into equation (2)

$$2=-3-v_1$$
$\therefore v_1=-5$ [Check in equation (1).]

The speed of A and B are 5 m s^{-1} and 3 m s^{-1} both now in the direction that B was moving before the impact.

Use impulse = change of momentum for each particle separately.

The impulse given to sphere A = change in momentum of sphere A

$$=2\times(-5)-2\times4$$
$$=-18\text{ Ns}$$

The impulse given to sphere B = change in momentum of sphere B

$$=6\times(-3)-6\times(-6)$$
$$=18\text{ Ns}$$

$\therefore$ A and B experience equal and opposite impulses – to the left on A and to the right on B.

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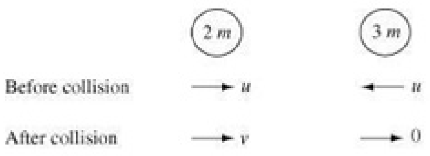
Exercise B, Question 5

Question:

Two particles of mass $2m$ and $3m$ respectively are moving towards each other with speed u. If the $3m$ mass is brought to rest by the collision, find the speed of the $2m$ mass after the collision and the coefficient of restitution between the particles.

Solution:



Let the speed of the $2m$ mass be $v\ \mathrm{m\,s^{-1}}$ after the collision.

Using conservation of linear momentum:

$$2mu - 3mu = 2mv + 3m \times 0$$
$$\therefore -mu = 2mv$$
$$\therefore v = -\frac{u}{2}$$

Using Newton's Law of Restitution:

$$e = \frac{0 - v}{u - (-u)}$$
$$= \frac{\frac{u}{2}}{2u}$$
$$= \frac{1}{4}$$

$\therefore$ The $2m$ mass moves with speed $\frac{u}{2}\ \mathrm{m\,s^{-1}}$ after the collision, having changed direction.

The coefficient of restitution is $\frac{1}{4}$.

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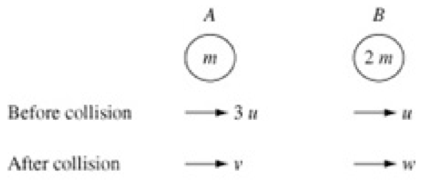
Exercise B, Question 6

Question:

Two particles A and B are travelling along the same straight line in the same direction on as smooth horizontal surface with speeds $3u$ and u respectively. Particle A catches up and collides with particle B. If the mass of B is twice that of A and the coefficient of restitution is e find, in terms of e and u, expressions for the speeds of A and B after the collision.

Solution:



Let the speeds of A and B, after the collision, be v and w respectively.
Using conservation of linear momentum:

$m \times 3u + 2m \times u = mv + 2mw$
$\div$ through by m
$\therefore v + 2w = 5u \quad (1)$

Using Newton's Law of Restitution:

$$\frac{w - v}{3u - u} = e$$
$\therefore w - v = 2ue \quad (2)$

Add equations (1) and (2)

$3w = u(5 + 2e)$

$$\therefore w = \frac{u}{3}(5 + 2e)$$

Subtract twice equation (2) from equation (1)

$\therefore 3v = 5u - 4ue$

$$\therefore v = \frac{u}{3}(5 - 4e)$$

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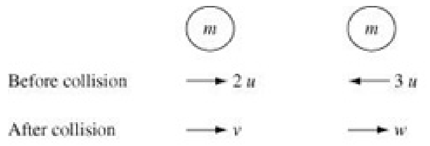
Exercise B, Question 7

Question:

Two identical particles of mass m are projected towards each other along the same straight line on a smooth horizontal surface with speeds of $2u$ and $3u$. After the collision the directions of motion of both particles are reversed. Show that this implies that the coefficient of restitution e satisfies the inequality $e > \frac{1}{5}$.

Solution:



Using conservation of linear momentum:

$$m \times 2u + m(-3u) = mv + mw$$
$$\therefore v + w = -u \quad (1)$$

Using Newton's Law of Restitution:

$$\frac{w - v}{2u + 3u} = e$$
$$\therefore w - v = 5eu \quad (2)$$

Add equations (1) and (2)

$$2w = 5eu - u$$
$$\therefore w = \frac{u}{2}(5e - 1)$$

Subtract equation (2) from equation (1)

$$2v = -u - 5eu.$$
$$\therefore v = \frac{u}{2}(-1 - 5e).$$

As directions of both particles are reversed, $v < 0$ and $w > 0$.

As $v = -\frac{u}{2}(1 + 5e)$, then $v < 0$ for all values of e.

$w = \frac{u}{2}(5e - 1)$, then $w > 0$ implies $5e - 1 > 0$

i.e. $e > \frac{1}{5}$

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Exercise B, Question 8

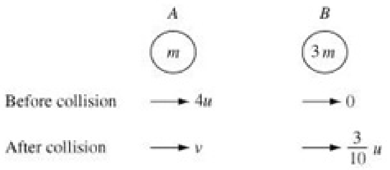
Question:

Two particles A and B of mass m and km respectively are placed on a smooth horizontal plane. Particle A is made to move on the plane with speed u so as to collide directly with B which is at rest. After the collision B moves with speed $\frac{3}{10}u$.

a Find, in terms of u and the constant k, the speed of A after the collision.

b By using Newton's Law of Restitution show that $\frac{7}{3} \le k \le \frac{17}{3}$.

Solution:



a Let the speed of A after the collision be v.

Using conservation of linear momentum:

$$mu + km \times 0 = mv + km \times \frac{3}{10}u$$

$$\therefore mv = mu - \frac{3}{10}kmu$$

$\div$ through by m

$$\therefore v = u - \frac{3}{10}ku \quad \text{or } \frac{u}{10}(10 - 3k)$$

b Using Newton's Law of Restitution:

$$\frac{\frac{3}{10}u - v}{u - 0} = e$$

$$\therefore \frac{3}{10}u - (u - \frac{3}{10}ku) = eu$$

$$\therefore \frac{3}{10}ku - \frac{7}{10}u = eu$$

Multiply both sides by 10 and $\div$ by u

$$\therefore 3k = 7 + 10e$$

$$\therefore k = \frac{7 + 10e}{3}$$

As $0 \le e \le 1$, then $\frac{7}{3} \le k \le \frac{17}{3}$.

To show the inequality, you will need to use $0 \le e \le 1$.

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Exercise B, Question 9

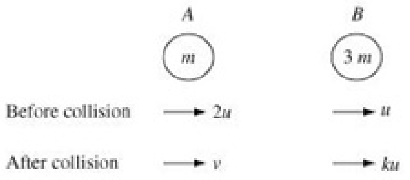
Question:

Two particles A and B of mass m and $3m$ respectively are placed on a smooth horizontal plane. Particle A is made to move on the plane with speed $2u$ so as to collide directly with B which is moving in the same direction with speed u. After the collision B moves with speed ku, where k is a positive constant.

a Find, in terms of u and the constant k, the speed of A after the collision.

b By using Newton's Law of Restitution show that $\frac{5}{4} \leq k \leq \frac{3}{2}$.

Solution:



a Let the velocity of A after the collision be v.

Use conservation of linear momentum:

$$m2u + 3mu = mv + 3mku$$
$$\therefore v + 3ku = 5u$$
$$\text{i.e. } v = u(5 - 3k)$$

b Using Newton's Law of Restitution:

$$\frac{ku - v}{2u - u} = e$$
$$\therefore ku - v = eu$$
$$\therefore ku - u(5 - 3k) = eu$$
$$\therefore 4ku - 5u = eu$$
$$\therefore k = \frac{e + 5}{4}$$

But $0 \leq e \leq 1$ $\therefore \frac{5}{4} \leq k \leq \frac{3}{2}$

You will need to use the condition $0 \leq e \leq 1$.

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Exercise B, Question 10

Question:

A particle P of mass m is moving with speed $4u$ on a smooth horizontal plane. The particle collides directly with a particle Q of mass $3m$ moving with speed $2u$ in the same direction as P. The coefficient of restitution between P and Q is e.

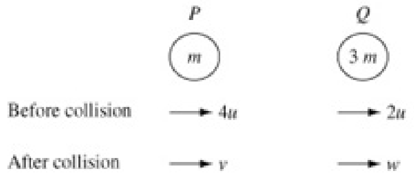
a Show that the speed of Q after the collision is $\frac{u}{2}(5+e)$.

b Find the speed of P after the collision, giving your answer in terms of e.

c Show that the direction of motion of P is unchanged by the collision, provided that $e < \frac{3}{5}$.

d Given that the magnitude of the impulse of P on Q is $2mu$, find the value of e.

Solution:



a Let the speeds of P and Q after the collision be v and w respectively.

Using conservation of linear momentum:

$$m \times 4u + 3m \times 2u = mv + 3mw$$
$$\therefore v + 3w = 10u \quad (1)$$

Using Newton's Law of Restitution:

$$\frac{w - v}{4u - 2u} = e$$
$$\therefore w - v = 2ue \quad (2)$$

Add equations (1) and (2).

$$4w = 10u + 2ue$$
$$\therefore w = \frac{2u}{4}(5 + e) = \frac{u}{2}(5 + e)$$

b Substitute into equation (1).

$$\therefore \quad v + \frac{3u}{2}(5 + e) = 10u$$
$$\therefore \quad v = 10u - \frac{15u}{2} - \frac{3ue}{2}$$
$$\therefore \quad v = \frac{u}{2}(5 - 3e)$$

[Check that w and v satisfy equation (2).]

c The direction of motion of P is unchanged provided that $\frac{u}{2}(5 - 3e) > 0$

i.e. $e < \frac{3}{5}$.

Use $v > 0$, where v is velocity of P after the collision.

d Change of momentum of Q is

Use impulse = change in momentum of Q.

$$3m(w - 2u) = 3m\left(\frac{5u}{2} + \frac{eu}{2} - 2u\right)$$
$$= \frac{3mu}{2}(1 + e)$$

But impulse of P on Q is $2mu$

$$\therefore 2mu = \frac{3mu}{2}(1 + e)$$
$$\therefore 1 + e = \frac{4}{3}$$
$$\therefore e = \frac{1}{3}$$

Solutionbank M2

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Exercise C, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8 \text{ m s}^{-2}$.

A smooth sphere collides normally with a fixed vertical wall. The two diagrams show the speeds of the sphere before and after collision. In each case find the value of the coefficient of restitution e.

a	Before impact		After impact	
	10 m s^{-1} $\rightarrow$	Wall	4 m s^{-1} $\leftarrow$	Wall

b	Before impact		After impact	
	6 m s^{-1} $\rightarrow$	Wall	3 m s^{-1} $\leftarrow$	Wall

Solution:

a $e = \dfrac{\text{speed of rebound}}{\text{speed of approach}}$

$= \dfrac{4}{10}$

$\therefore e = \dfrac{2}{5}$

Use $e = \dfrac{\text{speed of rebound}}{\text{speed of approach}}$.

b $e = \dfrac{\text{speed of rebound}}{\text{speed of approach}}$

$= \dfrac{3}{6}$

$\therefore e = \dfrac{1}{2}$

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Exercise C, Question 2

Question:

A smooth sphere collides normally with a fixed vertical wall. The two diagrams show the speed of the sphere before and after the collision. The value of e is given in each case.

Find the speed of the sphere after the collision in each case.

a $e=\frac{1}{2}$	Before impact		After impact	
	7 ms^{-1} $\rightarrow$ ○	Wall	$v\text{ ms}^{-1}$ $\leftarrow$ ○	Wall
b $e=\frac{1}{4}$	Before impact		After impact	
	12 ms^{-1} $\rightarrow$ ○	Wall	$V\text{ ms}^{-1}$ $\leftarrow$ ○	Wall

Solution:

a Use

Use $e=\dfrac{\text{speed of rebound}}{\text{speed of approach}}$ and make speed of rebound the subject of the formula.

$$e=\frac{\text{speed of rebound}}{\text{speed of approach}}$$

$$\therefore \frac{1}{2}=\frac{v}{7}$$

$$\therefore v=7\times\frac{1}{2}$$

$$=3.5$$

Speed of sphere after collision is 3.5 m s^{-1}.

b Use

$$e=\frac{\text{speed of rebound}}{\text{speed of approach}}$$

$$\therefore \frac{1}{4}=\frac{V}{12}$$

$$\therefore V=\frac{1}{4}\times 12$$

$$=3$$

$\therefore$ speed after collision is 3 m s^{-1}.

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Exercise C, Question 3

Question:

A smooth sphere collides normally with a fixed vertical wall. The two diagrams show the speed of the sphere before and after the collision. The value of e is given in each case.

Find the speed of the sphere before the collision in each case.

	Before impact		After impact	
a $e=\frac{1}{2}$				
	$u\ \text{ms}^{-1}$ $\rightarrow$ ○	Wall	$4\ \text{ms}^{-1}$ $\leftarrow$ ○	Wall
b $e=\frac{3}{4}$	Before impact		After impact	
	$u\ \text{ms}^{-1}$ $\rightarrow$ ○	Wall	$6\ \text{ms}^{-1}$ $\leftarrow$ ○	Wall

Solution:

a Use

$$e=\frac{\text{speed of rebound}}{\text{speed of approach}}$$

$$\therefore \frac{1}{2}=\frac{4}{u}$$

$$\therefore u\times\frac{1}{2}=4$$

$$\therefore u=4\times2$$

$$=8$$

Speed before the collision is $8\ \text{m s}^{-1}$.

b Use

$$e=\frac{\text{speed of rebound}}{\text{speed of approach}}$$

$$\therefore \frac{1}{4}=\frac{V}{12}$$

$$\therefore V=\frac{1}{4}\times12$$

$$=3$$

$\therefore$ speed after collision is $3\ \text{m s}^{-1}$.

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Exercise C, Question 4

Question:

A small smooth sphere of mass 0.3 kg is moving on a smooth horizontal table with a speed of 10 m s^{-1} when it collides normally with a fixed smooth wall. It rebounds with a speed of 7.5 m s^{-1}. Find the coefficient of restitution between the sphere and the wall.

Solution:

Use

$$e = \frac{\text{speed of rebound}}{\text{speed of approach}}$$

$$\therefore e = \frac{7.5}{10}$$

i.e. $e = 0.75$

The coefficient of restitution is 0.75 or $\frac{3}{4}$.

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Exercise C, Question 5

Question:

A particle falls 2.5 m from rest on to a smooth horizontal plane. It then rebounds to a height of 1.5 m. Find the coefficient of restitution between the particle and the plane. Give your answer to 2 s.f.

Solution:

The particle falls under gravity:

Find velocity when particle hits plane by using constant acceleration formula.

$s = 2.5, a = g, u = 0, v = ?$

Use $v^2 = u^2 + 2as$ as motion is under constant acceleration.

$$\therefore v^2 = 2 \times g \times 2.5$$
$$= 5g$$
$$\therefore v = \sqrt{5g} = \sqrt{5 \times 9.8} = \sqrt{49} = 7$$

Particle strikes the plane with velocity 7 m s^{-1}.

After it rebounds it moves under gravity to a height of 1.5 m.

After the impact the motion is again under gravity.

$u = ?\ s = 1.5\ a = -g\ v = 0$

Use

$$v^2 = u^2 + 2as$$
$$\therefore 0 = u^2 - 2g \times 1.5$$
$$\therefore u^2 = 3g$$
$$= 3 \times 9.8$$
$$= 29.4$$
$$\therefore u = \sqrt{29.4}$$
$$= 5.422$$

The velocity after impact is 5.422 m s^{-1}.

Using

$$e = \frac{\text{speed of rebound}}{\text{speed of approach}}$$
$$e = \frac{5.422}{7} = 0.78 \text{ (2 s.f.)}$$

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Exercise C, Question 6

Question:

A particle falls 3 m from rest onto a smooth horizontal plane. It then rebounds to a height h m. The coefficient of restitution between the particle and the plane is 0.25. Find the value of h.

Solution:

The particle falls under gravity

$s=3, u=0, a=g, v=?$

Use

$$\therefore v^2 = u^2 + 2as$$
$$v^2 = 2\times g\times 3$$
$$= 6g$$
$$v = \sqrt{6g}$$

Use $v^2 = u^2 + 2as$ for the motion before impact and for the motion after impact.

It hits the ground and rebounds. The velocity after the impact is ev.
i.e. new velocity is $0.25\sqrt{6g}$.
It rebounds and moves under gravity.
$u=0.25\sqrt{6g}, a=-g, s=h, v=0$
Use

$$v^2 = u^2 + 2as$$
$$\therefore 0 = \left(0.25\sqrt{6g}\right)^2 - 2gh$$
$$\therefore 2gh = \frac{1}{16}\times 6g$$
$$\therefore h = \frac{6g}{16}\div 2g$$
$$\therefore h = \frac{3}{16}$$

So the particle rebounds to a height of $\frac{3}{16}$ m $= 18.75$ cm .

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Edexcel AS and A Level Modular Mathematics

Exercise C, Question 7

Question:

A small smooth sphere falls from rest onto a smooth horizontal plane. It takes 2 seconds to reach the plane then another 2 seconds to reach the plane a second time. Find the coefficient of restitution between the particle and the plane.

Solution:

The sphere falls under gravity.

$u = 0, t = 2, a = g, v = ?$

Use

$$v = u + at$$

Use $v = u + at$ to find the sphere's velocity when it hits the plane.

$$\therefore v = 2g$$

This is its velocity when it hits the plane.

The sphere then bounces and its new velocity is $2ge$, where e is the coefficient of restitution.

The sphere then moves under gravity for 2 seconds.

$u = 2ge, a = -g, t = 2, s = 0$

Use

$$s = ut + \frac{1}{2}at^2$$

For the motion after impact, use $s = ut + \frac{1}{2}at^2$ with $s = 0$ and $a = -g$.

$$\therefore 0 = 2ge \times 2 - \frac{1}{2}g \times 4$$
$$= 4ge - 2g.$$
$$\therefore 4ge = 2g$$
$$e = \frac{2g}{4g} = \frac{1}{2}$$

The coefficient of restitution is $\frac{1}{2}$.

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Exercise C, Question 8

Question:

A small smooth sphere falls from rest onto a smooth horizontal plane. It takes 3 seconds to reach the plane. The coefficient of restitution between the particle and the plane is 0.49.

Find the time it takes for the sphere to reach the plane a second time.

Solution:

The sphere falls under gravity.

$u = 0, t = 3, v = ?, a = g$

Use

$$v = u + at$$
$$\therefore v = 3g$$

After impact the new velocity is $0.49 \times 3g$.

The sphere then moves again under gravity.

$u = 0.49 \times 3g, t = ?, a = -g, s = 0$

Use

$$s = ut + \frac{1}{2}at^2$$
$$\therefore 0 = 1.47gt - \frac{1}{2}gt^2$$
$$\therefore t = \frac{2 \times 1.47g}{g}$$
$$= 2.94 \text{ s}$$

So the time is 2.9 s (2 s.f.)

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Exercise D, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8 \text{ m s}^{-2}$.

Three small smooth spheres A, B and C of equal radius move along the same straight line on a horizontal plane. Sphere A collides with sphere B and then sphere B collides with sphere C.

The diagrams show the velocities before the first collision, after the first collision between A and B and then after the collision between B and C.

a Find the values of u, v, x and y if $e = \frac{1}{2}$ for both collisions.

Before collision			After A and B have collided			After B and C have collided		
$5 \text{ ms}^{-1} \rightarrow$	$1 \text{ ms}^{-1} \rightarrow$	$4 \text{ ms}^{-1} \rightarrow$	$u \text{ ms}^{-1} \rightarrow$	$v \text{ ms}^{-1} \rightarrow$	$4 \text{ ms}^{-1} \rightarrow$	$u \text{ ms}^{-1} \rightarrow$	$x \text{ ms}^{-1} \rightarrow$	$y \text{ ms}^{-1} \rightarrow$
A (2kg)	B (1kg)	C (2kg)	A (2kg)	B (1kg)	C (2kg)	A (2kg)	B (1kg)	C (2kg)

b Find the values of u, v, x and y if $e = \frac{1}{6}$ for the collision between A and B and $e = \frac{1}{2}$ for the collision between B and C.

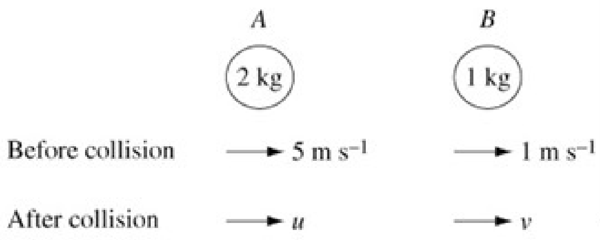
Before collision			After A and B have collided			After B and C have collided		
$10 \text{ ms}^{-1} \rightarrow$	$2 \text{ ms}^{-1} \leftarrow$	$3 \text{ ms}^{-1} \rightarrow$	$u \text{ ms}^{-1} \rightarrow$	$v \text{ ms}^{-1} \rightarrow$	$3 \text{ ms}^{-1} \rightarrow$	$u \text{ ms}^{-1} \rightarrow$	$x \text{ ms}^{-1} \rightarrow$	$y \text{ ms}^{-1} \rightarrow$
A (1.5kg)	B (2kg)	C (1kg)	A (1.5kg)	B (2kg)	C (1kg)	A (1.5kg)	B (2kg)	C (1kg)

Solution:

a

First collision (between *A* and *B*)

> First consider the collision between A and B and use conservation of linear momentum and Newton's Law of Restitution to find u and v. Then repeat the process for the collision between B and C.



Use conservation of linear momentum:

$$2\times5+1\times1=2u+1v$$
$$\therefore 11=2u+v \quad (1)$$

Use Newton's Law of Restitution:

$$e=\frac{1}{2}=\frac{v-u}{5-1}$$
$$\therefore v-u=\tfrac{1}{2}\times4$$
$$v-u=2 \quad (2)$$

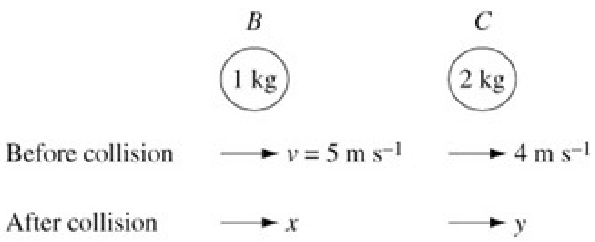
Subtract equation (2) from equation (1).

$$9=3u$$
$$\therefore u=3$$

Substitute into equation (1)

$$11=6+v$$
$$\therefore v = 5$$ [Check in equation (2).]

Second collision (between *B* and *C*)



Use conservation of linear momentum:

$$1\times5+2\times4=1\times x+2\times y$$
$$\therefore 13 = x+2y \quad (3)$$

Use Newton's Law of Restitution:

$$e=\frac{1}{2}=\frac{y-x}{5-4}$$
$$\therefore \frac{1}{2}=y-x \quad (4)$$

Add equations (3) and (4)

$13\frac{1}{2} = 3y$

$\therefore y = 4\frac{1}{2}$

Substitute back into equation (4)

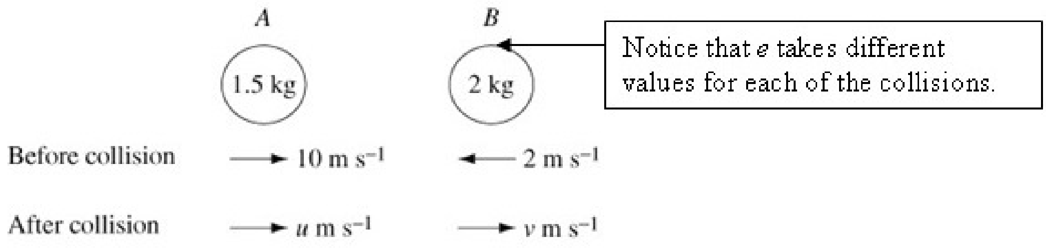
$\frac{1}{2} = 4\frac{1}{2} - x$

$\therefore x = 4$ [Check in equation (3).]

$\therefore u = 3, v = 5, x = 4, y = 4\frac{1}{2}$

b

First collision



Use conservation of linear momentum:

$1.5 \times 10 - 2 \times 2 = 1.5u + 2v$

$\therefore 11 = 1.5u + 2v \quad (1)$

Use Newton's Law of Restitution:

$e = \frac{1}{6} = \frac{v-u}{10-(-2)}$

$\therefore \frac{1}{6} \times 12 = v - u$

i.e. $2 = v - u \quad (2)$

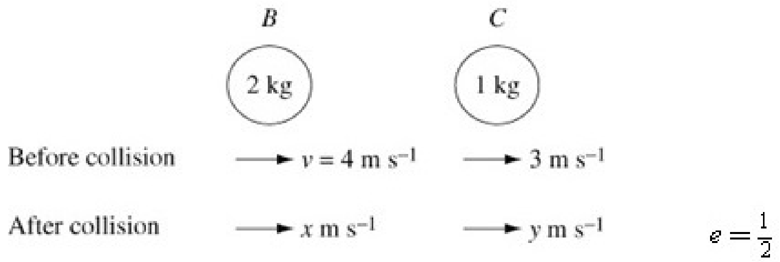
Add (1) to $1.5 \times (2)$.

$14 = 3.5v$

$\therefore v = 4$

Substitute into equation (1) to give $u = 2$.
[Check your answers in equation (2).]

Second collision



Use conservation of linear momentum:

$$2 \times 4 + 1 \times 3 = 2x + y$$
$$\therefore 11 = 2x + y \quad (1)$$

Use Newton's Law of Restitution:

$$e = \frac{1}{2} = \frac{y - x}{4 - 3}$$
$$\therefore \frac{1}{2} = y - x \quad (2)$$

Subtract equation (2) from equation (1)

$$10\frac{1}{2} = 3x$$
$$\therefore x = 3\frac{1}{2}$$

Substitute into equation (2)

$$\therefore y = 4$$
$$u = 2, v = 4, x = 3\frac{1}{2}, y = 4$$

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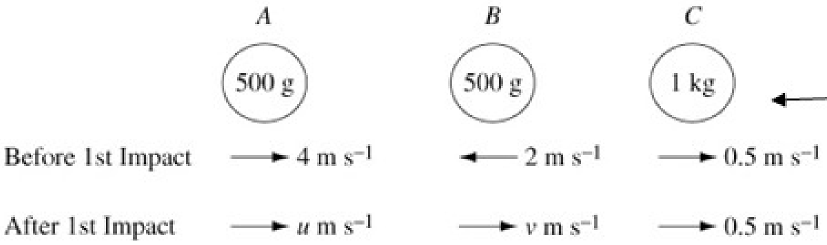
Exercise D, Question 2

Question:

Three small smooth spheres A, B and C of equal radius have masses 500 g, 500 g and 1 kg respectively. The spheres move along the same straight line on a horizontal plane with A following B which is following C. Initially the velocities of A, B and C are $4\mathbf{i}\text{ m s}^{-1}, -2\mathbf{i}\text{ m s}^{-1}$ and $0.5\mathbf{i}\text{ m s}^{-1}$ respectively, where $\mathbf{i}$ is a unit vector in the direction ABC. Sphere A collides with sphere B and then sphere B collides with sphere C. The coefficient of restitution between A and B is $\frac{2}{3}$ and between B and C is $\frac{1}{2}$. Find the velocities of the three spheres after all of the collisions have taken place.

Solution:



Draw clearly labelled diagrams for each collision separately. Make it clear which speed refers to which particle.

Impact between A and B

Using conservation of linear momentum:

$$0.5\times4-0.5\times2=0.5u+0.5v$$
$$1=0.5u+0.5v \quad (1)$$

Using Newton's Law of Restitution:

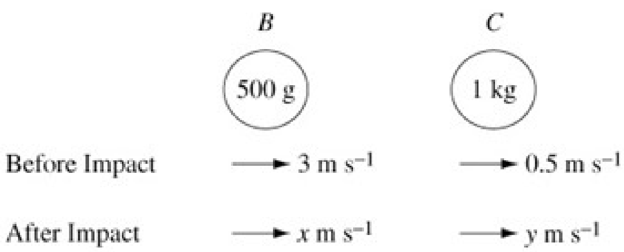
$$e=\frac{2}{3}=\frac{v-u}{4-(-2)}$$
$$\therefore \frac{2}{3}\times6=v-u$$
$$\text{i.e. } 4=v-u \qquad (2)$$

Multiply equation (1) by 2 and add to equation (2)

$$\therefore 6=2v$$
$$\therefore v=3$$

Substitute into equation (2) $\therefore 4=3-u$ and $u=-1$
[Check u and v in equation (1).]

Impact between B and C



Using conservation of linear momentum:

$$0.5\times3+1\times0.5=0.5x+1y$$
$$\therefore 2=0.5x+y \quad (1)'$$

Using Newton's Law of Restitution:

$$e = \frac{1}{2} = \frac{y - x}{3 - 0.5}$$
$$\therefore \frac{1}{2} \times 2.5 = y - x$$
$$\therefore 1.25 = y - x \quad (2)'$$

Subtract equation $(2)'$ from equation $(1)'$

$$0.75 = 1.5x$$
$$x = \frac{0.75}{1.5}$$
$$= 0.5$$

Substitute back into equation $(2)'$

$$\therefore y = 0.5 + 1.25 = 1.75$$

So $u = -1, v = 3, x = 0.5, y = 1.75$

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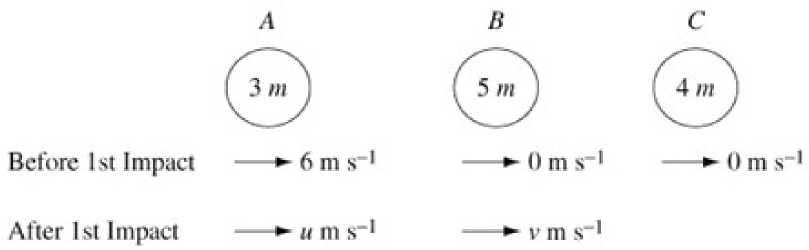
Exercise D, Question 3

Question:

Three perfectly elastic particles A, B and C of masses $3m$, $5m$ and $4m$ respectively lie at rest on a straight line on a smooth horizontal table with B between A and C. Particle A is projected directly towards B with speed 6 m s^{-1} and after A has collided with B, B then collides with C. Find the speed of each particle after the second impact.

Solution:



Use conservation of linear momentum:

$$3m \times 6 + 5m \times 0 = 3mu + 5mv$$
$$\therefore 18 = 3u + 5v \quad (1)$$

Use Newton's Law of Restitution:

$$e = 1 = \frac{v - u}{6}$$
$$\therefore 6 = v - u \quad (2)$$

'Perfectly elastic' means $e = 1$.

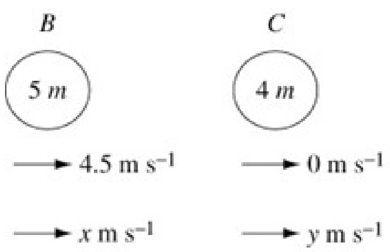
Add equation (1) to 3 times equation (2).

$$\therefore 36 = 8v$$
$$\therefore v = \frac{36}{8} = 4.5$$

Substitute into equation (2):

$$\therefore 6 = 4.5 - u$$
$$\therefore u = -1.5$$

Second impact:



Use conservation of linear momentum:

$$5m \times 4.5 + 4m \times 0 = 5mx + 4my$$
$$\therefore 22.5 = 5x + 4y \quad (1)'$$

Use Newton's Law of Restitution:

$$e = 1 = \frac{y - x}{4.5 - 0}$$

$\therefore 4.5 = y - x \quad (2)'$

Add equation $(1)'$ to 5 times equation $(2)'$

$\therefore 45 = 9y$

$\therefore y = 5$

Substitute into equation $(2)'$

Then

$4.5 = 5 - x$

$\therefore x = 0.5$

$\therefore u = -1.5, v = 4.5, x = 0.5, y = 5$

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Exercise D, Question 4

Question:

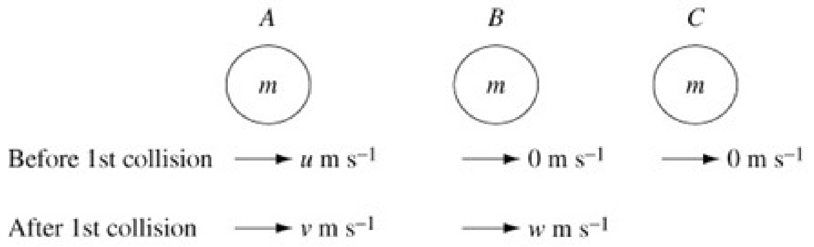
Three identical smooth spheres A, B and C, each of mass m, lie at rest on a straight line on a smooth horizontal table. Sphere A is projected with speed u to strike sphere B directly. Sphere B then strikes sphere C directly. The coefficient of restitution between any two sphere is e, $e \neq 1$.

a Find the speeds in terms of u and e of the spheres after these two collision.

b Show that A will catch up with B and there will be a further collision.

Solution:

a



Using conservation of linear momentum:

$$mu = mv + mw$$
$$\therefore u = v + w \quad (1)$$
$$e = \frac{w - v}{u}$$
$$\therefore eu = w - v \quad (2)$$

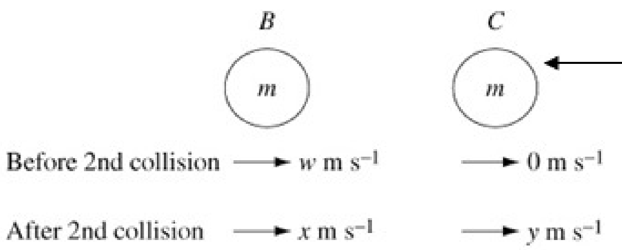
Add $(1) + (2)$

$$\therefore u(1+e) = 2w \rightarrow w = \frac{1}{2}u(1+e)$$

Subtract $(1) - (2)$

$$\therefore u(1-e) = 2v \rightarrow v = \frac{1}{2}u(1-e)$$

Consider second collision



The second collision here is similar to the first. The only difference is that the moving sphere has a different velocity. Replacing 'u' by this velocity will give expressions for velocities of B and C after the second impact.

By similar reasoning to above $y = \frac{1}{2}w(1+e)$ and $x = \frac{1}{2}w(1-e)$

And as $w = \frac{1}{2}u(1+e)$

$y = \frac{1}{4}u(1+e)^2$ and $x = \frac{1}{4}u(1+e)(1-e)$

$\therefore$ The speeds of A, B and C after the two collisions are

$\frac{1}{2}u(1-e), \frac{1}{4}u(1+e)(1-e)$ and $\frac{1}{4}u(1+e)^2$ respectively.

b A will catch up with B provided that

$\frac{1}{2}u(1-e) > \frac{1}{4}u(1+e)(1-e)$

i.e. provided that $2 > 1+e$

Since $e < 1$ this condition holds and A will catch up with B resulting in a further collision.

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Exercise D, Question 5

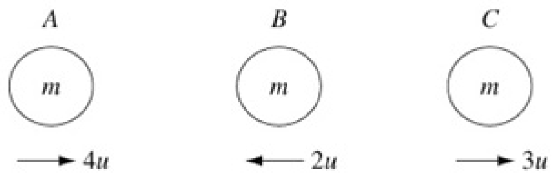
Question:

Three identical spheres A, B and C of equal mass m, and equal radius move along the same straight line on a horizontal plane. B is between A and C. A and B are moving towards each other with velocities $4u$ and $2u$ respectively while C moves away from B with velocity $3u$.

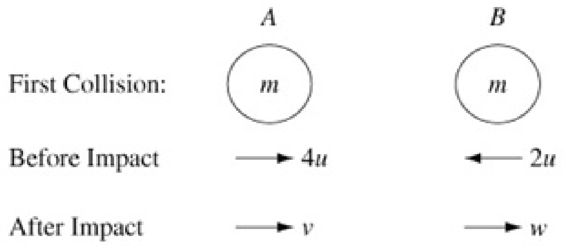
a If the coefficient of restitution between any two of the spheres is e, show that B will only collide with C if $e > \frac{2}{3}$.

b Find the direction of motion of A after collision, if $e > \frac{2}{3}$.

Solution:



First collision:



a Let velocities of A and B be v and w after impact.

Using conservation of linear momentum:

$$m\times 4u - m\times 2u = mv + mw$$

$$\therefore 2u = v + w \qquad (1)$$

Using Newton's Law of Restitution:

$$e = \frac{w - v}{4u - (-2u)}$$

$$\therefore 6ue = w - v \qquad (2)$$

Add equations (1) and (2)

$$2w = 2u + 6ue$$

$$\therefore w = u(1 + 3e)$$

B will then collide with C if

$$w > 3u$$

$$u(1 + 3e) > 3u$$

i.e. $3e > 2 \rightarrow e > \frac{2}{3}$

B collides with C if the velocity of B after the collision is greater than $3u$.

b Subtract (2) from (1)

$$2u - 6ue = 2v$$

$$\therefore v = 2u(1 - 3e)$$

If $e > \frac{2}{3}$ then $v < 0 \therefore A$ moves to the left.

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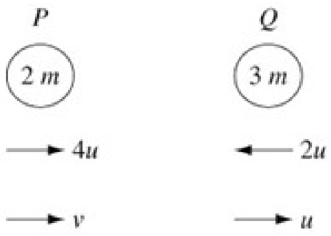
Exercise D, Question 6

Question:

Two particles P of mass $2m$ and Q of mass $3m$ are moving towards each other with speeds $4u$ and $2u$ respectively. The direction of motion of Q is reversed by the impact and its speed after impact is u. This particle then hits a smooth vertical wall perpendicular to its direction of motion. The coefficient of restitution between Q and the wall is $\frac{2}{3}$. In the subsequent motion, there is a further collision between Q and P. Find the speeds of P and Q after this collision.

Solution:



The **first collision** is between P and Q. Let the velocity of P after the collision be v.

Using conservation of momentum:

$$\begin{aligned} 2m\times 4u-3m\times 2u &= 2mv+3mu \\ \therefore 2mu &= 2mv+3mu \\ \therefore 2v &= -u \\ \therefore v &= -\tfrac{1}{2}u \end{aligned}$$

Using Newton's Law of Restitution:

From the information given find the coefficient of restitution between particles P and Q.

$$\begin{aligned} e &= \frac{u-v}{4u-(-2u)} \\ &= \frac{1\frac{1}{2}u}{6u} \\ &= \frac{1}{4} \end{aligned}$$

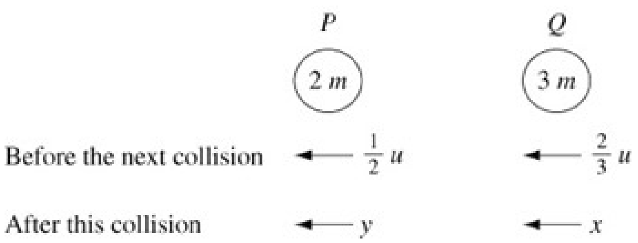
$\therefore$ coefficient of restitution between P and Q is $\frac{1}{4}$.

The **second collision** is between Q and the wall.

Q rebounds from the wall with velocity $e'u=\frac{2}{3}u$.

(as e', the coefficient of restitution between Q and the wall, is $\frac{2}{3}$).

The **next collision** is between P and Q.



Taking direction to left as positive: let velocities of Q and P after this collision be x and y respectively.

Using conservation of linear momentum $\leftarrow$:

$$\begin{aligned} 2m\times\tfrac{1}{2}u+3m\times\tfrac{2}{3}u &= 2my+3mx \\ \therefore mu+2mu &= 2my+3mx \\ \text{i.e. } 3u &= 2y+3x \quad (1) \end{aligned}$$

Using Newton's Law of Restitution:

$$e = \frac{1}{4} = \frac{y - x}{\frac{2}{3}u - \frac{1}{2}u}$$

$$\therefore \frac{1}{4} \times \frac{1}{6}u = y - x$$

$$\text{i.e. } \frac{u}{24} = y - x \quad (2)$$

Add (1) to 3 times (2)

Then

$$3\frac{3}{24}u = 5y$$

$$\text{i.e. } 5y = \frac{75u}{24}$$

$$\therefore y = \frac{15u}{24} = \frac{5u}{8}$$

Substitute into equation (2) $x = \frac{5u}{8} - \frac{u}{24} = \frac{7u}{12}$

$\therefore P$ has speed $\frac{5u}{8}$ and Q has speed $\frac{7u}{12}$ both to the left.

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Exercise D, Question 7

Question:

Two small smooth spheres P and Q of equal radius have masses m and $3m$ respectively.

Sphere P is moving with speed $12u$ on a smooth horizontal table when it collides directly with Q which is at rest on the table. The coefficient of restitution between P and Q is $\frac{2}{3}$.

a Find the speeds of P and Q immediately after the collision.

After the collision Q hits a smooth vertical wall perpendicular to the direction of its motion.

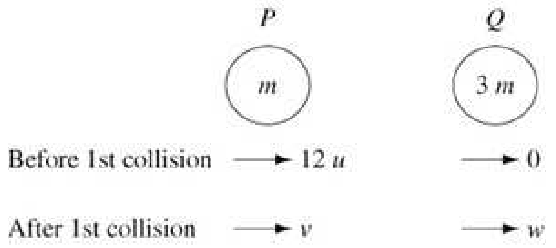
The coefficient of restitution between Q and the wall is $\frac{4}{5}$.

Q then collides with P a second time.

b Find the speeds of P and Q after the second collision between P and Q.

Solution:

a



Use conservation of linear momentum:

$$m \times 12u = mv + 3mw$$
$$\therefore 12u = v + 3w \quad (1)$$

Using Newton's Law of Restitution

$$e = \frac{2}{3} = \frac{w - v}{12u - 0}$$
$$\therefore \frac{2}{3} \times 12u = w - v$$
$$\therefore 8u = w - v \quad (2)$$

Add $(1) + (2)$

$\therefore 20u = 4w \rightarrow w = 5u$
Substitute into equation (2):

$$\therefore 8u = 5u - v$$
$$\text{i.e. } v = -3u$$

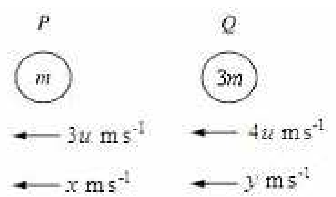
The speeds of P and Q after the first collision are $3u$ m s^{-1} to the left and $5u$ m s^{-1} to the right, respectively.

b

Q then hits a wall and rebounds with speed $4u$ m s^{-1}$(= \frac{4}{5} \times 5u)$.

Let the speeds of P and Q be x m s^{-1} and y m s^{-1} after the second collision between them.

Second collision



Use conservation of linear momentum $\leftarrow$:

$$m \times 3u + 3m \times 4u = mx + 3my$$
$$15mu = mx + 3my$$
$$\text{i.e. } 15u = x + 3y \quad (1)'$$

Use Newton's Law of Restitution:

$$e = \frac{2}{3} = \frac{x - y}{4u - 3u}$$

$$\therefore \frac{2}{3}u = x - y \qquad (2)'$$

Subtract $(1)' - (2)'$

$$\therefore 14\frac{1}{3}u = 4y$$

$$\therefore y = \frac{43}{3}u \div 4 = \frac{43u}{12}$$

Substitute into $(2)'$ to give

$$x = \frac{2}{3}u + \frac{43u}{12}$$

$$= \frac{51u}{12} = \frac{17u}{4}$$

$\therefore$ After the second collision the speeds of P and Q are $\frac{17u}{4}$ m s^{-1} and $\frac{43u}{12}$ m s^{-1} respectively. Both particles are moving away from the wall.

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Exercise D, Question 8

Question:

A small smooth table tennis ball, which may be modelled as a particle, falls from rest at a height 40 cm onto a smooth horizontal plane. The coefficient of restitution between the ball and the plane is 0.7.

a Find the height to which the ball rebounds after the first bounce.

b Find the height to which the ball rebounds after the second bounce.

c Find the total distance travelled by the ball before it comes to rest.

Solution:

a First stage of motion under gravity:
$u = 0$, $a = g = 9.8$, $s = 0.4$, $v = ?$
Use $v^2 = u^2 + 2as$. Then
$$v^2 = 2 \times 9.8 \times 0.4$$
$$\therefore v = 2.8$$
The ball hits the plane with velocity $2.8\ \text{m s}^{-1}$ and rebounds with velocity $0.7 \times 2.8\ \text{m s}^{-1}$ i.e. $1.96\ \text{m s}^{-1}$.

The ball then moves up under gravity to a height h.

This time

$u = 1.96$, $a = -9.8$, $s = h$, $v = 0$.

Use

$$v^2 = u^2 + 2as$$
$$\therefore 0 = 1.96^2 - 2 \times 9.8 \times h$$
$$\therefore h = \frac{1.96 \times 1.96}{2 \times 9.8}$$
$$= 0.196$$

i.e. it rebounds to a height of 19.6 cm $[= 40 \times 0.7 \times 0.7]$.

b After the second bounce the ball rebounds to a height $19.6 \times 0.7 \times 0.7 = 9.604$ cm.

c The total distance travelled is

$$0.4 + 2 \times 0.4 \times 0.7^2 + 2 \times 0.4 \times 0.7^4 + \ldots$$
$$= 0.4 + 2 \times 0.4 \times 0.7^2 (1 + 0.7^2 + 0.7^4 + \ldots)$$

You will need to use the sum of an infinite G. P. to find the answer.

The sum in the bracket is an infinite G. P. with sum $= \dfrac{1}{1 - 0.7^2}$

$$\therefore \text{Total distance} = 0.4 + 0.392(1.96\ldots)$$
$$= 1.169\ \text{m or } 1.17\ \text{m (3 s.f.)}$$

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Edexcel AS and A Level Modular Mathematics

Exercise D, Question 9

Question:

A small smooth ball, which may be modelled as a particle, falls from rest at a height H onto a smooth horizontal plane. The coefficient of restitution between the ball and the plane is e.

a Find in terms of H and e the height to which the ball rebounds after the first bounce.

b Find in terms of H and e the height to which the ball rebounds after the second bounce.

c Find an expression for the total distance travelled by the ball before it comes to rest.

Solution:

a First part of motion is under gravity:

$u = 0$ $s = H$ $a = g$ $v = ?$

Using $v^2 = u^2 + 2as \rightarrow v^2 = 2gH$

$\therefore$ The ball hits the plane with velocity $\sqrt{2gH}$.

It rebounds with velocity $e\sqrt{2gH}$.

Then it moves under gravity to height H'.

Using $u = e\sqrt{2gH}$, $s = H'$, $a = -g$, $v = 0$ with

$v^2 = u^2 + 2as$
$0 = e^2(2gH) - 2gH'$

$\therefore H' = e^2 H$.

The ball rebounds to a height $e^2 H$.

b After the second bounce it rebounds to a height $e^4 H$.

You can deduce the answer to **b** without repeating all the work done in **a**.

c Total distance travelled is:

$H + 2e^2H + 2e^4H + 2e^6H + \ldots$
$= H + 2e^2H(1 + e^2 + e^4 + \ldots)$

You need to use the sum to infinity of a Geometric Progression.

The expression in the bracket is a Geometric Progression. Using $S_\infty = \dfrac{a}{1-r}$, the

expression $= \dfrac{1}{1-e^2}$

$\therefore$ Total distance travelled is $H + 2e^2H \times \dfrac{1}{1-e^2} = \dfrac{H(1+e^2)}{(1-e^2)}$

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Exercise E, Question 1

Question:

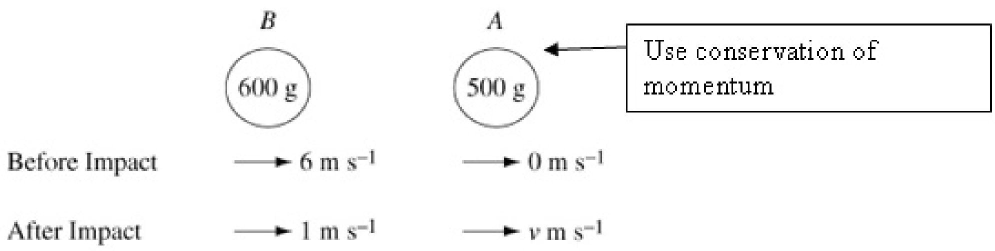
A particle A of mass 500 g lies at rest on a smooth horizontal table. A second particle B of mass 600 g is projected along the table with velocity 6 m s^{-1} and collides directly with A.

If the collision reduces the speed of B to 1 m s^{-1}, without changing its direction, find

a the speed of A after the collision,

b the loss of kinetic energy due to the collision.

Solution:



a Using conservation of momentum $\rightarrow$:

$$0.6 \times 6 + 0.5 \times 0 = 0.6 \times 1 + 0.5 \times v$$
$$\therefore 3.6 - 0.6 = 0.5v$$
$$\therefore v = 6$$

The speed of A after the collision is 6 m s^{-1}.

b Total kinetic energy before collision

$$= \frac{1}{2} \times 0.6 \times 6^2$$
$$= 10.8 \text{ J}$$

Use loss of K.E. = K.E. before impact − K.E. after impact

Total kinetic energy after collision

$$= \frac{1}{2} \times 0.6 \times 1^2 + \frac{1}{2} \times 0.5 \times v^2$$
$$= 0.3 + 9$$
$$= 9.3 \text{ J}$$

$$\text{The loss of K.E} = (10.8 - 9.3) \text{ J}$$
$$= 1.5 \text{ J}$$

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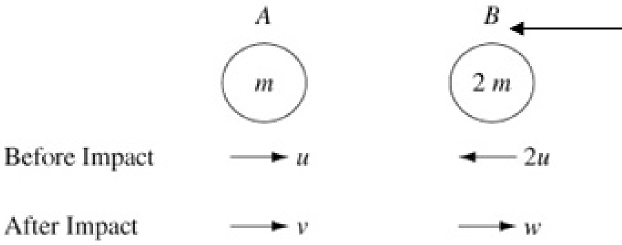
Exercise E, Question 2

Question:

Two particles A and B of mass m and $2m$ respectively move toward each other with speeds u and $2u$. If the coefficient of restitution between the spheres is $\frac{2}{3}$, find the speeds of A and of B after the collision. Find also, in terms of m and u, the loss of kinetic energy due to the collision.

Solution:



Use conservation of linear momentum and Newton's Law of Restitution to find the velocities after the impact.

Let the speeds of A and B after the collision be v and w respectively.

Use conservation of linear momentum $\rightarrow$:

$$mu - 2m \times 2u = mv + 2mw$$

$$\therefore -3u = v + 2w \qquad (1)$$

Use Newton's Law of Restitution:

$$\frac{w - v}{u - (-2u)} = \frac{2}{3}$$

$$\therefore w - v = \frac{2}{3} \times 3u$$

$$\therefore 2u = w - v \qquad (2)$$

Add equations (1) and (2)

$$\therefore -u = 3w$$

i.e. $w = -\dfrac{u}{3}$

Substitute into equation (2)

$$\therefore v = w - 2u = -\frac{7u}{3}.$$

The speed of A after impact is $\frac{7u}{3}$ in the direction away from B.

The speed of B after impact is $\frac{u}{3}$ towards A.

The kinetic energy before impact

$$= \frac{1}{2}mu^2 + \frac{1}{2} \times 2m(-2u)^2$$

$$= \frac{9}{2}mu^2$$

The kinetic energy after impact

$$= \frac{1}{2}mv^2 + \frac{1}{2} \times 2mw^2$$

$$= \frac{1}{2}\left(\frac{49}{9}u^2\right) + \frac{1}{2} \times 2m\left(\frac{u^2}{9}\right)$$

$$= \frac{51}{18}mu^2$$

$$\therefore \text{The loss of kinetic energy} = \frac{9}{2}mu^2 - \frac{51}{18}mu^2$$

$$= \frac{5}{3}mu^2$$

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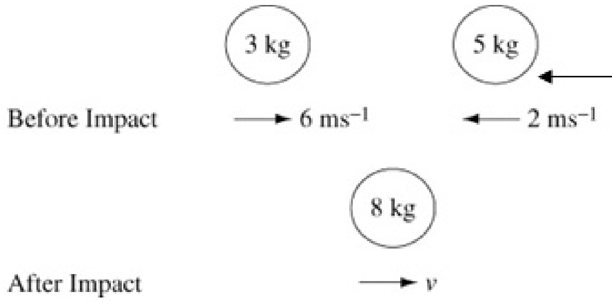
Exercise E, Question 3

Question:

A particle of mass 3 kg moving with velocity 6 m s^{-1} collides directly with a particle of mass 5 kg moving in the opposite direction with velocity 2 m s^{-1}. The particles coalesce and move with velocity v after the collision. Find the loss of kinetic energy due to the impact.

Solution:



Use conservation of linear momentum to find the velocity after the impact.

Let the velocity of the combined particle be v.

Using conservation of momentum:

$$\begin{aligned} 3\times 6+5\times(-2) &= 8\times v \\ \therefore 8 &= 8v \\ \therefore v &= 1 \end{aligned}$$

i.e. the combined particle moves with velocity 1 m s^{-1}.

$$\begin{aligned} \text{The total K.E. before impact} &= \tfrac{1}{2}\times 3\times 6^2+\tfrac{1}{2}\times 5\times(-2)^2 \\ &= 54+10 \\ &= 64 \text{ J} \\ \text{The total K.E. after impact} &= \tfrac{1}{2}\times 8\times v^2 \\ &= 4\times 1^2 \\ &= 4 \text{ J} \end{aligned}$$

$\therefore$ Loss of kinetic energy due to the impact is $64 \text{ J} - 4 \text{ J} = 60 \text{ J}$.

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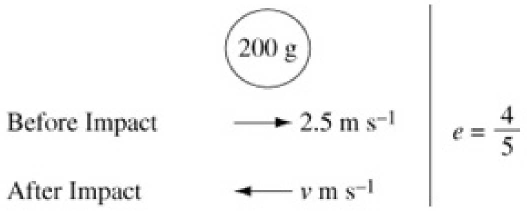
Exercise E, Question 4

Question:

A billiard ball of mass 200 g strikes a smooth cushion at right angles. Its velocity before the impact is 2.5 m s^{-1} and the coefficient of restitution is $\frac{4}{5}$. Find the loss in kinetic energy of the billiard ball due to the impact.

Solution:



After impact with the cushion the velocity of the billiard ball is $v \text{ m s}^{-1}$, where

$$\frac{v}{2.5} = \frac{4}{5}$$
$$\therefore v = 2$$

$\therefore$ The loss in kinetic energy is:

$$\frac{1}{2}\times 0.2\times 2.5^2 - \frac{1}{2}\times 0.2\times 2^2$$
$$= 0.625 - 0.4$$
$$= 0.225 \text{ J}$$

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Exercise E, Question 5

Question:

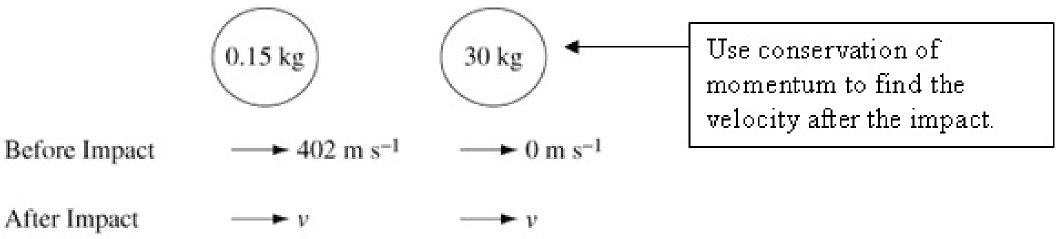
A bullet of mass 0.15 kg moving horizontally at 402 m s^{-1} embeds itself in a sandbag of mass 30 kg, which is suspended freely. Assuming that the sandbag is stationary before the impact, find

a the common velocity of the bullet and the sandbag,

b the loss of kinetic energy due to the impact.

Solution:

a



The common velocity after the impact is v.

Using conservation of linear momentum:

$$0.15 \times 402 + 0 = 30.15v$$

$$\therefore v = \frac{0.15 \times 402}{30.15}$$

$$= 2$$

$\therefore$ Common velocity is 2 m s^{-1}.

b Kinetic energy before the impact is $\frac{1}{2} \times 0.15 \times 402^2 = 12\,120.3 \text{ J}$

Kinetic energy after the impact is $\frac{1}{2} \times 30.15 \times 2^2 = 60.3 \text{ J}$

$\therefore$ Loss of kinetic energy $= 12\,060 \text{ J} = 12.06 \text{ kJ}$.

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Exercise E, Question 6

Question:

A particle of mass 0.4 kg is moving with velocity $(\mathbf{i}-4\mathbf{j})$ m s^{-1} when it receives an impulse $(3\mathbf{i}+2\mathbf{j})$ Ns. Find the new velocity of the particle and the change in kinetic energy of the particle as a result of the impulse.

Solution:

Let the velocity after the impulse be $\mathbf{v}$.

Using impulse = change in momentum

Use impulse = change in momentum.

$$\begin{aligned} 3\mathbf{i}+2\mathbf{j} &= 0.4\mathbf{v}-0.4(\mathbf{i}-4\mathbf{j}) \\ \therefore 0.4\mathbf{v} &= 3\mathbf{i}+2\mathbf{j}+0.4\mathbf{i}-1.6\mathbf{j} \\ &= 3.4\mathbf{i}+0.4\mathbf{j} \\ \therefore \mathbf{v} &= 8.5\mathbf{i}+\mathbf{j} \end{aligned}$$

$\therefore$ The new velocity of the particle is $(8.5\mathbf{i}+\mathbf{j})$ m s^{-1}.

The K.E. before the impulse $= \frac{1}{2}\times 0.4\times(1^2+(-4)^2)$
$= 3.4$ J

The K.E. after the impulse $= \frac{1}{2}\times 0.4\times(8.5^2+1^2)$
$= 14.65$ J

$\therefore$ The change in K.E. is an increase of 11.25 J.

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Exercise E, Question 7

Question:

A squash ball of mass 0.025 kg is moving with velocity $(22\mathbf{i}+37\mathbf{j})$ m s^{-1} when it hits a wall.

It rebounds with velocity $(10\mathbf{i}-11\mathbf{j})$ m s^{-1}. Find the change in kinetic energy of the squash ball.

Solution:

Kinetic energy before it hits the wall is
$\frac{1}{2}\times 0.25\times(22^2+37^2)=231.625$ J

A mass m kg moving with velocity $(v_1\mathbf{i}+v_2\mathbf{j})$ m s^{-1} has K.E $=\frac{1}{2}m(v_1^2+v_2^2)$ J.

Kinetic energy after rebound is $\frac{1}{2}\times 0.25\times(10^2+(-11)^2)=27.625$ J

$\therefore$ The loss in K.E. = 204 J

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Exercise E, Question 8

Question:

A particle of mass 0.2 kg is moving with velocity $(5\mathbf{i}+25\mathbf{j})$ m s^{-1} when it collides with a particle of mass 0.1 kg moving with velocity $(2\mathbf{i}+10\mathbf{j})$ m s^{-1}. The two particles coalesce and form one particle of mass 0.3 kg. Find the velocity of the combined particle and find the loss in kinetic energy as a result of the collision.

Solution:

Let the velocity of the combined particle be $(v_1\mathbf{i}+v_2\mathbf{j})$ m s^{-1}.

Using conservation of momentum:

Use conservation of momentum to find the velocity of the combined particle.

$$0.2(5\mathbf{i}+25\mathbf{j})+0.1(2\mathbf{i}+10\mathbf{j})=0.3(v_1\mathbf{i}+v_2\mathbf{j})$$

Equate $\mathbf{i}$ components:

$$1\mathbf{i}+0.2\mathbf{i}=0.3v_1\mathbf{i}$$
$$\therefore 1.2=0.3v_1$$
$$\Rightarrow v_1=4$$

Equate $\mathbf{j}$ components:

$$5\mathbf{j}+\mathbf{j}=0.3v_2\mathbf{j}$$
$$\therefore 6=0.3v_2$$
$$\Rightarrow v_2=20$$

$\therefore$ The required velocity is $(4\mathbf{i}+20\mathbf{j})$ m s^{-1}.

$$\begin{aligned}\text{The initial K.E.} &= \tfrac{1}{2}\times 0.2(5^2+25^2)+\tfrac{1}{2}\times 0.1(2^2+10^2)\\ &= 65+5.2\\ &= 70.2\text{ J}\\ \text{The K.E. after impact} &= \tfrac{1}{2}\times 0.3\times(4^2+20^2)\\ &= 62.4\text{ J}\\ \therefore\text{The loss in K.E.} &= 70.2\text{ J}-62.4\text{ J}\\ &= 7.8\text{ J}\end{aligned}$$

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Exercise E, Question 9

Question:

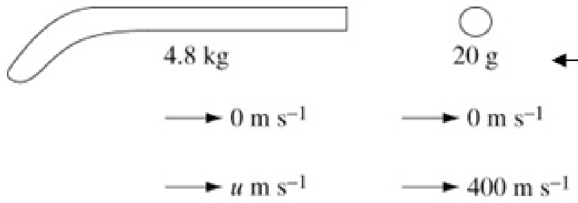
A bullet is fired horizontally from a rifle. The rifle has mass 4.8 kg and the bullet has mass 20 g. The initial speed of the bullet is 400 m s^{-1}. Find

a the initial speed with which the rifle recoils,

b the total kinetic energy generated as a result of firing the bullet.

Solution:

a



Convert the mass of the bullet into kg for your calculations.

Use conservation of momentum:

$$0 = 4.8u + 0.02 \times 400$$

$$\therefore u = \frac{-0.02 \times 400}{4.8}$$

$$= -\frac{5}{3}$$

$\therefore$ The rifle recoils (moves back) with a speed of $\frac{5}{3} \text{ m s}^{-1}$.

b The total K.E. before firing $= 0$

Total K.E. after firing $= \frac{1}{2} \times 4.8 \times \left(\frac{5}{3}\right)^2 + \frac{1}{2} \times (0.02) \times 400^2$

$= 6.\dot{6} + 1600$

$= 1606.\dot{6} \text{ J}$

$\therefore$ K.E. generated is $1606.\dot{6}$ J.

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Exercise E, Question 10

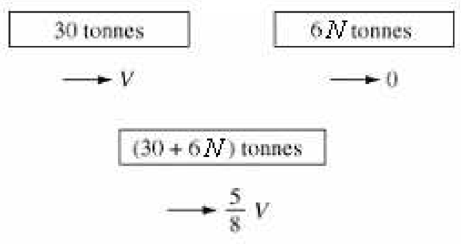
Question:

A train of mass 30 tonnes moving with a small velocity V impacts upon a number of stationary carriages each weighing 6 tonnes. The complete train and carriages now move forward with a velocity of $\frac{5}{8}V$. Find

a the number of stationary carriages,

b the fraction of the original kinetic energy lost in the impact.

Solution:



a Let the number of carriages be N.

Using conservation of momentum:

$$30V + 6N \times 0 = (30 + 6N)\frac{5}{8}V$$

Let the mass of the train and carriages be $(30 + 6N)$ tonnes.

$$\therefore 30 = \frac{150}{8} + \frac{30N}{8}$$

$$\therefore \frac{90}{8} = \frac{30N}{8}$$

$$\therefore N = 3$$

b

$$\text{Energy before impact} = \frac{1}{2} \times 30\,000V^2 = 15\,000V^2$$

$$\text{Energy after impact} = \frac{1}{2}(48\,000) \times \left(\frac{5V}{8}\right)^2$$

$$= 9375V^2$$

$$\therefore \text{Energy lost} = 5625V^2$$

$$\therefore \text{Fraction of original K.E. lost} = \frac{5625V^2}{15\,000V^2}$$

$$= \frac{3}{8}$$

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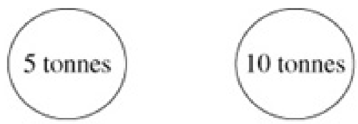
Exercise E, Question 11

Question:

A truck of mass 5 tonnes is moving at 1.5 m s^{-1} when it hits a second truck of mass 10 tonnes which is at rest. After the impact the second truck moves at 0.6 m s^{-1}. Find the speed of the first truck after the impact and the total loss of kinetic energy due to the impact.

Solution:



Before Impact $\longrightarrow 1.5 \text{ m s}^{-1}$ $\longrightarrow 0 \text{ m s}^{-1}$

After Impact $\longrightarrow v \text{ m s}^{-1}$ $\longrightarrow 0.6 \text{ m s}^{-1}$

Let the speed of the first truck be $v \text{ m s}^{-1}$, after the impact.

Use conservation of momentum:

$$5 \times 1.5 + 0 = 5v + 10 \times 0.6$$
$$\therefore 5v = 7.5 - 6$$
$$= 1.5$$
$$\therefore v = 0.3$$

$\therefore$ The first truck moves with a speed of 0.3 m s^{-1}.

The total K.E. before impact $= \frac{1}{2} \times 5000 \times 1.5^2 = 5625 \text{ J}$

The total K.E. after impact $= \frac{1}{2} \times 5000 \times 0.3^2 + \frac{1}{2} \times 10\,000 \times 0.6^2$
$= 2025 \text{ J}$

Convert the masses of the trucks into kg to calculate the K.E. in Joules.

$\therefore$ The loss of K.E. due to the impact $= 3600 \text{ J}$

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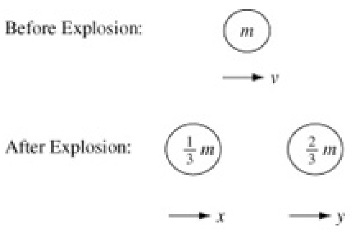
Exercise E, Question 12

Question:

A particle of mass m moves in a straight line with velocity v when it explodes into two parts, one of mass $\frac{1}{3}m$ and the other of mass $\frac{2}{3}m$ both moving in the same direction as before.
If the explosion increases the energy of the system by $\frac{1}{4}mu^2$, where u is a positive constant, find the velocities of the particles immediately after the explosion. Give your answers in terms of u and v.

Solution:



Let the velocities of the $\frac{1}{3}m$ mass and the $\frac{2}{3}m$ mass be x and y respectively.
Use conservation of momentum:

Use conservation of momentum, and energy equation to give simultaneous equations–one linear and one quadratic.

$$mv = \frac{1}{3}mx + \frac{2}{3}my$$

$$\therefore x + 2y = 3v \quad (1)$$

The energy of the system is increased by $\frac{1}{4}mu^2$.

$$\therefore \frac{1}{2}mv^2 + \frac{1}{4}mu^2 = \frac{1}{2}\left(\frac{1}{3}m\right)x^2 + \frac{1}{2}\left(\frac{2}{3}m\right)y^2$$

$$\therefore x^2 + 2y^2 = 3v^2 + \frac{3}{2}u^2 \quad (2)$$

From equation (1), $x = 3v - 2y$
Substitute into equation (2)

$$\therefore (3v - 2y)^2 + 2y^2 = 3v^2 + \frac{3}{2}u^2$$

$$\therefore 9v^2 - 12vy + 6y^2 = 3v^2 + \frac{3}{2}u^2$$

$$\therefore 6y^2 - 12vy + 6v^2 = \frac{3}{2}u^2$$

$$\therefore y^2 - 2vy + v^2 = \frac{1}{4}u^2$$

$$\therefore (y - v)^2 = \left(\pm\frac{1}{2}u\right)^2$$

$$\therefore y = v \pm \frac{1}{2}u$$

Substitute back into (1) $\therefore x = v \mp u$
But $x < y$, $\therefore x = v - u$ and $y = v + \frac{1}{2}u$

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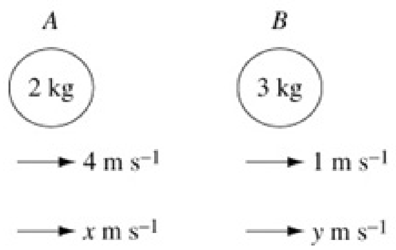
Exercise E, Question 13

Question:

A small smooth sphere A of mass 2 kg moves at 4 m s^{-1} on a smooth horizontal table. It collides directly with a second equal-sized smooth sphere B of mass 3 kg, which is moving away from A in the same direction at a speed of 1 m s^{-1}. If the loss of kinetic energy due to the collision is 3 J find the speeds and the directions of the two spheres after the collision.

Solution:



Use conservation of momentum:

$$2\times4+3\times1=2x+3y$$
$$\therefore 2x+3y = 11 \quad (1)$$

Use conservation of momentum and a change in energy equation to give simultaneous equations – one linear and one quadratic.

$$\text{The energy before the collision} = \tfrac{1}{2}\times2\times4^2+\tfrac{1}{2}\times3\times1^2$$
$$= 17.5\text{ J}$$
$$\text{The energy after collision} = \tfrac{1}{2}\times2x^2+\tfrac{1}{2}\times3y^2$$
$$= x^2+1.5y^2$$

As the loss of K.E. $=3$ J

$$\therefore \quad x^2+1.5y^2 = 14.5 \quad (2)$$

From (1) $x=\dfrac{11-3y}{2}$ – substitute into equation (2).

$$\therefore \frac{121-66y+9y^2}{4}+1.5y^2=14.5$$

Multiply by 4

$$\therefore 15y^2-66y+121=58$$
$$\therefore 5y^2-22y+21=0$$
$$\therefore (5y-7)(y-3)=0$$
$$\therefore y=3 \text{ or } \tfrac{7}{5}$$

Substitute value of y into equation (1)

$\therefore x=1$ or $x=3.4$.

But $x<y \quad \therefore x=1$ and $y=3$

$\therefore$ The 2 kg mass moves with speed 1 m s^{-1} and the 3 kg mass moves with speed 3 m s^{-1}.

They both move in the same direction as before the impact.

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Exercise E, Question 14

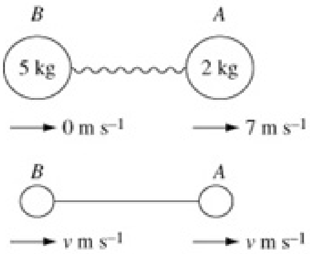
Question:

Two particles, A and B, of masses 2 kg and 5 kg respectively, are connected by a light inextensible string. The particles are side by side on a smooth floor and A is projected with speed 7 m s^{-1} directly away from B. When the string becomes taut particle B is jerked into motion and A and B then move with a common speed in the direction of the original velocity of A. Find

a the common speed of the particles after the string becomes taut,

b the loss of total kinetic energy due to the jerk.

Solution:



a Using conservation of momentum $\rightarrow$:

Use conservation of momentum to find the common speed after the string becomes taut.

$$2\times 7 = 2v + 5v$$
$$= 7v$$
$$\therefore v = 2$$

b
$$\text{K.E. before the jerk} = \frac{1}{2}\times 2\times 7^2 = 49\text{ J}$$
$$\text{K.E. after the jerk} = \frac{1}{2}\times 5v^2 + \frac{1}{2}\times 2v^2$$
$$= \frac{7}{2}\times 2^2$$
$$= 14\text{ J}$$
$$\therefore \text{Loss of K.E. due to the jerk} = 14\text{ J}$$

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Exercise E, Question 15

Question:

Two particles, A and B, of masses m and M respectively, are connected by a light inextensible string. The particles are side by side on a smooth floor and A is projected with speed u directly away from B. When the string becomes taut particle B is jerked into motion and A and B then move with a common speed in the direction of the original velocity of A.

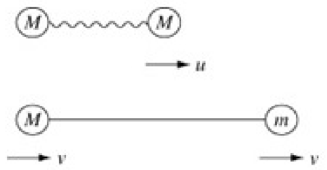
Find the common speed of the particles after the string becomes taut, and show that the loss of total kinetic energy due to the jerk is $\frac{mMu^2}{2(m+M)}$.

Solution:

String slack:

After string becomes taut:

Let the speed after the string becomes taut be v.



Use conservation of momentum $\rightarrow$:

$\therefore mu = Mv + mv$

$\therefore mu = v(M + m)$

$\therefore v = \frac{mu}{M+m}$ ← This is the common speed required.

Energy before the jerk $= \frac{1}{2}mu^2$

Energy after the jerk $= \frac{1}{2}Mv^2 + \frac{1}{2}mv^2$

$$= \frac{1}{2}(M+m)\left[\frac{mu}{M+m}\right]^2$$

$$\therefore \text{Loss of energy} = \frac{1}{2}mu^2 - \frac{1}{2}\frac{m^2u^2}{(M+m)}$$

$$= \frac{1}{2}mu^2\left[1 - \frac{m}{M+m}\right]$$

$$= \frac{1}{2}mu^2\left[\frac{M}{M+m}\right]$$

$$= \frac{mMu^2}{2(m+M)} \text{ as required}$$

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Exercise E, Question 16

Question:

Two particles of masses 3 kg and 5 kg lie on a smooth table and are connected by a slack inextensible string. The first particle is projected along the table with a velocity of 20 m s^{-1} directly away from the second particle.

a Find the velocity of each particle after the string has become taut.

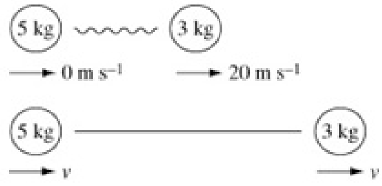
b Find the difference between the kinetic energies of the system when the string is slack and when it is taut.

The second particle is attached to a third particle of unknown mass by another slack string, and the velocity of the whole system after both strings have become taut is 6 m s^{-1}.

c Find the mass of the third particle.

Solution:

a String slack:



After string becomes taut:

Let the speed after the string becomes taut be v.

Using conservation of momentum $\rightarrow$:

$3 \times 20 = 5v + 3v$

$\therefore 8v = 60$

$\therefore v = \frac{60}{8}$

$= 7.5$

i.e. the common velocity is 7.5 m s^{-1}.

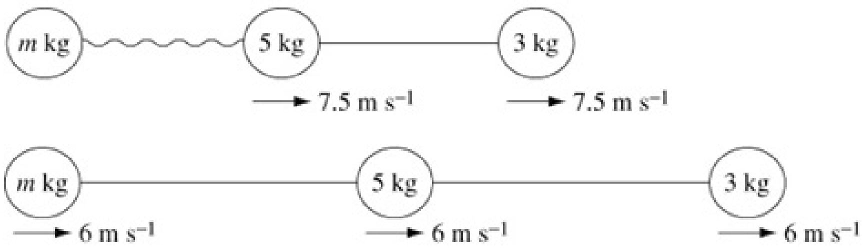
Consider conservation of momentum before and after first string becomes taut.

b Difference in K.E. $= \frac{1}{2} \times 3 \times 20^2 - \left[\frac{1}{2} \times 5 \times 7.5^2 + \frac{1}{2} \times 3 \times 7.5^2\right]$

$= 600 - [225]$

$= 375 \text{ J}$

c Second string slack.



After second string becomes taut

Let mass of third particle be m kg.

Use conservation of momentum:

$5 \times 7.5 + 3 \times 7.5 = 6m + 5 \times 6 + 3 \times 6$

$\therefore 60 = 6m + 48$

$\therefore 6m = 12$

$\therefore m = 2$

The mass of the third particle is 2 kg.

Use conservation of momentum again before and after second string becomes taut.

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Exercise E, Question 17

Question:

Three small spheres of mass 20 g, 40 g and 60 g respectively lie in order in a straight line on a large smooth table. The distance between adjacent spheres is 10 cm. Two slack strings, each 70 cm in length, connect the first sphere with the second, and the second sphere with the third. The 60 g sphere is projected with a speed of 5 m s^{-1}, directly away from the other two.

a Find the time which elapses before the 20 g sphere begins to move and the speed with which it starts.

b Find the loss in kinetic energy resulting from the two jerks.

Solution:

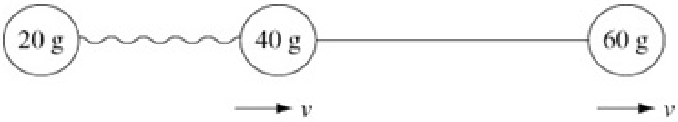
a

Stage 1: 60g particle moves 60 cm at 5 m s^{-1}

The particles are 10 cm apart. The strings are each 70 cm long. So the particle moves 60 cm at constant speed before the string becomes taut.

This takes time $t_1 = 0.6 \div 5 = 0.12 \text{ s}$

Stage 2: 40 g mass is jerked into motion.



Let speed after jerk be v.

Use conservation of momentum.

$$\therefore 60 \times 5 = 40 \times v + 60 \times v$$
$$300 = 100v$$
$$\therefore v = 3$$

Stage 3: 60 g and 40 g particle move 60 cm at 3 m s^{-1}.

This takes time $t_2 = 0.6 \div 3 = 0.2 \text{ s}$

Stage 4: 20 g mass is jerked into motion.

Let speed after jerk be w.

Use conservation of momentum.

$$\therefore 60 \times 3 + 40 \times 3 = 20w + 40w + 60w$$
$$300 = 120w$$
$$\therefore w = 2.5$$

i.e. after a time $t_1 + t_2 = 0.32 \text{ s}$ the 20 g mass moves with a velocity 2.5 m s^{-1}.

b

$$\text{Final K.E.} = \tfrac{1}{2} \times 0.02w^2 + \tfrac{1}{2} \times 0.04w^2 + \tfrac{1}{2} \times 0.06w^2$$
$$= 0.06w^2$$
$$= 0.375 \text{ J}$$
$$\text{Initial K.E.} = \tfrac{1}{2} \times 0.06 \times 5^2$$
$$= 0.75 \text{ J}$$
$$\therefore \text{Loss in K.E.} = 0.75 \text{ J} - 0.375 \text{ J}$$
$$= 0.375 \text{ J}$$

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Exercise F, Question 1

Question:

A cricket ball of mass 0.5 kg is struck by a bat. Immediately before being struck the velocity of the ball is $-25\mathbf{i}$ m s^{-1}. Immediately after being struck the velocity of the ball is $(23\mathbf{i}+20\mathbf{j})$ m s^{-1}. Find the magnitude of the impulse exerted on the ball by the bat and the angle between the impulse and the direction of $\mathbf{i}$.

Solution:

$$\begin{aligned}\text{Impulse} &= \text{change in momentum}\\ &= 0.5(23\mathbf{i}+20\mathbf{j})-0.5(-25\mathbf{i})\\ &= (24\mathbf{i}+10\mathbf{j})\text{ Ns}\end{aligned}$$

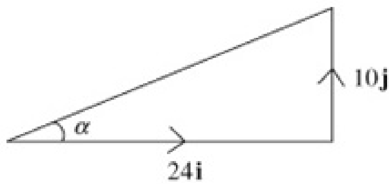
Use impulse = change in momentum.

$\therefore$ Magnitude of the impulse

$$=\sqrt{24^2+10^2}\text{ Ns}$$
$$= 26\text{ Ns}$$

Angle between the impulse and the direction $\mathbf{i}$ is α where

$$\tan\alpha = \frac{10}{24}$$
$$\therefore \alpha = 23^\circ \text{ (nearest degree)}$$



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Exercise F, Question 2

Question:

A ball of mass 0.2 kg is hit by a bat which gives it an impulse of $(2.4\mathbf{i}+3.6\mathbf{j})$ Ns.

The velocity of the ball immediately after being hit is $(12\mathbf{i}+5\mathbf{j})$ m s^{-1}. Find the velocity of the ball immediately before it is hit.

Solution:

Let velocity before being hit be $\mathbf{u}$ m s^{-1}.

Use impulse = change in momentum.

$$\begin{aligned} \text{impulse} &= \text{change in momentum} \\ \therefore 2.4\mathbf{i}+3.6\mathbf{j} &= 0.2(12\mathbf{i}+5\mathbf{j}) - 0.2\mathbf{u} \\ \therefore 0.2\mathbf{u} &= 2.4\mathbf{i}+\mathbf{j}-2.4\mathbf{i}-3.6\mathbf{j} \\ &= -2.6\mathbf{j} \\ \therefore \quad \mathbf{u} &= -13\mathbf{j} \end{aligned}$$

The velocity of the ball immediately before it is hit is $-13\mathbf{j}$ m s^{-1}.

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Exercise F, Question 3

Question:

A particle P of mass 0.3 kg is moving so that its position vector $\mathbf{r}$ metres at time t seconds is given by

$$\mathbf{r} = (t^3 + t^2 + 4t)\mathbf{i} + (11t)\mathbf{j}$$

a Calculate the speed of P when $t = 4$.

When $t = 4$, the particle is given an impulse $(2.4\mathbf{i} + 3.6\mathbf{j})$ Ns.

b Find the velocity of P immediately after the impulse.

Solution:

a
$$\mathbf{r} = (t^3 + t^2 + 4t)\mathbf{i} + 11t\mathbf{j}$$
$$\therefore \mathbf{v} = \dot{\mathbf{r}} = (3t^2 + 2t + 4)\mathbf{i} + 11\mathbf{j}$$

Differentiate displacement vector to give the velocity vector.

When $t = 4, \mathbf{v} = (60)\mathbf{i} + 11\mathbf{j}$

the magnitude of $\mathbf{v} = \sqrt{60^2 + 11^2}$
$= 61$

$\therefore$ The speed when $t = 4$ is 61 m s^{-1}.

b Let the velocity immediately after the impulse be $\mathbf{V} \text{ m s}^{-1}$.

Then as impulse = change in momentum

$$\therefore 2.4\mathbf{i} + 3.6\mathbf{j} = 0.3\mathbf{V} - 0.3(60\mathbf{i} + 11\mathbf{j})$$
$$\therefore 0.3\mathbf{V} = 2.4\mathbf{i} + 3.6\mathbf{j} + 18\mathbf{i} + 3.3\mathbf{j}$$
$$= 20.4\mathbf{i} + 6.9\mathbf{j}$$
$$\therefore \mathbf{V} = 68\mathbf{i} + 23\mathbf{j}$$

Use impulse = change in momentum.

$\therefore$ velocity after the impulse is $(68\mathbf{i} + 23\mathbf{j}) \text{ m s}^{-1}$.

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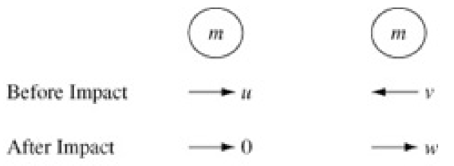
Exercise F, Question 4

Question:

Two identical spheres, moving in opposite directions, collide directly. As a result of the impact one of the spheres is brought to rest. The coefficient of restitution between the spheres is $\frac{1}{3}$. Show that the ratio of the speeds of the spheres before the impact is 2 : 1.

Solution:



$e = \frac{1}{3}$

Let the spheres each have mass m and let their speeds before the impact be u m s^{-1} and v m s^{-1} (towards each other).

Let the speeds after impact be 0 and w m s^{-1}.

Using conservation of momentum $\rightarrow$:

Let the speeds before impact be u and v and after impact be 0 and w, then use conservation of momentum and the restitution equation.

$mu - mv = mw$

i.e. $u - v = w$ (1)

Using Newton's Law of Restitution:

$$e = \frac{1}{3} = \frac{w - 0}{u - (-v)}$$

$\therefore u + v = 3w$ (2)

Add (1) + (2)

$\therefore 2u = 4w \rightarrow u = 2w$

Substitute into (1)

$\therefore v = w$

$\therefore$ The ratio of the speeds before impact is $u : v = 2w : w = 2 : 1$ as required.

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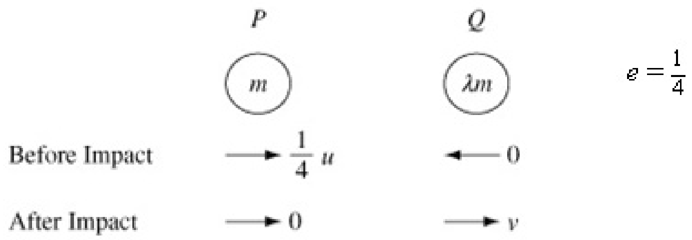
Exercise F, Question 5

Question:

A particle P of mass m is moving in a straight line with speed $\frac{1}{4}u$ at the instant when it collides directly with a particle Q of mass λm, which is at rest. The coefficient of restitution between P and Q is $\frac{1}{4}$. Given that P comes to rest immediately after hitting Q find the value of λ.

Solution:



Let the velocity of Q after impact be v.

Use conservation of linear momentum:

$$m\times\frac{1}{4}u+0=0+\lambda mv$$

$$\therefore \lambda v=\frac{1}{4}u \quad (1)$$

Use Newton's Law of Restitution:

$$e=\frac{1}{4}=\frac{v-0}{\frac{1}{4}u}$$

$$\therefore v=\frac{1}{16}u \quad (2)$$

Solving equations (1) and (2) $\lambda=4$

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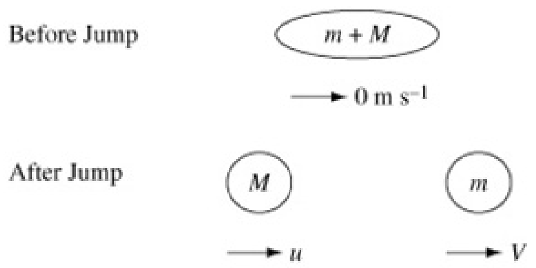
Exercise F, Question 6

Question:

A boy of mass m dives off a boat of mass M which was previously at rest. Immediately after diving off, the boy has a horizontal speed of V. Calculate the speed with which the boat begins to move. Prove that the total kinetic energy of the boy and the boat is $\frac{m(m+M)V^2}{2M}$.

Solution:



Let the boat move with velocity u.

Using conservation of linear momentum:

Use conservation of linear momentum to find the velocity of the boat.

$$Mu + mV = 0$$

$$\therefore u = -\frac{mV}{M}$$

(Note that the boat moves in the opposite direction to the boy.)

Total K.E. of boy and boat

$$= \frac{1}{2}mV^2 + \frac{1}{2}M\left(-\frac{mV}{M}\right)^2$$

$$= \frac{mMV^2 + m^2V^2}{2M}$$

$$= \frac{m(m+M)V^2}{2M}, \text{ as required}$$

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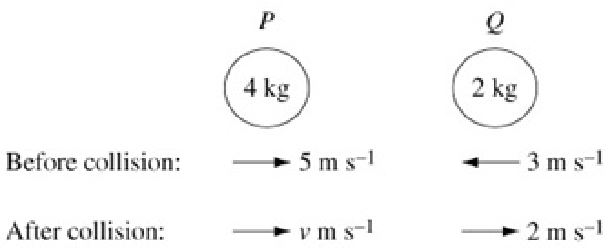
Exercise F, Question 7

Question:

Two spheres P and Q of equal radius and masses 4 kg and 2 kg respectively are travelling towards each other along a straight line on a smooth horizontal surface. Initially, P has a speed of 5 m s^{-1} and Q has a speed of 3 m s^{-1}. After the collision the direction of Q is reversed and it is travelling at a speed of 2 m s^{-1}. Find the speed of P after the collision and the loss of kinetic energy due to the collision.

Solution:



Use conservation of linear momentum:

$$4\times5-2\times3=4v+2\times2$$
$$\therefore 4v=10$$
$$\therefore v=2.5\text{ m s}^{-1}$$

Use conservation of linear momentum to find the speed of P after the collision.

$$\text{The total K.E. before collision}=\frac{1}{2}\times4\times5^2+\frac{1}{2}\times2\times(-3)^2$$
$$=50+9$$
$$=59\text{ J}$$

$$\text{The total K.E. after collision}=\frac{1}{2}\times4\times2.5^2+\frac{1}{2}\times2\times2^2$$
$$=12.5+4$$
$$=16.5\text{ J}$$

$\therefore$ Loss of K.E. due to the collision is 42.5 J.

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Exercise F, Question 8

Question:

A body P of mass 4 kg is moving with velocity $(2\mathbf{i}+16\mathbf{j})$ m s^{-1} when it collides with a body Q of mass 3 kg moving with velocity $(-\mathbf{i}-8\mathbf{j})$ m s^{-1}. Immediately after the collision the velocity of P is $(-4\mathbf{i}-32\mathbf{j})$ m s^{-1}. Find the velocity of Q immediately after the collision.

Solution:

Let the velocity of Q after the collision be v m s^{-1}.

Use conservation of momentum:

Use conservation of momentum.

$$
\begin{aligned}
4(2\mathbf{i}+16\mathbf{j})+3(-\mathbf{i}-8\mathbf{j}) &= 4(-4\mathbf{i}-32\mathbf{j})+3\mathbf{v} \\
\therefore 5\mathbf{i}+40\mathbf{j} &= -16\mathbf{i}-128\mathbf{j}+3\mathbf{v} \\
\therefore 3\mathbf{v} &= 21\mathbf{i}+168\mathbf{j} \\
\therefore \mathbf{v} &= 7\mathbf{i}+56\mathbf{j}
\end{aligned}
$$

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Exercise F, Question 9

Question:

A particle P of mass $3m$ is moving in a straight line with speed u at the instant when it collides directly with a particle Q of mass m which is at rest. The coefficient of restitution between P and Q is e.

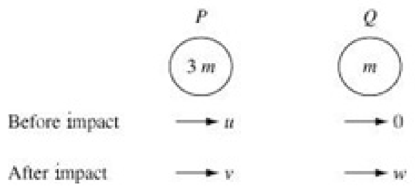
a Show that after the collision P is moving with speed $\frac{u(3-e)}{4}$.

b Show that the loss of kinetic energy due to the collision is $\frac{3mu^2(1-e^2)}{8}$.

c Find in terms of m, u and e the impulse exerted on Q by P in the collision.

Solution:

a



Let the velocities of P and Q be v and w respectively, after the impact.

Use conservation of momentum:

$$3mu+0=3mv+mw$$
$$\therefore 3v+w=3u \quad (1)$$

Use Newton's Law of Restitution:

$$e=\frac{w-v}{u-0}$$
$$\therefore w-v=eu \quad (2)$$

Subtract $(1)-(2)$

$$\therefore 4v=3u-eu$$
$$\therefore v=\frac{u(3-e)}{4}$$

Substitute into equation (2) to give

$$w=eu+\frac{u(3-e)}{4}$$
$$w=\frac{3u}{4}(1+e).$$

b Loss of K.E.

$$=\frac{1}{2}\times 3mu^2-\frac{1}{2}\times 3mv^2-\frac{1}{2}mw^2$$

Find the velocity of Q as well as the velocity of P so that you can calculate the total kinetic energy.

$$=\frac{m}{2}\left[3u^2-3\frac{u^2(3-e)^2}{16}-9\frac{u^2(1+e)^2}{16}\right]$$
$$=\frac{3mu^2}{32}\left[16-(9-6e+e^2)-(3+6e+3e^2)\right]$$
$$=\frac{3mu^2}{32}\left[4-4e^2\right]$$
$$=\frac{3mu^2}{8}(1-e^2)$$

c Impulse exerted on Q is change of momentum of Q

$$=\frac{3mu(1+e)}{4}\text{ Ns}.$$

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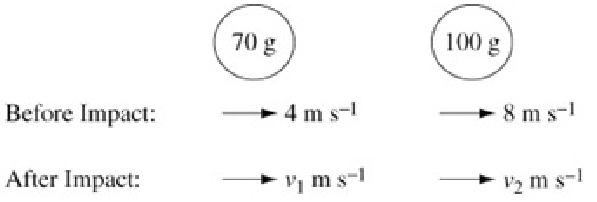
Exercise F, Question 10

Question:

Two spheres of mass 70 g and 100 g are moving along a straight line towards each other with velocities 4 m s^{-1} and 8 m s^{-1} respectively. Their coefficient of restitution is $\frac{5}{12}$. Find their velocities after impact and the amount of kinetic energy lost in the collision.

Solution:



Let the velocities after impact be $v_1 \text{ m s}^{-1}$ and $v_2 \text{ m s}^{-1}$.

Using conservation of momentum:

$$0.07 \times 4 + 0.1 \times (-8) = 0.07 v_1 + 0.1 v_2$$
$$\therefore 7v_1 + 10v_2 = -52 \quad (1)$$

Use conservation of momentum and Newton's Law of Restitution to find the velocities after the impact.

Using Newton's Law of Restitution:

$$e = \frac{5}{12} = \frac{v_2 - v_1}{4 - (-8)}$$
$$\therefore v_2 - v_1 = 5 \quad (2)$$

Solving equation (1) and (2)

$v_1 = -6$ and $v_2 = -1$

So the velocities after impact are 6 m s^{-1} and 1 m s^{-1} in the direction of the 100 g mass prior to the impact.

Loss of K.E.

$$= \frac{1}{2} \times 0.07 \times 4^2 + \frac{1}{2} \times 0.1 \times (-8)^2 - \left[\frac{1}{2} \times 0.07 \times (-6)^2 + \frac{1}{2} \times 0.1 \times (-1)^2\right]$$
$$= [0.56 + 3.2] - [1.26 + 0.05]$$
$$= 2.45 \text{ J}$$

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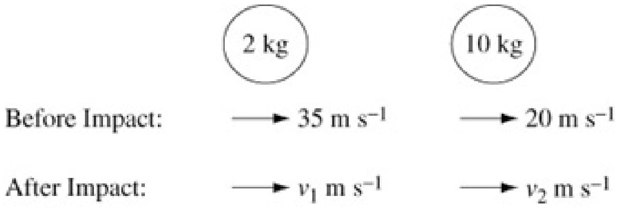
Exercise F, Question 11

Question:

A mass of 2 kg moving at 35 m s^{-1} catches up and collides with a mass of 10 kg moving in the same direction at 20 m s^{-1}. Five seconds after the impact the 10 kg mass encounters a fixed barrier which reduces it to rest. Assuming the coefficient of restitution between the masses is $\frac{3}{5}$, find the further time that will elapse before the 2 kg mass strikes the 10 kg mass again.

You may assume that the masses are moving on a smooth surface and have constant velocity between collisions.

Solution:



Using conservation of momentum $\rightarrow$:

$$2\times35+10\times20=2v_1+10v_2$$
$$\therefore 2v_1+10v_2=270$$
$$\text{or } v_1+5v_2=135 \quad (1)$$

Using Newton's Law of Restitution:

$$e=\frac{3}{5}=\frac{v_2-v_1}{35-20}$$
$$\therefore v_2-v_1=9 \quad (2)$$

Add equations $(1)+(2)$

$$6v_2=144$$
$$\therefore v_2=24$$

Substitute into (1) $\therefore v_1=15$

Five seconds after the impact the 10 kg mass has moved a distance $24\times5=120\text{ m}$.

It takes the 2 kg mass a time of $\frac{120}{15}$ to travel 120 m, i.e. 8 seconds.

$\therefore$ The *further* time required $=8\text{ s}-5\text{ s}=3\text{ s}$

After the impact, assume that the particles move at constant speed and use speed $\times$ time $=$ distance.

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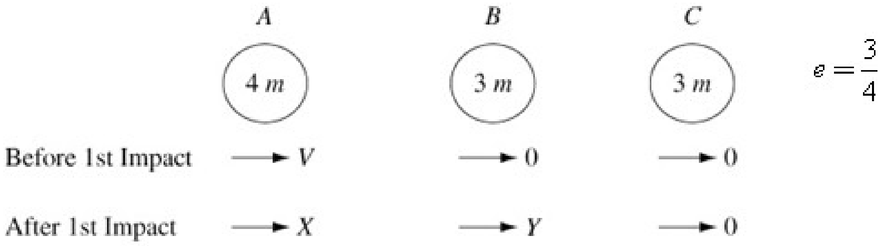
Exercise F, Question 12

Question:

Three balls A, B and C of masses $4m$, $3m$ and $3m$, respectively and of equal radius lie at rest on a smooth horizontal table with their centres in a straight line. Their coefficient of restitution is $\frac{3}{4}$. Show that if A is projected towards B with speed V there are three impacts and the final velocities are $\frac{5}{32}V, \frac{1}{4}V$ and $\frac{7}{8}V$ respectively.

Solution:



First consider impact of A with B, then of B with C, then of A with B again.

Use conservation of momentum for 1st impact:

$$4mV = 4mX + 3mY$$

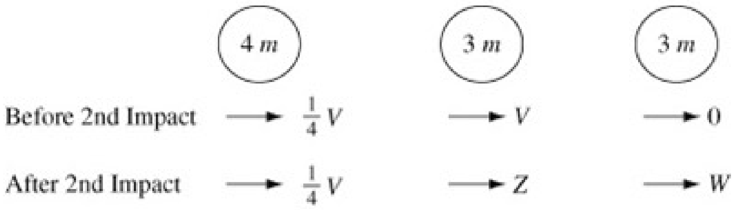
$\therefore 4X + 3Y = 4V \quad (1)$

Use Newton's Law of Restitution for 1st impact:

$$\frac{3}{4} = \frac{Y - X}{V}$$

$\therefore Y - X = \frac{3}{4}V \quad (2)$

Solve (1) and (2) to give $Y = V$, $X = \frac{1}{4}V$.



Use conservation of momentum for impact between B and C:

$$3mV = 3mZ + 3mW$$

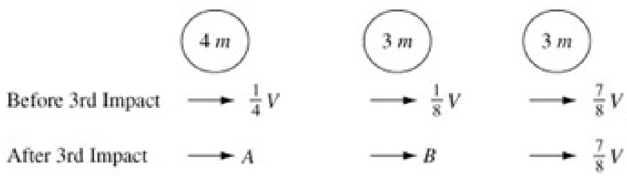
$\therefore Z + W = V \quad (3)$

Use Newton's Law of Restitution for this impact:

$$\frac{3}{4} = \frac{W - Z}{V}$$

$\therefore W - Z = \frac{3}{4}V \quad (4)$

Solve (3) and (4) to give $W = \frac{7}{8}V$, $Z = \frac{1}{8}V$.



Use conservation of momentum for 3rd impact between A and B.

$$4m\times\frac{1}{4}V+3m\times\frac{1}{8}V=4mA+3mB$$

$$\therefore 4A+3B=\frac{11}{8}V \quad (5)$$

Use Newton's Law of Restitution for this impact:

$$\frac{3}{4}=\frac{B-A}{\frac{1}{4}V-\frac{1}{8}V}$$

$$\therefore B-A=\frac{3}{32}V \quad (6)$$

Equation $(5)+4\times$ Equation $(6)\Rightarrow 7B=\frac{14}{8}V\Rightarrow B=\frac{1}{4}V$

Substitute into (6) give $A=\frac{1}{4}V-\frac{3}{32}V=\frac{5}{32}V$

$\therefore$ After 3 collisions the velocities are $\frac{5}{32}V, \frac{1}{4}V$ and $\frac{7}{8}V$ for the particles A, B and C respectively.

As $\frac{5}{32}V<\frac{1}{4}V<\frac{7}{8}V$ there are no further collisions.

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Exercise F, Question 13

Question:

A bullet of mass 60 g is fired horizontally at a fixed vertical metal barrier. The bullet hits the barrier when it is travelling at 600 m s^{-1} and then rebounds.

a Find the kinetic energy lost at the impact if $e = 0.4$.

b Give one possible form of energy into which the lost kinetic energy has been transformed.

Solution:

a Velocity of bullet after hitting the barrier

$$= 600 \times 0.4$$
$$= 240 \text{ m s}^{-1}.$$

Kinetic energy lost

$$= \frac{1}{2} \times 0.06 \times 600^2 - \frac{1}{2} \times 0.06 \times 240^2$$
$$= 9072 \text{ N}$$

b *Either* heat *or* sound.

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Exercise F, Question 14

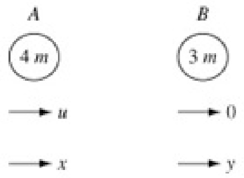
Question:

A particle A of mass $4m$ moving with speed u on a horizontal plane strikes directly a particle B of mass $3m$ which is at rest on the plane. The coefficient of restitution between A and B is e.

a Find, in terms of e and u, the speeds of A and B immediately after the collision.

b Given that the magnitude of the impulse exerted by A on B is $2mu$ show that $e = \frac{1}{6}$.

Solution:



a Let the speeds after the collision be x and y.

Use conservation of momentum:

$$4mu + 0 = 4mx + 3my$$
$$\therefore 4x + 3y = 4u \quad (1)$$

Use Newton's Law of Restitution:

$$e = \frac{y - x}{u}$$
$$\therefore y - x = eu \quad (2)$$

Add $(1) + 4 \times (2)$

$$\Rightarrow \quad 7y = 4u + 4eu$$
$$\therefore y = \frac{4}{7}u(1+e)$$

Substitute into (2)

$$\therefore x = \frac{4}{7}u(1+e) - eu$$
$$= \frac{4u}{7} - \frac{3ue}{7}$$
$$x = \frac{u}{7}(4 - 3e)$$

b Impulse = change in momentum of B

$$\therefore 2mu = 3m \times \frac{4}{7}u(1+e)$$
$$\therefore 1 + e = \frac{14}{12}$$
$$\therefore e = \frac{1}{6}$$

Use impulse = change in momentum of particle B.

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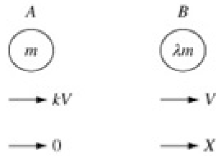
Exercise F, Question 15

Question:

A ball of mass m moving with speed kV on a smooth table hits another ball of mass λm moving with speed V travelling in the same direction on the table. The impact reduces the first ball to rest. Show that the coefficient of restitution is $\frac{\lambda+k}{\lambda(k-1)}$ and that λ must be greater than $\frac{k}{k-2}$ and k must be greater than 2.

Solution:



Let the speed of the second ball, after impact, be X.

Use conservation of momentum:

$$mkV + \lambda mV = \lambda mX$$

$$\therefore X = \frac{(\lambda + k)V}{\lambda} \quad *$$

Use Newton's Law of Restitution:

Let e = coefficient of restitution

Then $e = \dfrac{X}{kV - V}$

Substituting the value for X given in $*$:

$$e = \frac{(\lambda + k)V}{\lambda(k-1)V}$$

$$\therefore e = \frac{\lambda + k}{\lambda(k-1)}$$

As $e < 1$ $\therefore \dfrac{\lambda + k}{\lambda(k-1)} < 1$

$\therefore \lambda + k < \lambda k - \lambda$ (as $\lambda > 0$ and $k > 1$)

$\therefore 2\lambda + k < \lambda k$

$\lambda(k-2) > k$

Use the condition $e < 1$ to prove the inequalities.

As $k > 0$ and $\lambda > 0$ $\therefore k > 2$

$$\therefore \lambda > \frac{k}{k-2}$$

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Exercise F, Question 16

Question:

A ball is dropped from zero velocity and after falling for 1 s under gravity meets another equal ball which is moving upwards at 7 m s^{-1}.

a Taking the value of g as 9.8m s^{-2}, calculate the velocity of each ball after the impact, given that the coefficient of restitution is $\frac{1}{4}$.

b Find the percentage loss in kinetic energy due to the impact, giving your answer to 2 significant figures.

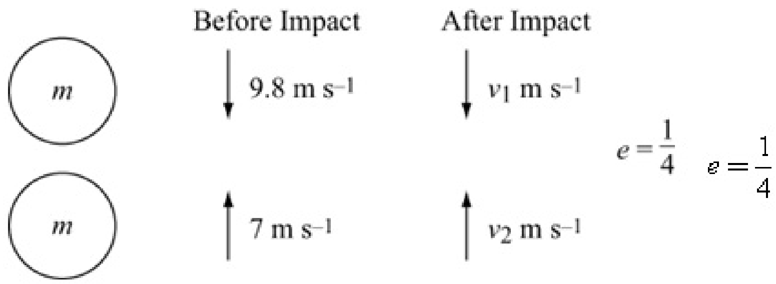
Solution:

a Find the velocity of the first ball before impact:

$u = 0, t = 1, a = 9.8, v = ?$

Use $v = u + at$
Then $v = 9.8$

Use constant acceleration formulae to find the velocity of the first ball prior to the impact.



Let mass of each ball be m.

Use conservation of momentum:

$9.8m - 7m = mv_1 + mv_2$

$\therefore v_1 + v_2 = 2.8 \quad (1)$

Use Newton's Law of Restitution:

$$e = \frac{1}{4} = \frac{v_2 - v_1}{9.8 - (-7)}$$

$\therefore \frac{1}{4} \times 16.8 = v_2 - v_1$

i.e. $v_2 - v_1 = 4.2 \quad (2)$

Add $(1) + (2)$

$\therefore 2v_2 = 7$

i.e. $v_2 = 3.5$

Substitute in (1)

$\therefore v_1 = -0.7$

Both balls change directions, the first moves up with speed 0.7 m s^{-1} and the second moves down with speed 3.5 m s^{-1}.

b

$$\text{K.E. before impact } = \frac{1}{2}m \times 9.8^2 + \frac{1}{2}m \times 7^2$$
$$= 72.52m \text{ J}$$
$$\text{K.E. after impact } = \frac{1}{2}m \times 0.7^2 + \frac{1}{2}m \times 3.5^2$$
$$= 6.37m \text{ J}$$

$$\therefore \% \text{loss in K.E.} = \frac{72.52 - 6.37}{72.52} = 91.2\%$$
$$= 91\% \text{ (2 s.f.)}$$

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Exercise F, Question 17

Question:

A particle falls from a height 8 m onto a fixed horizontal plane. The coefficient of restitution between the particle and the plane is $\frac{1}{4}$.

a Find the height to which the particle rises after impact.

b Find the time the particle takes from leaving the plane after impact to reach the plane again.

c What is the velocity of the particle after the second rebound?

You may leave your answers in terms of g.

Solution:

a Stage one: Particle falls under gravity ↓:

$u = 0, s = 8, a = g, v = ?$

Use constant acceleration formulae for the motion under gravity.

Use

$v^2 = u^2 + 2as$

$\therefore v^2 = 16g$

Stage two: First impact:

The particle rebounds with velocity $\frac{1}{4}\sqrt{16g}$.

Stage three: Particle moves under gravity ↑:

$u = \frac{1}{4}\sqrt{16g}, v = 0, s = ?, t = ?, a = -g$

Use

$v^2 = u^2 + 2as$

$\therefore 0 = \frac{1}{16} \times 16g - 2gs$

$\therefore s = 0.5\text{ m}$

i.e. the height to which particle rises is 0.5 m.

b Also use

$v = u + at$

$\therefore 0 = \frac{1}{4}\sqrt{16g} - gt$

$\therefore t = \frac{\sqrt{g}}{g} = 0.319$

$\therefore$ The time taken to reach the plane again

$= 2 \times 0.319$

$= 0.639\text{ s or } \frac{2}{\sqrt{g}}\text{ s}$

c The particle returns to the plane with velocity $\frac{1}{4}\sqrt{16g}$.

Stage four: Second impact

The velocity after the second rebound from the plane is $\frac{1}{4} \times \frac{1}{4}\sqrt{16g} = \frac{1}{4}\sqrt{g}\text{ m s}^{-1}$.

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Exercise F, Question 18

Question:

A particle falls from a height h onto a fixed horizontal plane. If e is the coefficient of restitution between the particle and the plane, show that the total time taken before the particle finishes bouncing is $\frac{1+e}{1-e} \times \sqrt{\frac{2h}{g}}$.

Solution:

Stage one: Particle falls under gravity $\downarrow$:

$u = 0, s = h, a = g, t = ?, v = ?$
Use

$$v^2 = u^2 + 2as$$
$$\therefore v^2 = 2gh$$

$\therefore$ Particle hits plane with velocity $\sqrt{2gh}$.
Use

$$s = ut + \frac{1}{2}at^2$$
$$\therefore h = \frac{1}{2}gt^2$$
$$\therefore t_1 = \sqrt{\frac{2h}{g}}$$

This is the time to the first bounce.

Stage two: Particle rebounds from plane.

The particle rebounds with velocity $e\sqrt{2gh}$.

Stage three: Particle moves under gravity until it hits the plane again $\uparrow$:

$u = e\sqrt{2gh}, s = 0, a = -g, t = ?, v = ?$
Use

$$s = ut + \frac{1}{2}at^2$$
$$\therefore 0 = e\sqrt{2gh}\,t - \frac{1}{2}gt^2$$
$$\therefore t = \frac{2e\sqrt{2gh}}{g}$$
$$\therefore t_2 = 2e\sqrt{\frac{2h}{g}}$$

This is time from first to second bounce.

and $v = -e\sqrt{2gh}$

This is velocity before second impact.

Stage four: Particle rebounds (again) from plane. It rebounds with velocity $e^2\sqrt{2gh}$.

By similar working you find that $t_3 = 2e^2\sqrt{\frac{2h}{g}}$ and $t_4 = 2e^3\sqrt{\frac{2h}{g}}, \ldots$ [Times between successive impacts form a G.P.]

$\therefore$ Total time taken by the particle is:

$$\sqrt{\frac{2h}{g}} + 2e\sqrt{\frac{2h}{g}} + 2e^2\sqrt{\frac{2h}{g}} + 2e^3\sqrt{\frac{2h}{g}} + \ldots$$
$$= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2h}{g}}\left(e + e^2 + e^3 + \ldots\right)$$

The times between successive impacts form a geometric progression.

The expression in the bracket is a G.P. with $a = e$ and $r = e$. It is an infinite G.P. and $S_\infty = \frac{a}{1-r} = \frac{e}{1-e}$.

Use the formula $\frac{a}{1-r}$ to find S_∞ for a G.P.

$$\therefore \text{Total time} = \sqrt{\frac{2h}{g}}\left[1 + \frac{2e}{1-e}\right]$$
$$= \frac{1+e}{1-e} \times \sqrt{\frac{2h}{g}}$$

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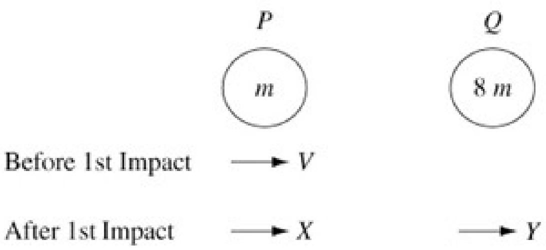
Exercise F, Question 19

Question:

A sphere P of mass m lies on a smooth table between a sphere Q of mass $8m$ and a fixed vertical plane. Sphere P is projected towards sphere Q. The coefficient of restitution between the two spheres is $\frac{7}{8}$. Given that sphere P is reduced to rest by the second impact with sphere Q find the coefficient of restitution between sphere P and the fixed vertical plane.

Solution:



Let initial velocity of P be V and use conservation of momentum and Newton's Law of Restitution to find velocities of P and Q after impact.

Let initial speed of P be V and let speeds of P and Q be X and Y after the 1st impact.

Use conservation of momentum:

$$\therefore mV = mX + 8mY$$

i.e. $X + 8Y = V$ (1)

Use Newton's Law of Restitution:

$$\frac{7}{8} = \frac{Y - X}{V}$$

$\therefore Y - X = \frac{7}{8}V$ (2)

Add equations (1) and (2)

$$\therefore 9Y = \frac{15}{8}V$$

i.e. $Y = \frac{5}{24}V$

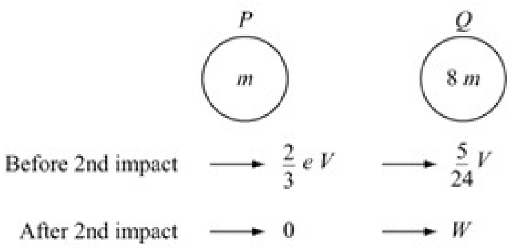
Substitute into (2)

$$\therefore X = \frac{5}{24}V - \frac{7}{8}V$$
$$= -\frac{2}{3}V$$

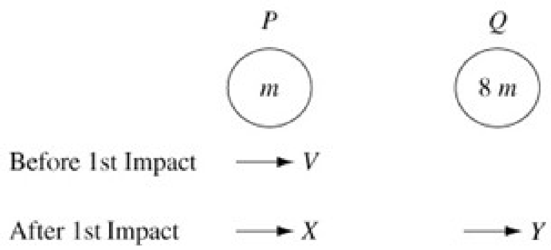
P then hits the vertical plane with speed $\frac{2}{3}V$ and rebounds with speed $\frac{2}{3}eV$.

Then consider impact of P with the vertical plane to find new velocity of P.

2nd impact between P and Q:



After the 2nd impact between P and Q let the velocity of Q be W and the velocity of P be 0.



Let initial velocity of P be V and use conservation of momentum and Newton's Law of Restitution to find velocities of P and Q after impact.

Use conservation of momentum:

$$m \times \frac{2}{3}eV + 8m \times \frac{5}{24}V = 8mW$$

$$\therefore 2eV + 5V = 24W \quad (1)$$

Use Newton's Law of Restitution:

$$\frac{7}{8} = \frac{W - 0}{\frac{2}{8}eV - \frac{5}{24}V}$$

$$\therefore \frac{7}{8}\left(\frac{2}{8}eV - \frac{5}{24}V\right) = W$$

$$\text{i.e. } 14eV - \frac{35}{8}V = 24W \quad (2)$$

Subtract $(1) - (2)$

Finally consider 2nd impact of P and Q.

$$\therefore -12eV + 5V + \frac{35}{8}V = 0$$

$$\therefore 12eV = \frac{75V}{8}$$

$$\therefore e = \frac{75}{96}$$

$$= \frac{25}{32}$$

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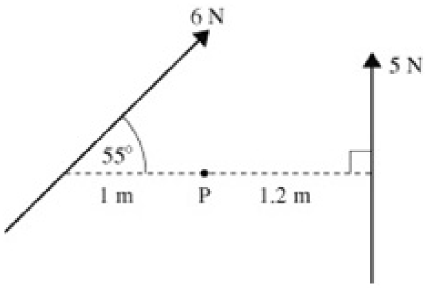
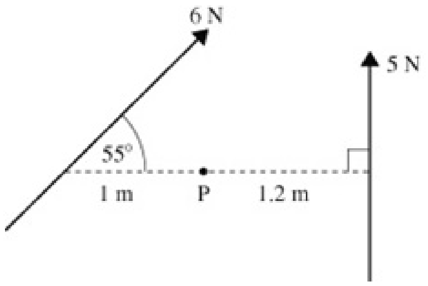
Exercise A, Question 1

Question:

Find the sum of the moments about P of the forces shown.



Solution:



↻ $1 \times 6 \times \sin 55 = 4.91\ldots$

↺ $1.2 \times 5 = 6$

$\Rightarrow$ $6 - 4.91 = 1.09$ Nm anticlockwise

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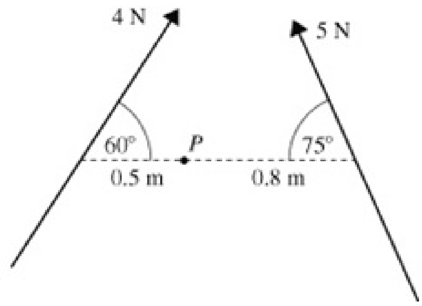
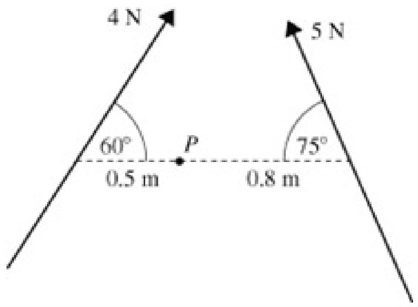
Exercise A, Question 2

Question:

Find the sum of the moments about P of the forces shown.



Solution:



↻ $0.5 \times 4 \times \sin 60 = 1.73\ldots$

↺ $0.8 \times 5 \times \sin 75 = 3.86\ldots$

$\Rightarrow$ $3.86 - 1.73$

$= 2.13$ Nm anticlockwise

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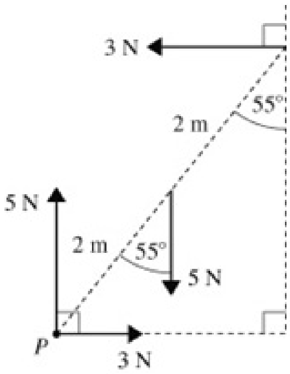
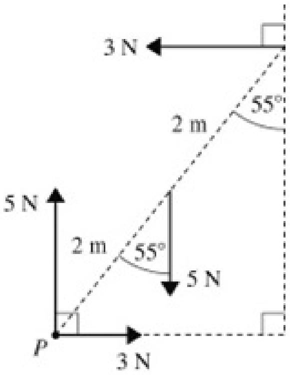
Exercise A, Question 3

Question:

Find the sum of the moments about P of the forces shown.



Solution:



↻ $5 \times 2 \times \sin 55 = 8.19\ldots$

↺ $3 \times 4 \times \cos 55 = 6.88\ldots$

$\Rightarrow$ $8.19 - 6.88$

$= 1.31$ Nm clockwise

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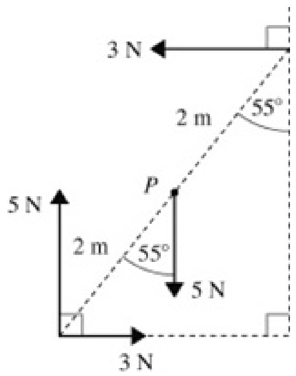
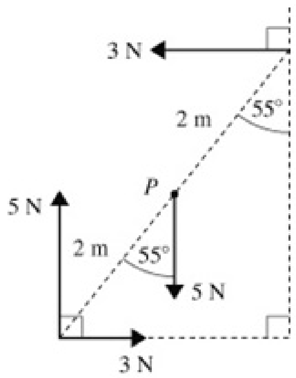
Exercise A, Question 4

Question:

Find the sum of the moments about P of the forces shown.



Solution:



↺ $5 \times 2 \times \sin 55° = 8.19\ldots$

↻ $3 \times 2 \times \cos 55° + 3 \times 2 \times \cos 55°$

$= 6.88\ldots$

$\Rightarrow$ 1.31 Nm clockwise

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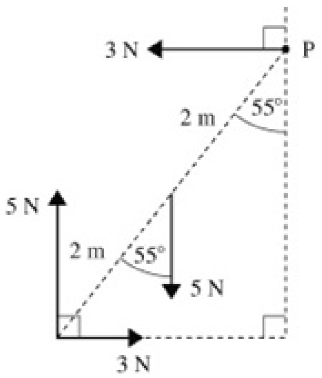
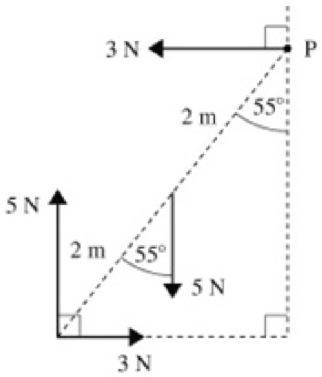
Exercise A, Question 5

Question:

Find the sum of the moments about P of the forces shown.



Solution:



↻ $3 \times 4 \times \cos 55° + 5 \times 2 \times \sin 55°$
$= 15.07\ldots$

↺ $5 \times 4 \times \sin 55° = 16.38\ldots$

$\Rightarrow$ 1.31 Nm anticlockwise

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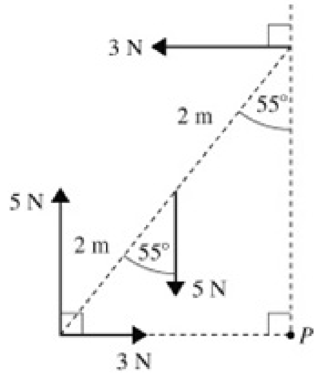
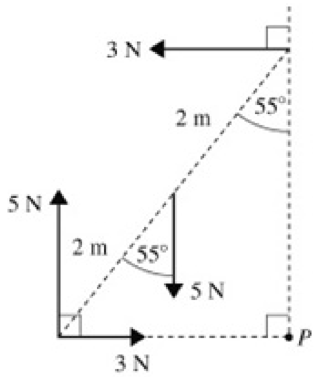
Exercise A, Question 6

Question:

Find the sum of the moments about P of the forces shown.



Solution:



↻ $5 \times 4 \times \sin 55° = 16.38\ldots$

↺ $3 \times 4 \times \cos 55° + 5 \times 2 \times \sin 55$
$= 15.07\ldots$

$\Rightarrow$ 1.31 Nm clockwise

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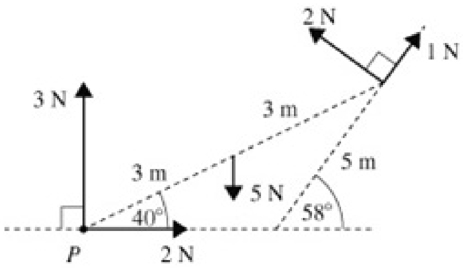
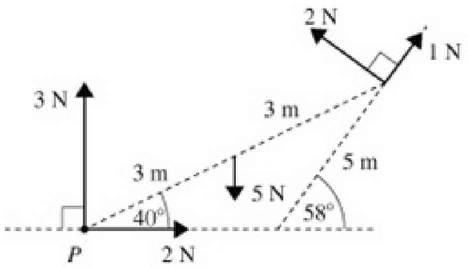
Exercise A, Question 7

Question:

Find the sum of the moments about P of the forces shown.



Solution:



$5\times3\times\cos40° = 11.49\ldots$

$2\times6\times\sin71° + 1\times6\times\sin19°$

$= 13.29\ldots$

$\Rightarrow$ 1.81 Nm anticlockwise

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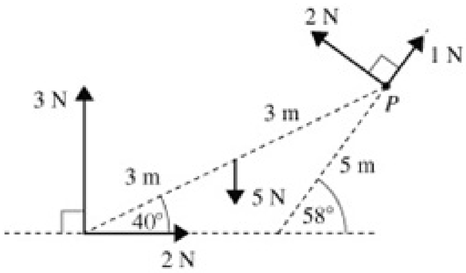
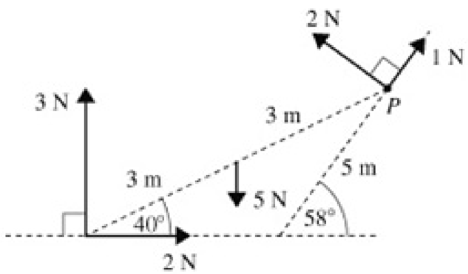
Exercise A, Question 8

Question:

Find the sum of the moments about P of the forces shown.



Solution:



$3 \times 6 \times \cos 40° = 13.7\ldots$

$5 \times 3 \times \cos 40° + 2 \times 6 \times \sin 40°$
$= 19.2\ldots$

$\Rightarrow$ 5.42 Nm anticlockwise

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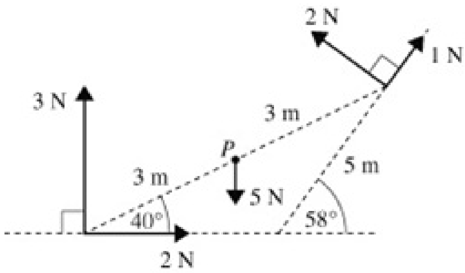
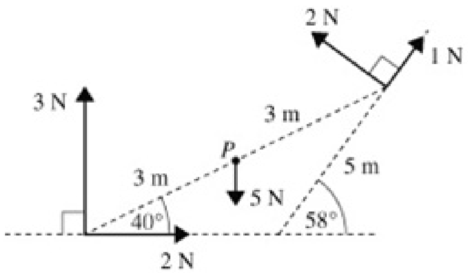
Exercise A, Question 9

Question:

Find the sum of the moments about P of the forces shown.



Solution:



↻ $3 \times 3 \times \cos 40° = 6.89\ldots$

↺ $2 \times 3 \times \sin 40° + 2 \times 3 \times \sin 71° + 1 \times 3 \times \sin 19°$

$= 10.5\ldots$

$\Rightarrow$ 3.61 Nm anticlockwise

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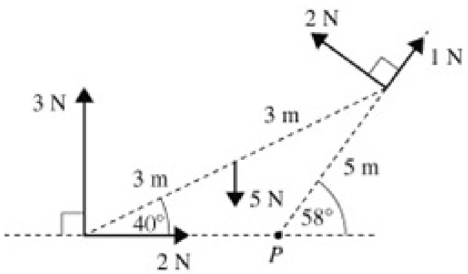
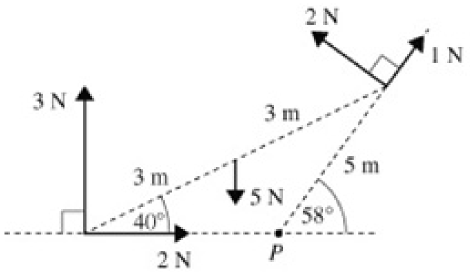
Exercise A, Question 10

Question:

Find the sum of the moments about P of the forces shown.



Solution:



↻ $3\times(6\times\cos 40^\circ - 4.5\times\cos 59^\circ) = 6.83\ldots$

↺ $2\times 4.5 + 5\times(3\times\cos 40^\circ - 4.5\times\cos 59^\circ)$
$= 8.90\ldots$

$\Rightarrow$ 2.07 Nm anticlockwise

NB Since $3\times\cos 40^\circ - 4.5\times\cos 59^\circ = -0.019\ldots$
We can deduce that the 5 N force has a clockwise moment about P. However, this does not mean that the working is invalid – the negative value compensates for the sense of the rotation.

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Exercise B, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8 \text{ m s}^{-2}$.

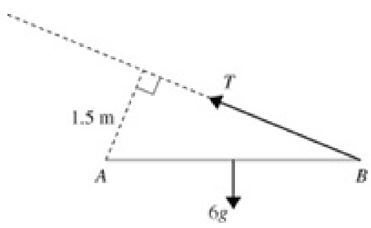
Each of the following diagrams shows a uniform beam AB of length 4 m and mass 6 kg.

The beam is freely hinged at A and resting horizontally in equilibrium. In each case find

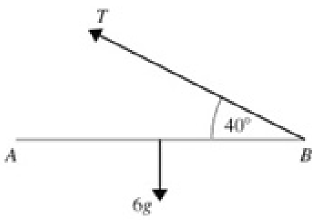
i the magnitude of the force T,

ii the magnitude and direction of the reaction at A.

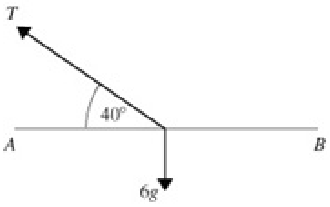
a



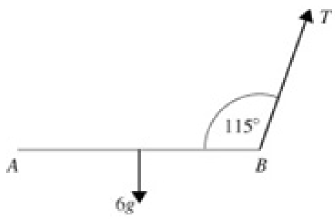
b



c

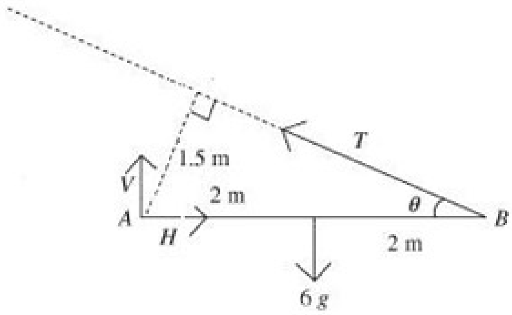


d



Solution:

a



Using moments:

$$\circlearrowleft A: 6g \times 2 = T \times 1.5$$
$$T = 8g = 78.4 \text{ N}$$

$$\text{R}(\uparrow)\ V + T\sin\theta = 6g$$
$$V = 6g - T \times \frac{1.5}{4} = 29.4 \text{ N}$$
$$\leftrightarrow H = T\cos\theta = T \times \frac{\sqrt{64-9}}{8} = 72.7 \text{ N}$$

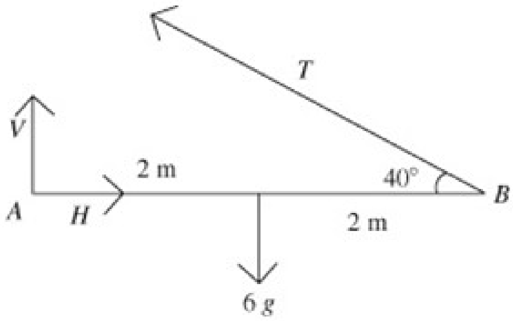
Using Pythagoras:

Magnitude of the force at A

$$= \sqrt{29.4^2 + 72.7^2} = 78.4 \text{ N}$$

The force acts at $\tan^{-1}\frac{29.4}{72.7} = 22^\circ$ above AB.

b



Using moments:

$$\circlearrowleft A: 6g \times 2 = T \times 4\sin 40^\circ$$
$$T = 45.7 \text{ N}$$
$$\text{R}(\uparrow)\ V + T\sin 40^\circ = 6g$$
$$V = 6g - T\sin 40^\circ = 29.4 \text{ N}$$
$$\text{R}(\rightarrow) H = T\cos 40^\circ = 35.0 \text{ N}$$

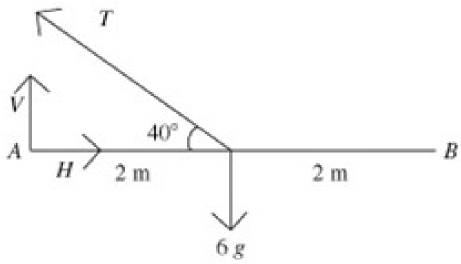
Using Pythagoras:

Magnitude of the force at A

$$= \sqrt{29.4^2 + 35.0^2} = 45.7 \text{ N}$$

The force acts at $\tan^{-1}\frac{29.4}{35.0} = 40^\circ$ above AB.

c



Using moments:

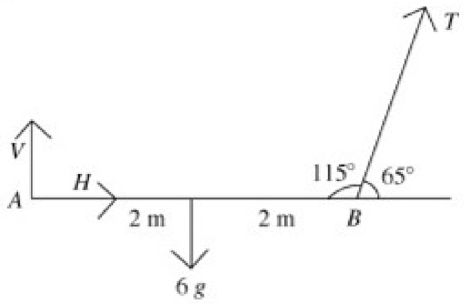
$$\circlearrowleft A: 6g \times 2 = T \times 2\sin 40^\circ$$
$$T = 91.5 \text{ N}$$

$$\text{R}(\uparrow)\ V + T\sin 40^\circ = 6g$$
$$V = 6g - T\sin 40^\circ = 0 \text{ N}$$
$$\text{R}(\rightarrow)\ H = T\cos 40^\circ = 70.1 \text{ N}$$

Magnitude of the force at $A = 70.1$ N.

The force acts parallel to AB.

d



Using moments:

$\circlearrowleft A: 6g \times 2 = T \sin 65° \times 4$

$T = 32.4 \text{ N}$

$R(\uparrow)\ V + T \sin 65° = 6g$

$V = 6g - T \sin 65° = 29.4 \text{ N}$

$R(\rightarrow)\ H + T \cos 65° = 0$

$H = -T \cos 65° = -13.7 \text{ N}$

Using Pythagoras:

Magnitude of the force at A

$= \sqrt{29.4^2 + 13.7^2} = 32.4 \text{ N}$

The force acts at $\tan^{-1} \frac{29.4}{13.7} = 115°$ above AB.

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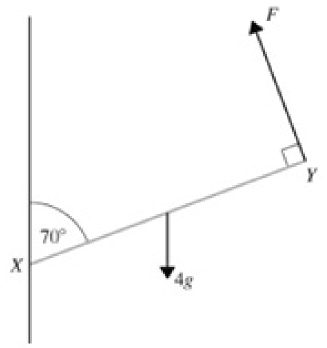
Exercise B, Question 2

Question:

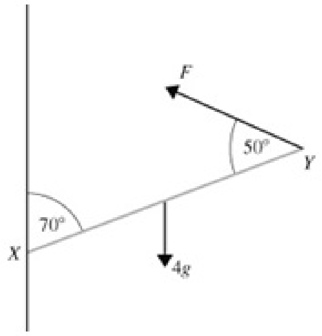
Each of the following diagrams shows a uniform rod XY of mass 4 kg and length 5 m.

The rod is freely hinged to a vertical wall at X. The rod rests in equilibrium at an angle to the horizontal. Find the magnitude of the force F in each case.

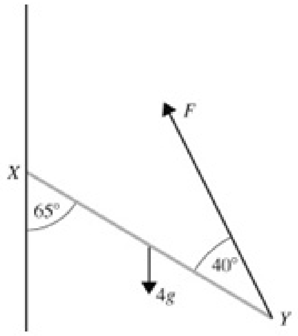
a



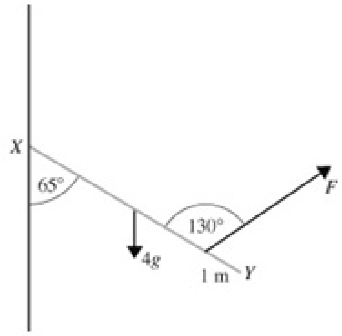
b



c

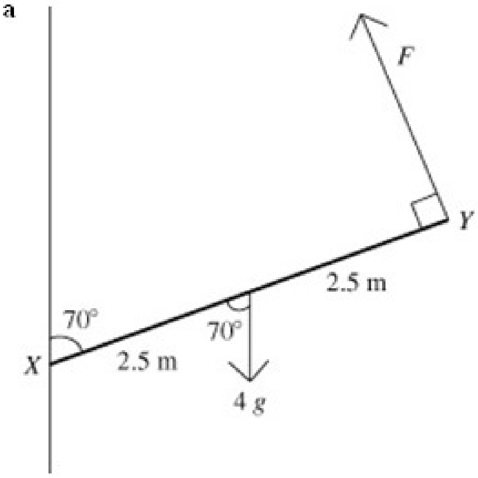


d



Solution:

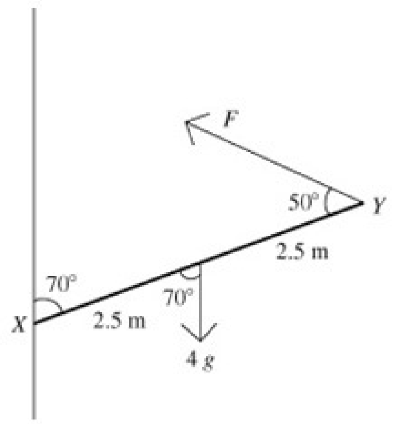
a



$$\mathrm{M}(X) \quad 2.5 \times 4g \times \sin 70^\circ = 5 \times F$$

$$F = \frac{10g \times \sin 70^\circ}{5} = 18.4 \text{ N}$$

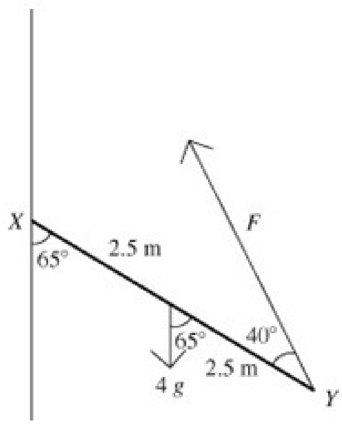
b



$$\mathrm{M}(X) \quad 2.5 \times 4g \times \sin 70^\circ = 5 \times F \times \sin 50^\circ$$

$$F = \frac{10g \times \sin 70^\circ}{5 \times \sin 50^\circ} = 24.0 \text{ N}$$

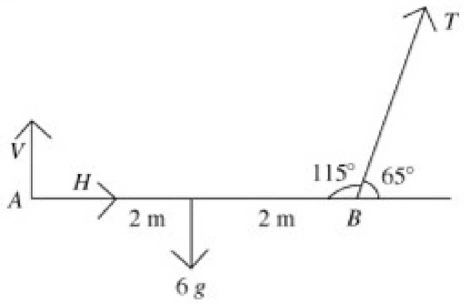
c



$$\mathrm{M}(X) \quad 2.5 \times 4g \times \sin 65^\circ = 5 \times F \times \sin 40^\circ$$

$$F = \frac{10g \times \sin 65^\circ}{5 \times \sin 40^\circ} = 27.6 \text{ N}$$

d



Using moments:

$\circlearrowleft A: 6g \times 2 \ = T\sin 65^\circ \times 4$

$T = 32.4 \text{ N}$

$R(\uparrow)\ V + T\sin 65^\circ = 6g$

$V = 6g - T\sin 65^\circ = 29.4 \text{ N}$

$R(\rightarrow)\ H + T\cos 65^\circ = 0$

$H \ = -T\cos 65^\circ = -13.7 \text{ N}$

Using Pythagoras:

Magnitude of the force at A

$= \sqrt{29.4^2 + 13.7^2} = 32.4 \text{ N}$

The force acts at $\tan^{-1}\frac{29.4}{13.7} = 115^\circ$ above AB.

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Exercise B, Question 3

Question:

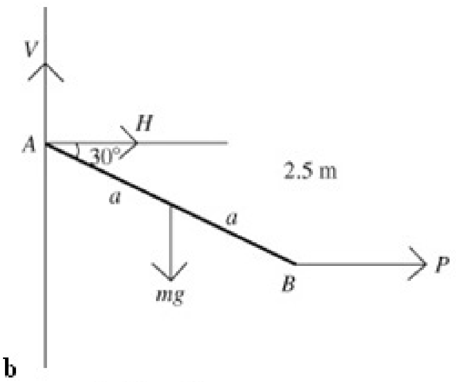
A uniform rod AB of length $2a$ m and mass m kg is smoothly hinged at A. It is maintained in equilibrium by a horizontal force of magnitude P acting at B. The rod is inclined at $30°$ to the horizontal with B below A.

a Show that $P = \frac{\sqrt{3}}{2}mg$.

b Find the magnitude and direction of the reaction at the hinge.

Solution:

a



M(A) $mg \times a\cos 30° = P \times 2a \times \sin 30°$

$$P = \frac{mga\cos 30°}{2a\sin 30°} = \frac{mga\frac{\sqrt{3}}{2}}{2a\frac{1}{2}} = \frac{\sqrt{3}}{2}mg$$

as required.

b

R(↑) $V = mg$

R(→) $H + P = 0$, $H = -\frac{\sqrt{3}}{2}mg$

$\Rightarrow$ The reaction at the hinge has magnitude $= mg\sqrt{1^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = mg\sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{2}mg$

and acts at $\tan^{-1}\frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = 49.1°$ to the vertical (away from AB).

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Exercise B, Question 4

Question:

A uniform beam AB of mass 10 kg and length 3 m is attached to a vertical wall by means of a smooth hinge at A. The beam is maintained in the horizontal position by means of a light inextensible string, one end of which is attached to the beam at B and the other end of which is attached to the wall at a point 2 m vertically above A.

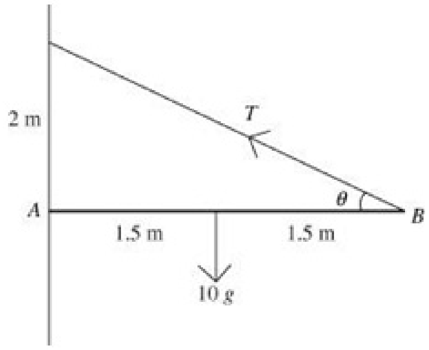
a Find the tension in the string.

A particle of mass M kg is now attached to the beam at B.

b Given that the tension in the string is now double its original value, find the value of M.

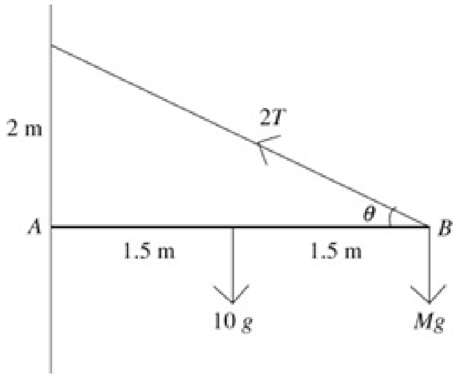
Solution:

a



So $T = 15g \times \frac{\sqrt{13}}{6} = 88.3$ N

b



$$\text{M}(A) \quad 10g \times 1.5 + Mg \times 3 = 2T\sin\theta \times 3 = 2 \times 15g$$
$$3Mg = 15g,$$
$$M = 5$$

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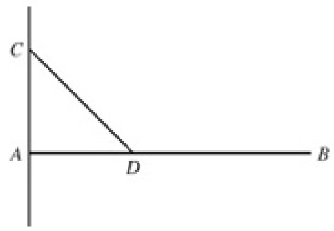
Exercise B, Question 5

Question:

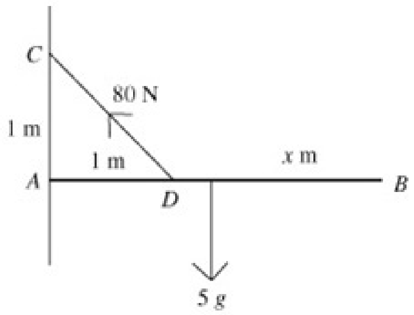
A uniform horizontal beam AB of mass 5 kg is freely hinged to a vertical wall and is supported by a rod CD as shown in the diagram.

Given that the tension in the rod is 80 N,

$AC = 1\text{ m}$ and the angle between the rod and the vertical is $45°$, find the length of the beam.



Solution:



Suppose that the length of AB is $2x$ m.

$$M(A) \quad 5g \times x = 80 \cos 45° \times 1$$

$$x = \frac{80 \cos 45°}{5g} = 1.15\ldots \text{ m}$$

The length of AB is 2.31 m.

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Exercise B, Question 6

Question:

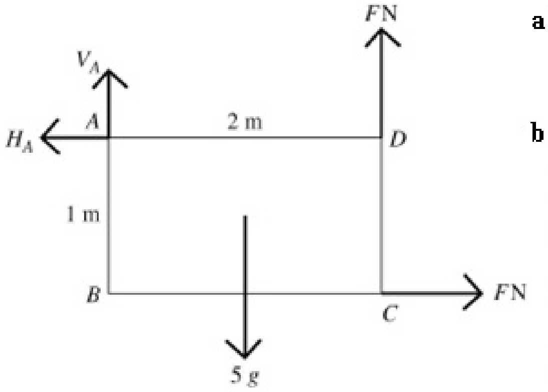
$ABCD$ is a uniform rectangular lamina with mass 5 kg, side $AB = 1$ m, and side $AD = 2$ m.

It is hinged at A so that it is free to move in a vertical plane. It is maintained in equilibrium, with B vertically below A, by a horizontal force acting at C and a vertical force acting at D, each of magnitude F N. Find

a the value of F,

b the magnitude and direction of the force exerted by the hinge on the lamina.

Solution:



a M(A) $5g \times 1 = F \times 1 + F \times 2 = 3F$

$F = \frac{5}{3}g = 16.3$ N

b R($\rightarrow$) $H_A = F = \frac{5g}{3}$ N

R($\uparrow$) $V_A + F = 5g$,

$V_A = 5g - F = \frac{10}{3}g$ N

Magnitude $= \frac{5g}{3}\sqrt{1^2 + 2^2} = 36.5$ N

Direction $\tan^{-1}\frac{1}{2} = 26.6°$ to the vertical away from the lamina.

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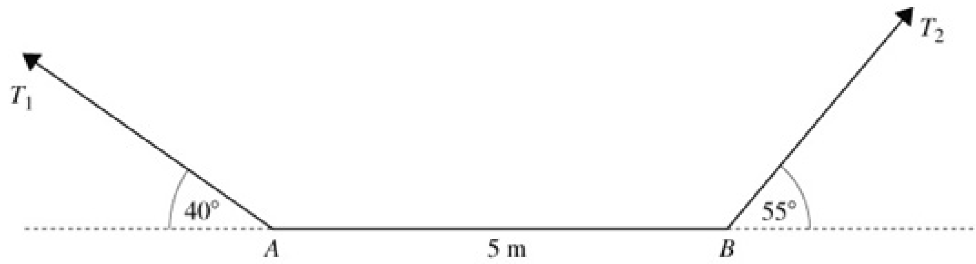
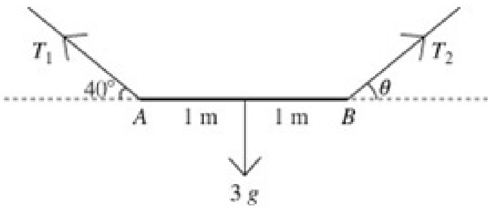
Exercise B, Question 7

Question:



A uniform rod AB of mass 3 kg and length 2 m rests horizontally in equilibrium supported by two strings attached at the ends of the rod. The strings make angles of $40°$ and θ with the horizontal, as shown in the diagram. Find the magnitudes of the tensions in the strings and the value of θ.

Solution:



M(B) $T_1 \times \sin 40° \times 2 = 3g \times 1$

$$T_1 = \frac{3g}{2\sin 40°} = 22.9 \text{ N}$$

R($\uparrow$) $T_1 \sin 40° + T_2 \sin \theta = 3g$

Using the moments equation:

$\frac{3}{2}g + T_2 \sin\theta = 3g \Rightarrow T_2 \sin\theta = \frac{3}{2}g$

R($\rightarrow$) $T_1 \cos 40° = T_2 \cos\theta = \dfrac{3g}{2\sin\theta}\cos\theta$

But we know that $T_1 = \dfrac{3g}{2\sin 40°}$ so we can deduce that $\cot 40° = \cot\theta, \theta = 40°$.

Thus $T_2 = 22.9$ N.

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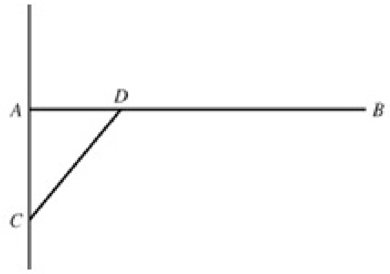
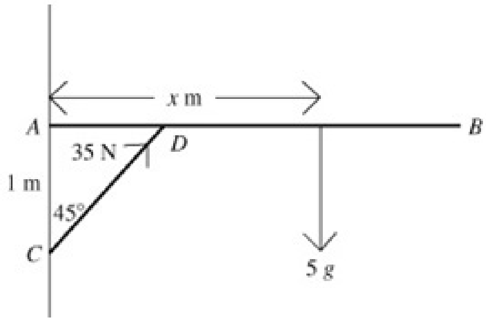
Exercise B, Question 8

Question:

A non-uniform horizontal beam AB of mass 5 kg and length 3 m is freely hinged to a vertical wall and is supported by a rod CD as shown in the diagram. Given that the thrust in the rod is 35 N, $AC = 1\text{ m}$ and the angle between the rod and the vertical is $45°$, find the distance of the centre of mass of the beam from A.



Solution:



Suppose that the centre of mass is x m from A. We are not told anything about any force(s) acting at A, but as they have zero moment about A this will not matter.

$$\text{M}(A) \quad 35\cos 45° \times 1 = 5g \times x$$

$$x \quad = \frac{35\cos 45°}{5g} \approx 0.51\text{ m}$$

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Exercise B, Question 9

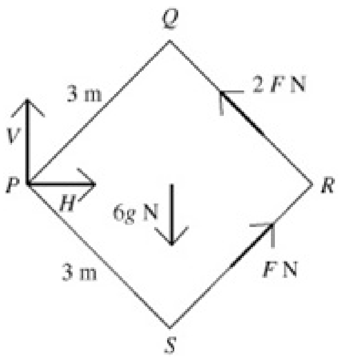
Question:

$PQRS$ is a uniform square lamina of side 3 m and mass 6 kg. It is freely hinged at P so that it is free to move in a vertical plane. It is maintained in equilibrium, with PR horizontal, and Q above S, by a force of magnitude FN acting along SR and a force of magnitude $2F$ N acting along RQ. Find

a the value of F,

b the magnitude and direction of the force exerted by the hinge on the lamina.

Solution:



a PQR is an isosceles triangle with two sides of 3 m, so the length of PR is $3\sqrt{2}$ m .

$$\text{M}(P)\ 6g \times \frac{3\sqrt{2}}{2} = 2F \times 3 + F \times 3 = 9F$$

$$F = g\sqrt{2} \approx 13.9 \text{ N}$$

b $\text{R}(\rightarrow)\quad H + F\cos 45° = 2F\cos 45°,$

$H = F\cos 45° = g = 9.8 \text{ N}$

$\text{R}(\uparrow)\quad V + 2F\cos 45° + F\cos 45° = 6g$

$V = 6g - 3F\cos 45° = 3g = 29.4 \text{ N}$

$\Rightarrow$ Magnitude of the combined force

$$= \sqrt{9.8^2 + 29.4^2} = \sqrt{960.4} = 31.0 \text{ N}$$

The resultant force is acting at angle

$\tan^{-1}\frac{9.8}{29.4} = \tan^{-1}\frac{1}{3} = 18.4°$ to the upward vertical at P.

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Exercise B, Question 10

Question:

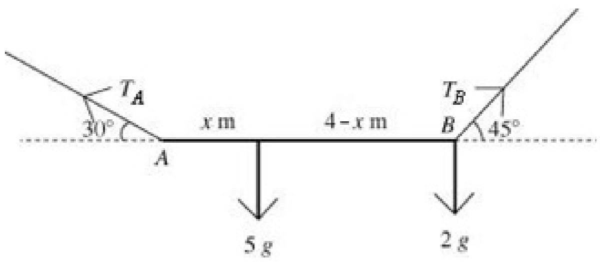
A non-uniform rod AB of mass 5 kg and length 4 m, with a particle of mass 2 kg attached at B, rests horizontally in equilibrium supported by two strings attached at the ends of the rod.

The strings make angles of 30° and 45° with the horizontal, as shown in the diagram. Find

a the tensions in each of the strings,

b the position of the centre of mass of the rod.

Solution:



a R(→) $T_A \cos 30° = T_B \cos 45°$,

$$T_A \frac{\sqrt{3}}{2} = T_B \frac{1}{\sqrt{2}}, T_A = \sqrt{\frac{2}{3}} T_B$$

R(↑) $T_A \sin 30° + T_B \sin 45° = 5g + 2g = 7g$

$$T_A \times \frac{1}{2} + T_B \times \frac{1}{\sqrt{2}} = T_B \times \left(\frac{1}{2} \times \sqrt{\frac{2}{3}} + \frac{1}{\sqrt{2}}\right) = 7g$$

$T_B = 61.5$ N and $T_A = 50.2$ N

b Suppose that the centre of mass is x m from A.

M(A) $5g \times x + 2g \times 4 = T_B \times \sin 45° \times 4$

$$x = \frac{T_B \times \frac{1}{\sqrt{2}} \times 4 - 8g}{5g}$$

$$x = 1.95 \text{ m}$$

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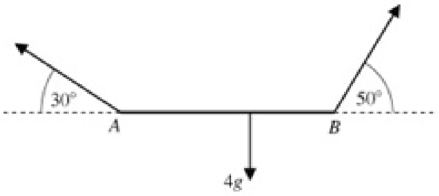
Exercise C, Question 1

Question:

Use the method described in Example 4 to answer the questions in this exercise.
Whenever a numerical value of g is required, take $g = 9.8 \text{ m s}^{-2}$.

A non-uniform rod AB, of mass 4 kg and length 6 m, rests horizontally in equilibrium supported by two strings attached at the ends of the rod. The strings make angles of 30° and 50° with the horizontal, as shown in the diagram. Find



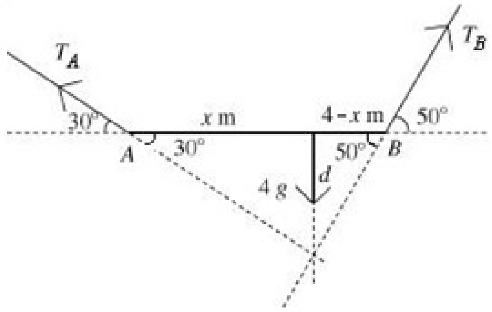
a the position of the centre of mass of the rod,

b the tensions in the two strings.

Solution:

Three forces – the lines of action must be concurrent:

a



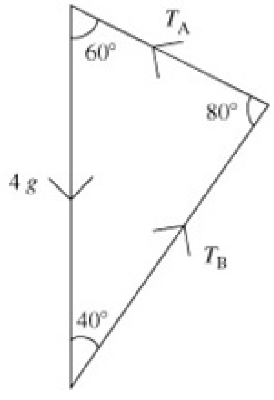
d is common to two triangles, so

$d = x\sin 30° = (4 - x)\sin 50°$

$x(\sin 30° + \sin 50°) = 4\sin 50°$

$$x = \frac{4\sin 50°}{\sin 30° + \sin 50°} = 2.42 \text{ m from } A$$

b Triangle of forces:



Using the sine rule:

$$\frac{T_A}{\sin 40°} = \frac{T_B}{\sin 60°} = \frac{4g}{\sin 80}$$

$$T_A = \frac{4g\sin 40°}{\sin 80°} = 25.6 \text{ N and}$$

$$T_B = \frac{4g\sin 60°}{\sin 80°} = 34.5 \text{ N}$$

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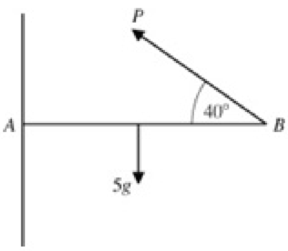
Exercise C, Question 2

Question:

A uniform rod AB of mass 5 kg and length 3 m is freely hinged to a vertical wall at A. The rod is maintained in horizontal equilibrium by a force P N acting at B, as shown in the diagram. Find

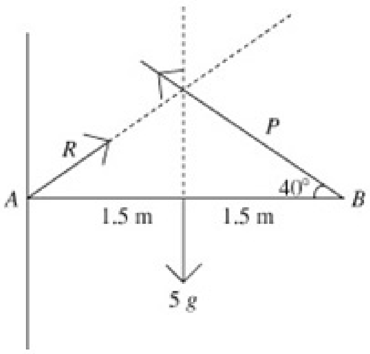
a the magnitude of P,

b the magnitude and direction of the reaction of the force exerted by the hinge on the rod.



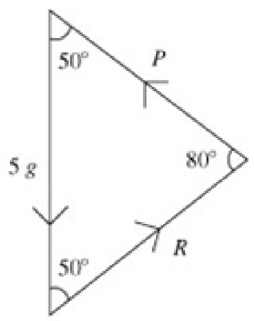
Solution:

a Three forces – the lines of action must be concurrent:



Because the rod is uniform we know that the centre of mass is at the mid-point of the rod. This implies a symmetrical diagram (isosceles triangle), so we know that R acts at $40°$ to the rod and the magnitudes of P and R are equal.

b Triangle of forces:



Using the sine rule:

$$\frac{R}{\sin 50°} = \frac{P}{\sin 50°} = \frac{5g}{\sin 80}$$

$$R = P = \frac{5g \sin 50°}{\sin 80°} = 38.1\,\text{N}$$

Alternatively, using the fact that the triangle is isosceles,

$$\frac{\frac{5g}{2}}{P} = \cos 50°, P = \frac{5g}{2\cos 50} = 38.1\,\text{N}$$

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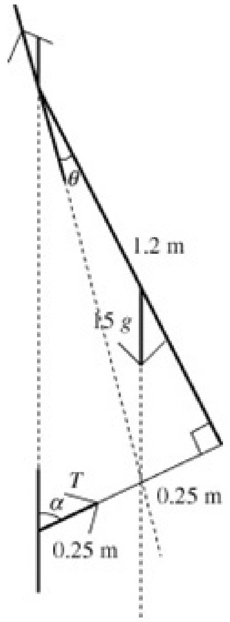
Exercise C, Question 3

Question:

A window of mass 15 kg and height 120 cm is hinged along its top edge. It is kept open by the thrust from a light strut of length 50 cm attached to the wall and perpendicular to the lower edge of the window. By modelling the window as a uniform lamina, calculate the thrust in the strut and the magnitude and direction of the force exerted on the window by the hinge.

Solution:

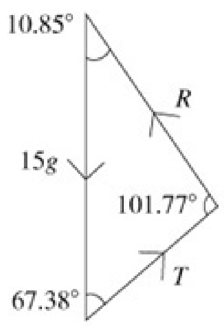
Three forces – the lines of action must be concurrent:



Using the right angled triangles in the diagram,

$$\tan\alpha = \frac{1.2}{0.5},\ \alpha = 67.38^\circ$$
$$\tan\theta = \frac{0.25}{1.2},\ \theta = 11.77^\circ$$
$$90^\circ - 67.38^\circ - 11.77^\circ = 10.85^\circ$$
$$90^\circ + 11.77^\circ = 101.77^\circ$$

Triangle of forces:



Using the sine rule:

$$\frac{R}{\sin 67.38^\circ} = \frac{T}{\sin 10.85^\circ} = \frac{15g}{\sin 101.77}$$
$$R = \frac{15g \sin 67.38^\circ}{\sin 101.77^\circ} = 139 \text{ N and}$$
$$T = \frac{15g \sin 10.85^\circ}{\sin 101.77^\circ} = 28.3 \text{ N}$$

R acts at 10.9° to the upward vertical.

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Exercise C, Question 4

Question:

A uniform rod AB of weight 20 N and length 3 m is freely hinged to a vertical wall at A. A force P is applied at B at right angles to the rod in order to keep the rod in equilibrium at an angle of 30° to the horizontal with B above A. Find

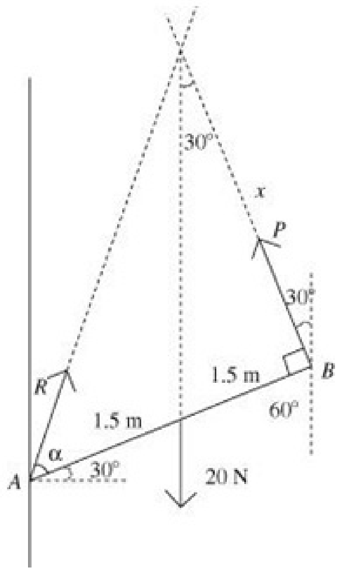
a the magnitude of P,

b the magnitude and direction of the reaction at the hinge.

Solution:

Three forces – the lines of action must be concurrent:

a



x lies in two right angled triangles:

$\frac{x}{3} = \tan\alpha$ and $\frac{1.5}{x} = \tan 30°$

So

$$x = \frac{1.5}{\tan 30°} \text{ and}$$

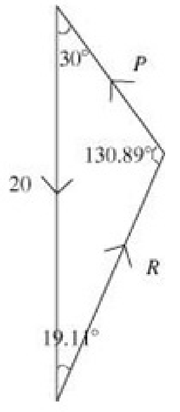
$$\tan\alpha = \frac{1.5}{3\tan 30°} = \frac{\sqrt{3}}{2}$$

$$\alpha = 40.89\ldots°$$

$$90° - 30° - 40.89° = 19.11°$$

$$180° - 30° - 19.11° = 130.89°$$

b Triangle of forces:



Using the sine rule:

$$\frac{P}{\sin 19.11°} = \frac{R}{\sin 30°} = \frac{20}{\sin 130.89°}$$

$$P = \frac{20\sin 19.11°}{\sin 130.89°} = 8.7 \text{ N and}$$

$$R = \frac{20\sin 30°}{\sin 130.89°} = 13.2 \text{ N acting at } 19.11°$$

to the upward vertical.

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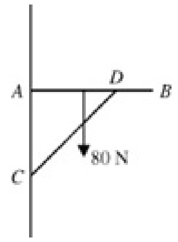
Exercise C, Question 5

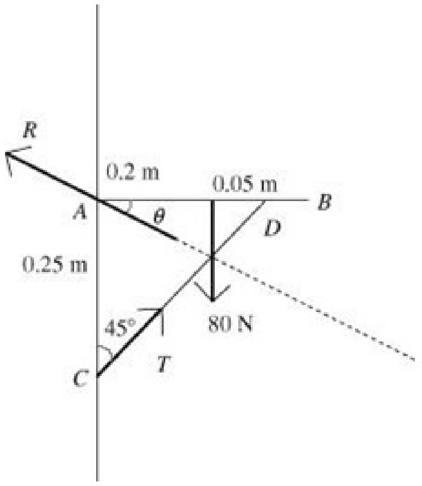
Question:

AB is a loaded shelf freely hinged to the wall at A, and supported in a horizontal position by a light strut CD, which is attached to the shelf at D, 25 cm from A, and attached to the wall at C, 25 cm below A. The total weight of the shelf and its load is 80 N, and the centre of mass is 20 cm from A. Find the thrust in the strut and the magnitude and direction of the force exerted by the hinge on the shelf.



Solution:

– the lines of action must be concurrent:



The isosceles right angled triangle ADC gives us angles of 45°.

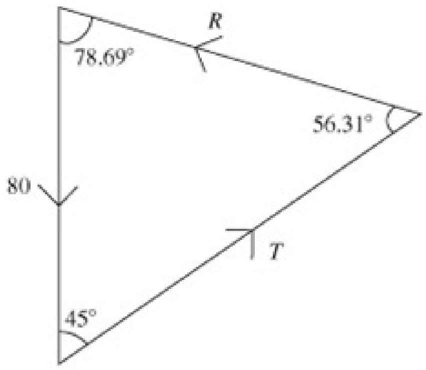
Therefore $\tan\theta = \frac{0.05}{0.2} = \frac{1}{4}$

$$\theta = \tan^{-1} 0.25 = 11.31^\circ$$

$$90^\circ - 11.31^\circ = 78.69^\circ$$

$$180^\circ - 78.69^\circ - 45^\circ = 56.31^\circ$$

Triangle of forces:



Using the sine rule:

$$\frac{R}{\sin 45^\circ} = \frac{T}{\sin 78.69^\circ} = \frac{80}{\sin 56.31^\circ}$$

$$R = \frac{80 \sin 45^\circ}{\sin 56.31^\circ} = 68.0 \text{ N at } 78.69^\circ \text{ to the}$$

upward vertical and

$$T = \frac{80 \sin 78.69^\circ}{\sin 56.31^\circ} = 94.3 \text{ N}$$

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Exercise C, Question 6

Question:

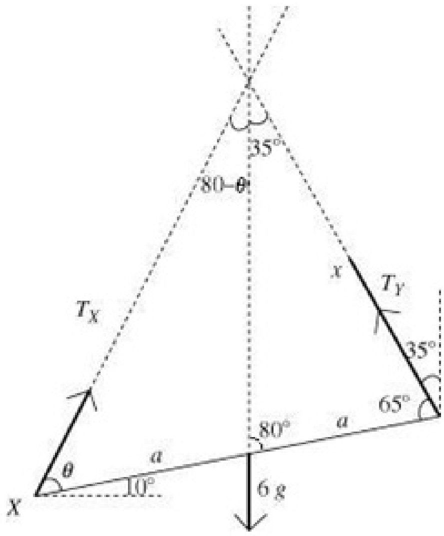
A uniform rod XY of mass 6 kg is suspended from the ceiling by light inextensible strings attached to its ends. The rod is resting in equilibrium at $10°$ to the horizontal with X below Y. The string attached to Y is at an angle of $35°$ to the vertical. Find

a the angle that the other string makes with the vertical,

b the tensions in the two strings.

Solution:

a Three forces – the lines of action must be concurrent:



Suppose that the rod has length $2a$.
x lies in two triangles:

$$\frac{x}{\sin 80^\circ} = \frac{a}{\sin 35^\circ},\ x = \frac{a\sin 80}{\sin 35} \text{ and}$$

$$\frac{x}{\sin\theta} = \frac{2a}{\sin(115-\theta)},\ x = \frac{2a\sin\theta}{\sin(115-\theta)}.$$

$$\frac{a\sin 80^\circ}{\sin 35^\circ} = \frac{2a\sin\theta}{\sin(115^\circ-\theta)}$$

$$\frac{\sin 80^\circ}{\sin 35^\circ} = \frac{2\sin\theta}{\sin 115^\circ\cos\theta - \cos 115^\circ\sin\theta}$$

(Cancelling the a and using the expansion for $\sin(A+B)$)

$$\sin 80^\circ(\sin 115^\circ\cos\theta - \cos 115^\circ\sin\theta) = 2\sin\theta\cos 35^\circ$$

(Cross-multiplying.)

$$\cos\theta(\sin 80^\circ\sin 115^\circ) = \sin\theta(2\sin 35^\circ + \sin 80^\circ\cos 115^\circ)$$

(Rearrange and collect like terms.)

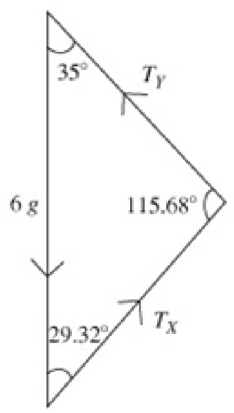
so

$$\tan\theta = \frac{\sin 80^\circ\sin 115^\circ}{2\sin 35^\circ + \sin 80^\circ\cos 115^\circ} = 1.22$$

$$\theta = 50.68^\circ$$

$\Rightarrow$ the other string is at $(90^\circ - 10^\circ - 50.68^\circ) = 29.3^\circ$ to the vertical

b Triangle of forces:



Using the sine rule:

$$\frac{T_X}{\sin 35^\circ} = \frac{T_Y}{\sin 29.32^\circ} = \frac{6g}{\sin 115.68^\circ}$$

$$T_X = \frac{6g\sin 35^\circ}{\sin 115.68^\circ} = 37.4 \text{ N and}$$

$$T_Y = \frac{6g\sin 29.32^\circ}{\sin 115.68^\circ} = 31.9 \text{ N}$$

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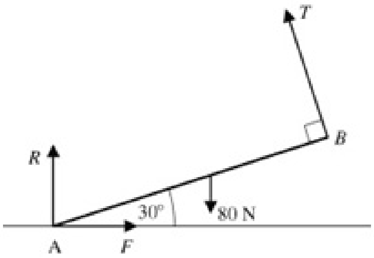
Exercise D, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8\text{ m s}^{-2}$.

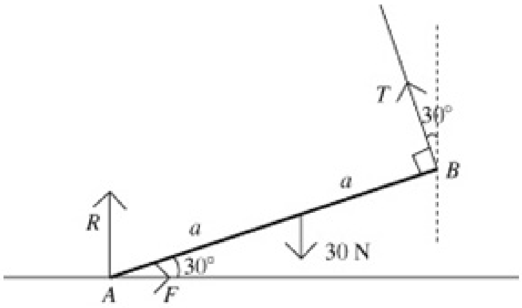
A uniform rod AB of weight 80 N rests with its lower end A on a rough horizontal floor. A string attached to end B keeps the rod in equilibrium.

The string is held at $90°$ to the rod. The tension in the string is T. The coefficient of friction between the rod and the ground is μ. R is the normal reaction at A and F is the frictional force at A.



Find the magnitudes of T, R and F, and the least possible value of μ.

Solution:



Suppose that the rod has length $2a$.

M(A) $\quad 2aT = 80 \times a \times \cos 30°$

Divide by a and substitute for $\cos 30°$

$2T = 80 \times \frac{\sqrt{3}}{2}$, $T = 20\sqrt{3}$ N

R($\rightarrow$) $\quad T\cos 60° \quad = F = 10\sqrt{3}$ N

R($\uparrow$) $\quad T\cos 30° + R = 80$

$R = 80 - T\cos 30° = 80 - 10\sqrt{3} \times \frac{\sqrt{3}}{2}$

$\quad = 65$ N

$F \le \mu R \Rightarrow 10\sqrt{3} \le \mu \times 65$

$\Rightarrow \mu \ge \frac{10\sqrt{3}}{65} = \frac{2\sqrt{3}}{13}$

$\Rightarrow$ minimum $\mu = 0.27$ (2 s.f.)

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Exercise D, Question 2

Question:

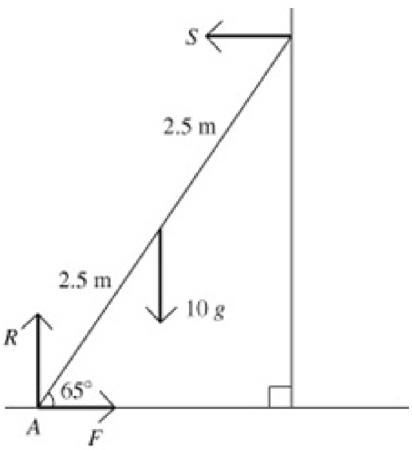
A uniform ladder of mass 10 kg and length 5 m rests against a smooth vertical wall with its lower end on rough horizontal ground. The ladder rests in equilibrium at an angle of 65° to the horizontal. Find

a the magnitude of the normal contact force S at the wall,

b the magnitude of the normal contact force R at the ground and the frictional force at the ground,

c the least possible value of the coefficient of friction between the ladder and the ground.

Solution:



Let A be the end of the ladder on the ground.
F is the frictional force at A

a

$$M(A)\ 10g \times 2.5\cos 65^\circ = S \times 5\sin 65^\circ$$

$$S = \frac{25g\cos 65^\circ}{5\sin 65^\circ} = 5g\frac{\cos 65^\circ}{\sin 65^\circ} = 22.8\text{ N}$$

b

$$\text{R}(\rightarrow)\ F = S = 22.8\text{ N}$$
$$\text{R}(\uparrow)\ R = 10g = 98\text{ N}$$

c

$$F \le \mu R \Rightarrow 22.8 \le \mu \times 98$$
$$\Rightarrow \mu \ge 0.233\ (3\text{ s.f.})$$

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Exercise D, Question 3

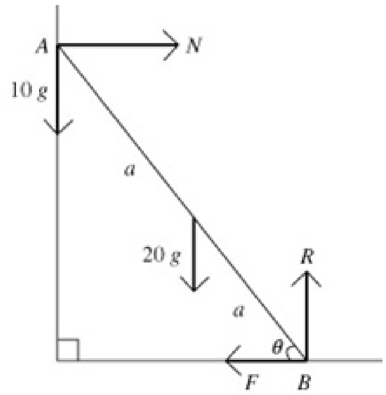
Question:

A uniform ladder AB of mass 20 kg rests with its top A against a smooth vertical wall and its base B on rough horizontal ground. The coefficient of friction between the ladder and the ground is $\frac{3}{4}$. A mass of 10 kg is attached to the ladder. Given that the ladder is about to slip, find the inclination of the ladder to the horizontal

a if the 10 kg mass is attached at A,

b if the 10 kg mass is attached at B.

Solution:



Let the ladder have length $2a$, and be inclined at θ to the horizontal.

a

R(↑) $R = 30g$

$M(A)$ $20g \times a\cos\theta + F \times 2a\sin\theta = R \times 2a\cos\theta$

Dividing through by a and substituting for R

$\Rightarrow 20g\cos\theta + 2F\sin\theta = 60g\cos\theta$

$2F\sin\theta = 40g\cos\theta$

$F = \dfrac{20g\cos\theta}{\sin\theta}$

N is the normal reaction at A,

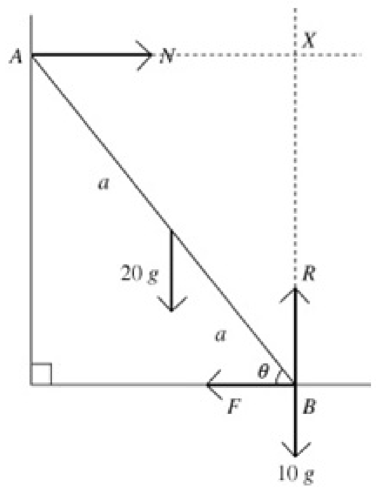
R is the normal reaction at B, and

F is the frictional force at B.

$F = \mu R \Rightarrow \dfrac{20g}{\tan\theta} = \frac{3}{4} \times 30g$

$\tan\theta = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$, $\theta = 41.6°$

Moving the 10 kg mass to B:



b

R(↑) $R = 30g$

X is the point where the lines of action of N and R meet.

M(X) $20g \times a\cos\theta = F \times 2a\sin\theta$

Dividing through by a and rearranging

$\Rightarrow \quad F = \dfrac{20g\cos\theta}{2\sin\theta} = \dfrac{10g}{\tan\theta}$

$F = \mu R \Rightarrow \dfrac{10g}{\tan\theta} = \frac{3}{4} \times 30g$

$\tan\theta = \frac{4}{9}$, $\theta = 24.0°$

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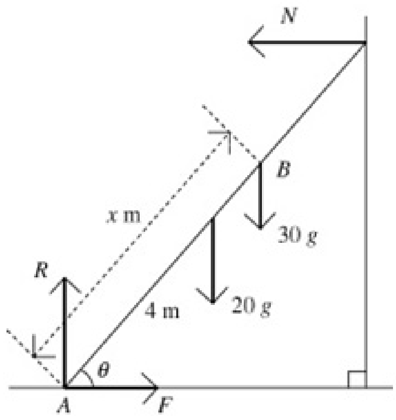
Exercise D, Question 4

Question:

A uniform ladder of mass 20 kg and length 8 m rests against a smooth vertical wall with its lower end on rough horizontal ground. The coefficient of friction between the ground and the ladder is 0.3. The ladder is inclined at an angle θ to the horizontal, where $\tan\theta = 2$.

A boy of mass 30 kg climbs up the ladder. Find how far up the ladder he can climb without it slipping.

Solution:



Suppose that the boy reaches the point B, distance x from A, the end of the ladder in contact with the ground.

When the ladder is in limiting equilibrium,

$\mathrm{F} = \mu R \Rightarrow F = 0.3R$

$\mathrm{R}(\rightarrow) \quad F = N, \mathrm{R}(\uparrow) \quad R = 50g$

$M(A) \quad 20g \times 4\cos\theta + 30g \times x\cos\theta = N \times 8\sin\theta$

Dividing through by $8\sin\theta$:

$$N\tan\theta = \frac{80g + 30gx}{8}$$

Substituting for $\tan\theta$:

$$N = \frac{80g + 30gx}{16} = F$$

$$\mu = 0.3 \Rightarrow \frac{80g + 30gx}{16} = 0.3 \times 50g$$

$$8 + 3x = 1.5 \times 16 = 24$$

$$3x = 16, x = 5\frac{1}{3}\text{ m}$$

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Exercise D, Question 5

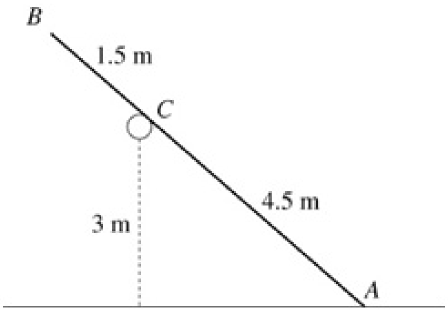
Question:

A smooth horizontal rail is fixed at a height of 3 m above a rough horizontal surface. A uniform pole AB of weight 4 N and length 6 m is resting with end A on the rough ground and touching the rail at point C.

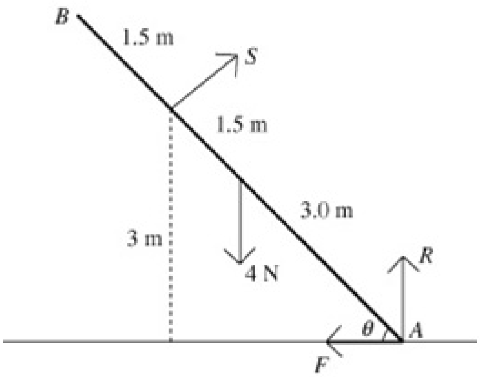
The vertical plane containing the pole is perpendicular to the rail. The distance AC is 4.5 m and the pole is in limiting equilibrium. Calculate

a the magnitude of the force exerted by the rail on the pole,

b the coefficient of friction between the pole and the ground.



Solution:



S is the normal reaction at C,
R is the normal reaction at A, and
F is the friction at A.
θ is the angle between the pole and the ground.

a

$M(A) \quad 4.5S = 4\times 3\cos\theta$

From the diagram, $\sin\theta = \frac{3}{4.5} = \frac{2}{3}$,

$\Rightarrow \cos\theta = \frac{\sqrt{9-4}}{3} = \frac{\sqrt{5}}{3}$

$4.5S = \frac{12\sqrt{5}}{3} = 4\sqrt{5}, S = \frac{8\sqrt{5}}{9}$ N

b

$R(\rightarrow) \quad F = S\sin\theta = \frac{8\sqrt{5}}{9}\times\frac{2}{3} = \frac{16\sqrt{5}}{27}$

$R(\uparrow) \quad R + S\cos\theta = 4$

$R \quad = 4 - \frac{8\sqrt{5}}{9}\times\frac{\sqrt{5}}{3} = 4 - \frac{40}{27} = \frac{68}{27}$

$F \quad = \mu R \Rightarrow \frac{16\sqrt{5}}{27} = \mu\times\frac{68}{27}$

$\mu \quad = \frac{16\sqrt{5}}{68} = \frac{4\sqrt{5}}{17} \approx 0.526$ (3 s.f.)

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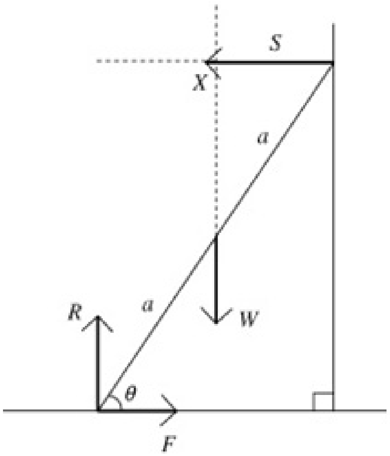
Exercise D, Question 6

Question:

A uniform ladder rests in limiting equilibrium with its top against a smooth vertical wall and its base on a rough horizontal floor. The coefficient of friction between the ladder and the floor is μ. Given that the ladder makes an angle θ with the floor, show that $2\mu \tan\theta = 1$.

Solution:



Suppose that the ladder has length $2a$ and weight W.
S is the normal reaction at the wall, R is the normal reaction at the floor, and F is the friction at the floor.
X is the point where the lines of action of W and S meet.

$$M(X)\ 2a\sin\theta \times F = R \times a\cos\theta,$$

$$2F\sin\theta = R\cos\theta$$

The ladder is in limiting equilibrium, so

$$F = \mu R$$

$$\Rightarrow 2\mu R\sin\theta = R\cos\theta$$

$$2\mu\sin\theta = \cos\theta$$

$$\frac{2\mu\sin\theta}{\cos\theta} = 1$$

$$2\mu\tan\theta = 1$$

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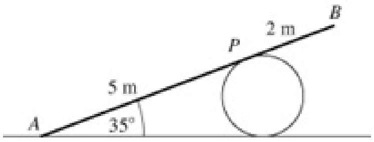
Exercise D, Question 7

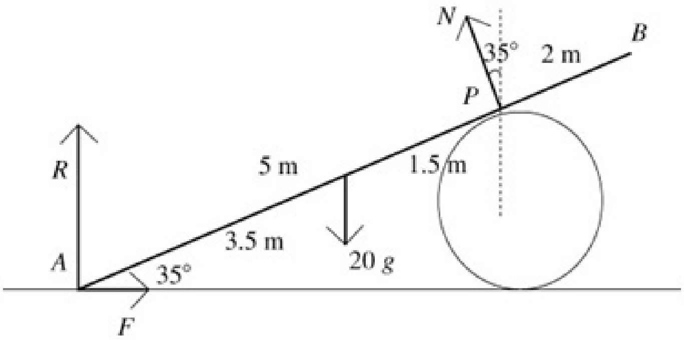
Question:

A uniform ladder AB has length 7 m and mass 20 kg. The ladder is resting against a smooth cylindrical drum at P, where AP is 5 m, with end A in contact with rough horizontal ground. The ladder is inclined at $35°$ to the horizontal.



Find the normal and frictional components of the contact force at A, and hence find the least possible value of the coefficient of friction between the ladder and the ground.

Solution:



N is the normal reaction at P, R is the normal reaction at A and F is the friction at A.

$M(A)\quad 20g\times 3.5\cos 35^\circ = 5N$

$\Rightarrow N = \dfrac{20g\times 3.5\cos 35^\circ}{5} = 14g\cos 35^\circ$

$R(\uparrow)\quad N\cos 35^\circ + R = 20g$

$R = 20g - 14g\cos 35^\circ \times \cos 35^\circ = 103.9\ldots\text{ N}$

$R(\rightarrow)\quad F = N\sin 35^\circ = 14g\cos 35^\circ\sin 35^\circ = 64.46\ldots\text{ N}$

$F \leq \mu R$

$\Rightarrow 14g\cos 35^\circ\sin 35^\circ \leq \mu(20g - 14g\cos^2 35^\circ)$

$\Rightarrow \mu \geq \dfrac{14\cos 35^\circ\sin 35^\circ}{20 - 14\cos^2 35^\circ}$

$\mu \geq 0.620\ (3\text{ s.f.})$

Least possible value is 0.620 (3 s.f.).

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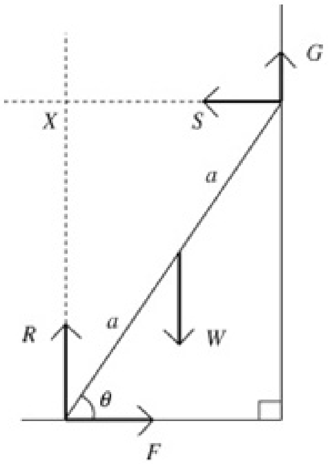
Exercise D, Question 8

Question:

A uniform ladder rests in limiting equilibrium with one end on rough horizontal ground and the other end against a rough vertical wall. The coefficient of friction between the ladder and the ground is μ_1 and the coefficient of friction between the ladder and the wall is μ_2. Given that the ladder makes an angle θ with the horizontal, show that $\tan\theta = \dfrac{1-\mu_1\mu_2}{2\mu_1}$.

Solution:



Suppose that the ladder has length $2a$ and weight W.

As the ladder rests in limiting equilibrium,
$F = \mu_1 R$ and $G = \mu_2 S$.
$\mathrm{M}(X)\; W \times a\cos\theta = F \times 2a\sin\theta + G \times 2a\cos\theta$
Dividing by $a\cos\theta$ gives
$W = 2F\tan\theta + 2G$

$\mathrm{R}(\rightarrow)\; F = S$
$\mathrm{R}(\uparrow)\; W = R + G$

Substituting for W and F in the moments equation:

R and F are the normal reaction and the friction where the ladder is in contact with the ground. S and G are the normal reaction and the friction at the wall.

X is the point where the lines of action of R and S meet.

$R + G = 2\mu_1 R\tan\theta + 2G$
$\Rightarrow R - G = 2\mu_1 R\tan\theta$
But $G = \mu_2 S = \mu_2 F = \mu_2\mu_1 R$ so
$R - \mu_1\mu_2 R = 2\mu_1 R\tan\theta$
Hence $\dfrac{1-\mu_1\mu_2}{2\mu_1} = \tan\theta$

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Exercise D, Question 9

Question:

A uniform ladder of weight W rests in equilibrium with one end on rough horizontal ground and the other resting against a smooth vertical wall. The vertical plane containing the ladder is at right angles to the wall and the ladder is inclined at $60°$ to the horizontal. The coefficient of friction between the ladder and the ground is μ.

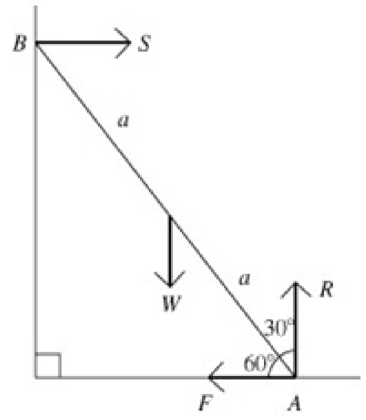
a Find, in terms of W, the magnitude of the force exerted by the wall on the ladder.

b Show that $\mu \geq \frac{1}{6}\sqrt{3}$.

A load of weight w is attached to the ladder at its upper end (resting against the wall).

c Given that $\mu = \frac{1}{5}\sqrt{3}$ and that the equilibrium is limiting, find w in terms of W.

Solution:



Let A and B be the ends of the ladder.
S is the normal reaction at B, R the normal reaction at A and F the friction at B.
The length of the ladder is $2a$.

a

$M(A) \quad W \times a\cos 60^\circ = S \times 2a\cos 30^\circ$

$$S = \frac{Wa\cos 60^\circ}{2a\cos 30^\circ} = \frac{W \times \frac{1}{2}}{2 \times \frac{\sqrt{3}}{2}} = \frac{W}{2\sqrt{3}}$$

b

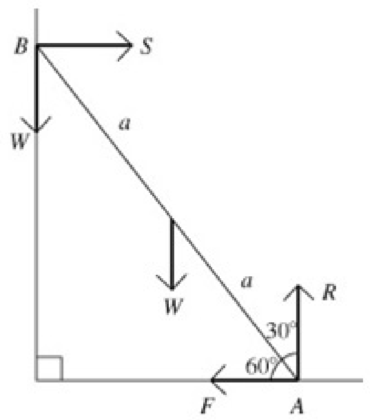
$\mathrm{R}(\rightarrow) \quad F = S = \dfrac{W}{2\sqrt{3}}, \mathrm{R}(\uparrow) \quad R = W$

The ladder is in equilibrium so $F \le \mu R$,

$$\Rightarrow \frac{W}{2\sqrt{3}} \le \mu W, \mu \ge \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{2 \times 3} = \frac{\sqrt{3}}{6}$$

$\mu \ge \dfrac{\sqrt{3}}{6}$, as required.

c



Adding the load at the top of the ladder and using the limiting equilibrium:

$\mathrm{R}(\rightarrow)\ S = F = \frac{\sqrt{3}R}{5}, \mathrm{R}(\uparrow)\ w + W = R$

$M(B)\ W \times a\cos 60^\circ + F \times 2a\sin 60^\circ = R \times 2a\cos 60^\circ$

i.e. $Wa \times \frac{1}{2} + 2aF \times \frac{\sqrt{3}}{2} = 2aR \times \frac{1}{2}$

Dividing by $\frac{a}{2}$ and substituting for F:

$$W + 2\sqrt{3} \times \frac{\sqrt{3}R}{5} = 2R$$

$$W = 2R - \frac{6}{5}R = \frac{4}{5}R = \frac{4}{5}(w + W)$$

$$\Rightarrow W - \frac{4}{5}W = \frac{4}{5}w, w = \frac{W}{4}$$

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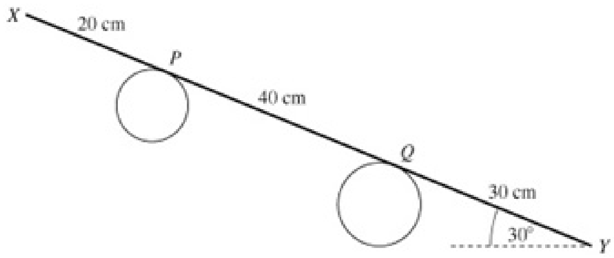
Exercise D, Question 10

Question:



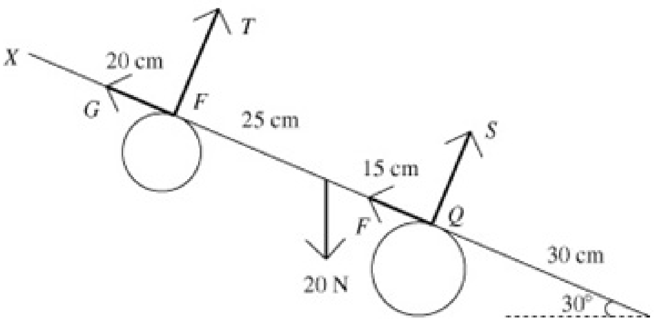
A uniform rod XY has weight 20 N and length 90 cm. The rod rests on two parallel pegs, with X above Y, in a vertical plane which is perpendicular to the axes of the pegs, as shown in the diagram. The rod makes an angle of 30° to the horizontal and touches the two pegs at P and Q, where $XP = 20$ cm and $XQ = 60$ cm.

a Calculate the normal components of the forces on the rod at P and at Q.

The coefficient of friction between the rod and each peg is μ.

b Given that the rod is about to slip, find μ.

Solution:



T is the normal force at P,
G is the friction at P,
S is the normal force at Q,
F is the friction at Q.

a

$\text{M}(P)\ S \times 4 = 20 \times 2.5 \times \cos 30°$

$$S = \frac{20 \times 2.5 \times \frac{\sqrt{3}}{2}}{4} = \frac{25\sqrt{3}}{4} \text{ N},$$

$\text{M}(Q)\ T \times 4 = 20 \times 1.5 \times \cos 30°$

$$T = \frac{20 \times 1.5 \times \frac{\sqrt{3}}{2}}{4} = \frac{15\sqrt{3}}{4} \text{ N}$$

b

$\text{R}(\searrow)\ G + F = 20 \cos 60° = 10$

$$= \mu T + \mu S = \mu \times \frac{40\sqrt{3}}{4} = 10\sqrt{3}\mu$$

as the rod is about to slip.

$$\Rightarrow \mu = \frac{1}{\sqrt{3}}$$

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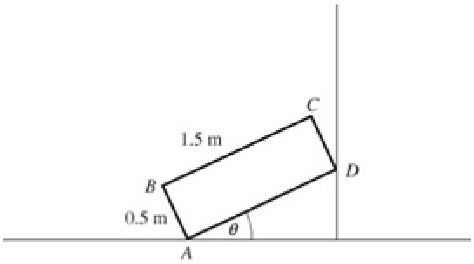
Exercise D, Question 11

Question:

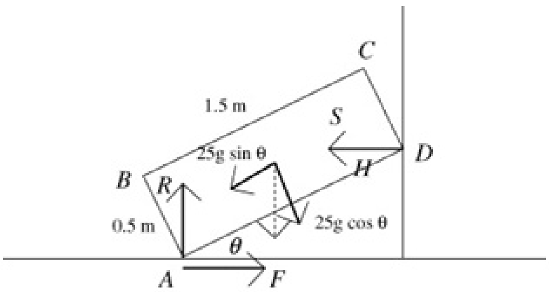
The diagram shows the vertical cross section $ABCD$ through the centre of mass of a uniform rectangular box. The box is resting on a rough horizontal floor and leaning against a smooth vertical wall.



The box has mass 25 kg. $AB = 0.5\text{ m}$, $BC = 1.5\text{ m}$ and AD is at an angle of θ to the horizontal. The coefficient of friction between the box and the ground is $\frac{1}{4}$.

Given that the box is about to slip, find the value of θ.

Solution:



R and S are the normal reactions at A and D respectively. F is the friction at A.

As the box is about to slip, $F = \frac{1}{4}R$.

In order to take moments about A, resolve the weight into components $25g\cos\theta$

R and S are the normal reactions at A and parallel to AB and $25g\sin\theta$ parallel to AD.

$$M(A) \quad S \times 1.5\sin\theta + 0.25 \times 25g\sin\theta = 25g\cos\theta \times 0.75$$

$$S = \frac{25g\cos\theta \times 0.75 - 25g\sin\theta \times 0.25}{1.5\sin\theta}$$

$$= \frac{25g(3\cos\theta - \sin\theta)}{6\sin\theta}$$

$$\text{R}(\rightarrow) \quad F = S = \frac{25g(3\cos\theta - \sin\theta)}{6\sin\theta}$$

$$\text{R}(\uparrow) \quad R = 25g$$

$$F = \frac{1}{4}R \Rightarrow \frac{3\cos\theta - \sin\theta}{6\sin\theta} = \frac{1}{4}$$

so

$$12\cos\theta - 4\sin\theta = 6\sin\theta$$

$$12\cos\theta = 10\sin\theta$$

$$\tan\theta = \frac{12}{10} = \frac{6}{5}$$

$$\theta = 50.2^\circ$$

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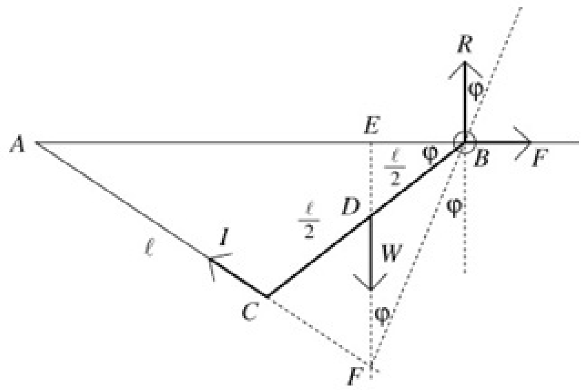
Exercise D, Question 12

Question:

A uniform rod of length l has a ring at one end which can slide along a rough horizontal pole. The coefficient of friction between the ring and the pole is 0.2. The other end of the rod is attached to the end of the pole by a light inextensible cord of length l. The rod rests in equilibrium at an angle of θ to the horizontal. Using a geometrical method, or otherwise, find the smallest possible value of θ.

Solution:



The diagram of forces has four forces acting, the weight, W, the tension, T, in the cord, the normal reaction, R, and the friction, F, at the ring end. Combining R and F to give a single contact force at the ring reduces the problem to one of an object in equilibrium with three forces acting. These must be concurrent.

Triangle ABC is isosceles (2 sides of length l).

Equilibrium $F \le \mu R, 0.2 \ge \frac{F}{R}$, but

$\frac{F}{R} = \tan\varphi \Rightarrow 0.2 \ge \tan\varphi = \frac{EB}{EF}$

$$\text{Triangle } EBD \Rightarrow EB = \frac{l}{2}\cos\theta$$

$$\text{Triangle } AEF \Rightarrow EF = \frac{3l}{2}\sin\theta$$

$$\Rightarrow \frac{EB}{EF} = \frac{\cos\theta}{3\sin\theta} = \frac{1}{3\tan\theta} \le 0.2$$

$$\Rightarrow \tan\theta \ge \frac{1}{3\times 0.2} = \frac{5}{3}$$

$$\Rightarrow \theta \ge 59.0^\circ$$

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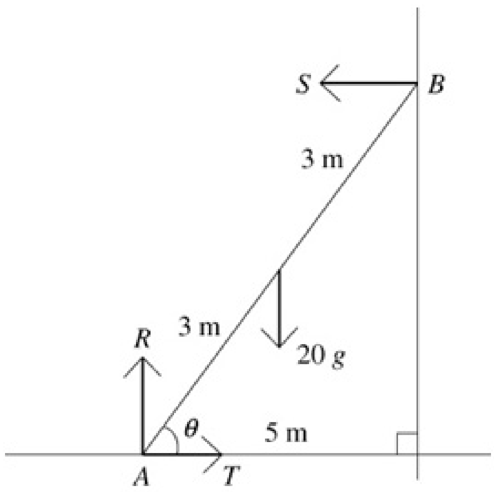
Exercise E, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8\text{ m s}^{-2}$.

A uniform ladder of mass 20 kg and length 6 m rests with one end on a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is held in this position by a light inextensible rope of length 5 m which has one end attached to the bottom of the ladder and the other end fastened to a point at the base of the wall, vertically below the top of the ladder. Find the tension in the rope.

Solution:



Let A and B be the ends of the ladder.
R is the normal reaction at A,
S is the normal reaction at B (both smooth surfaces).
T is the tension in the rope.
The angle between the ladder and the ground is θ.

$\text{R}(\rightarrow)\quad R = 20g$

$\text{R}(\uparrow)\quad S = T$

$$\text{M}(A)\; 20g \times 3 \times \cos\theta = S \times 6 \times \sin\theta$$

$$\text{so } 20g \times 3 \times \frac{5}{6} = S \times 6 \times \frac{\sqrt{6^2 - 5^2}}{6}$$

$$50g = \sqrt{11}S$$

$$T = S = \frac{50g}{\sqrt{11}} = 150\text{ N (2 s.f.)}$$

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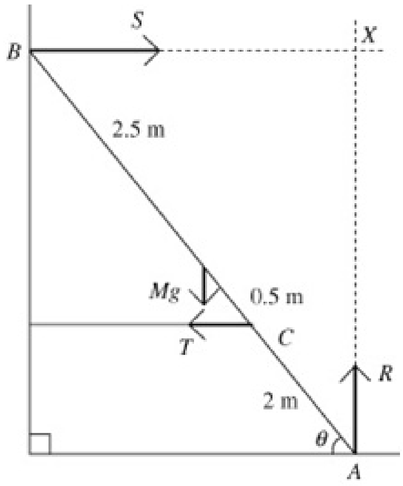
Exercise E, Question 2

Question:

A uniform ladder AB of mass M kg and length 5 m rests with end A on a smooth horizontal floor and end B against a smooth vertical wall. The ladder is held in equilibrium at an angle θ to the floor by a light horizontal string attached to the wall and to a point C on the ladder. If $\tan\theta = 2$, find the tension in the string when the length AC is 2 m.

Solution:



Given $\tan\theta = 2$.
The normal reactions at A and B are R and S respectively.
X is the point where the times of action of R and S meet.
T is the tension in the string.

M(X) $\quad Mg \times 2.5 \times \cos\theta = T \times 3 \times \sin\theta$
(Note that, by taking moments about X we do not need to find out anything about R and S.)
Dividing by $\cos\theta$:

$$\frac{5Mg}{2} = \frac{3T\sin\theta}{\cos\theta} = 3T\tan\theta = 6T$$

Therefore $T = \dfrac{5Mg}{12}$

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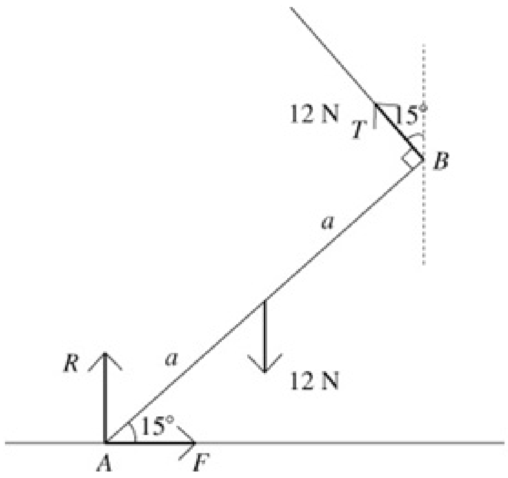
Exercise E, Question 3

Question:

A uniform pole AB of weight 12 N has its lower end A on rough horizontal ground. The pole is being raised into a vertical position by a rope attached to B. The rope and the pole lie in the same vertical plane and A does not slip across the ground. Find the horizontal and vertical components of the reaction at the ground when the rope is perpendicular to the pole and the pole is at $15°$ to the horizontal.

Solution:



Let the length of the pole be $2a$.
T is the tension in the string.
R is the normal reaction at A, and
F is the force due to friction at A.

$$\text{M}(A)\ 12\times a\times\cos 15° = T\times 2a$$
$$\Rightarrow T = 6\cos 15°$$
$$\text{R}(\rightarrow)\quad F = T\sin 15° = 6\cos 15°\sin 15° = 1.5\text{ N}$$
$$\text{R}(\uparrow)\quad R + T\cos 15° = 12$$
$$R = 12 - 6\cos 15°\times\cos 15° = 6.4\text{ N}\,(2\text{ s.f.})$$

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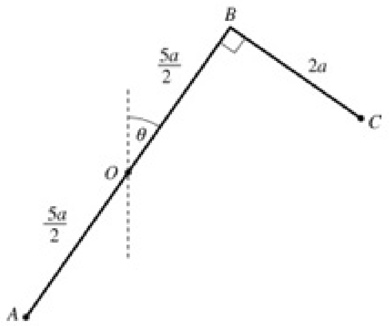
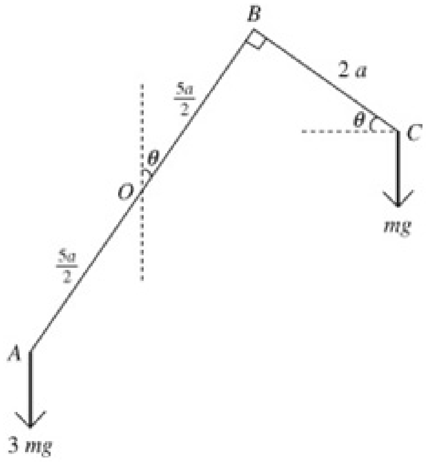
Exercise E, Question 4

Question:

AB is a light rod of length $5a$ rigidly joined to a light rod BC of length $2a$ so that the rods are perpendicular to each other and in the same vertical plane, as shown in the diagram. The centre, O, of AB is fixed and the rods can rotate freely about O in a vertical plane. A particle of mass $3m$ is attached at A and a particle of mass m is attached at C. The system rests in equilibrium with AB inclined at an acute angle θ to the vertical. Find the value of θ.



Solution:



There will be a reaction at O, but if we take moments about O we will not need to know anything about this force.

$\text{M}(O)\,3mg\times\frac{5a}{2}\sin\theta = mg\times(\frac{5a}{2}\sin\theta+2a\cos\theta)$

Dividing by mga:

$$\frac{15\sin\theta}{2}=\frac{5\sin\theta}{2}+2\cos\theta$$

$\Rightarrow 5\sin\theta = 2\cos\theta,\ \tan\theta=\frac{2}{5}$

$\Rightarrow\theta\ =\tan^{-1}0.4=21.8^\circ$

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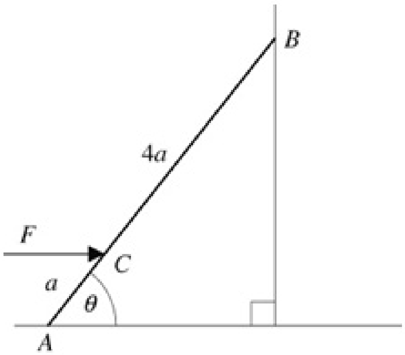
Exercise E, Question 5

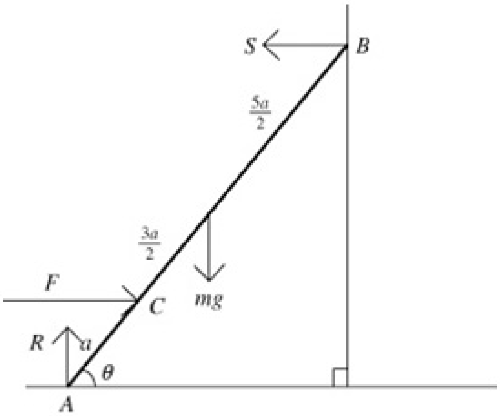
Question:

A uniform ladder AB has one end A on smooth horizontal ground. The other end B rests against a smooth vertical wall. The ladder is modelled as a uniform rod of mass m and length $5a$. The ladder is kept in equilibrium by a horizontal force F acting at a point C of the ladder where $AC = a$. The force F and the ladder lie in a vertical plane perpendicular to the wall. The ladder is inclined to the horizontal at an angle θ, where $\tan\theta = 1.8$, as shown in the diagram.



Show that $F = \dfrac{25mg}{32}$.

Solution:



Let S be the force between the ladder and the wall at B.
Let R be the normal reaction at A. (Both surfaces are smooth, so no friction.)

R($\rightarrow$) $\quad F = S$

M(A) $\quad mg \times \dfrac{5a}{2} \times \cos\theta + F \times a \times \sin\theta = S \times 5a \times \sin\theta$

Dividing through by $a\cos\theta$ gives

$$\frac{5mg}{2} + F\tan\theta = 5S = 5F,$$

so $\dfrac{5mg}{2} = 5F - 1.8F = 3.2F$ and

$$F = \frac{5mg}{2\times 3.2} = \frac{50mg}{64} = \frac{25mg}{32}$$ as required.

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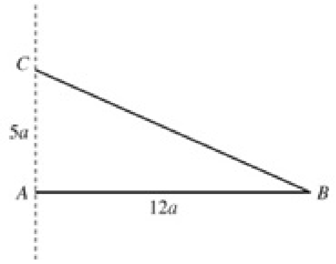
Exercise E, Question 6

Question:

A uniform rod AB, of length $12a$ and weight W, is free to rotate in a vertical plane about a smooth pivot at A.

One end of a light inextensible string is attached to B.

The other end is attached to point C which is vertically above A, with $AC = 5a$. The rod is in equilibrium with AB horizontal, as shown in the diagram.

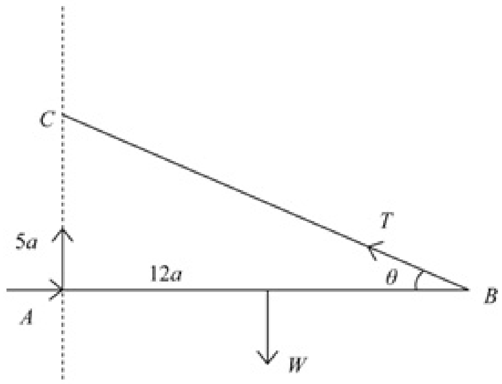


Find, in terms of W,

a the tension in the string,

b the magnitude of the horizontal component of the force exerted by the pivot on the rod.

Solution:



Let the angle between the string and the rod be θ, T be the tension in the string, X the horizontal component of the force at A and Y the vertical component of the force at A.

a

M(A) $W \times 6a = T \sin\theta \times 12a = T \times \frac{5}{13} \times 12a$

(since ABC is a 5, 12, 13 triangle)

Giving $T = \frac{W \times 6a \times 13}{5 \times 12a} = \frac{13W}{10}$

b

R($\rightarrow$) $X = T\cos\theta = \frac{13W}{10} \times \frac{12}{13} = \frac{6W}{5}$

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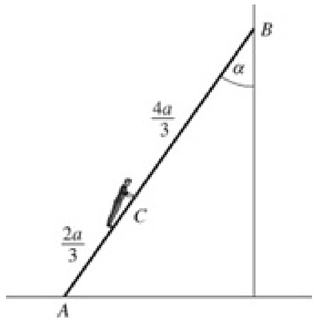
Exercise E, Question 7

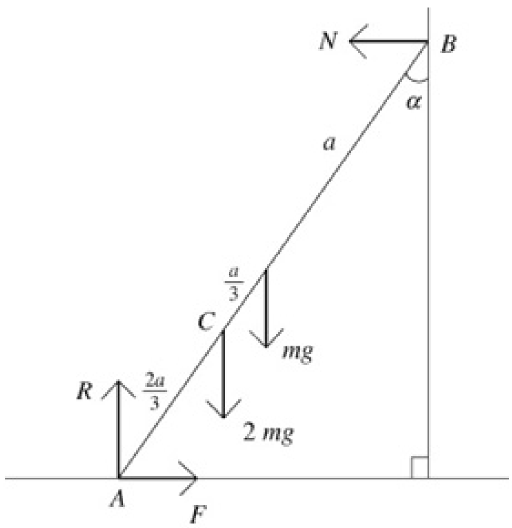
Question:

A uniform ladder AB, of mass m and length $2a$, has one end A on rough horizontal ground. The other end B rests against a smooth vertical wall. The ladder is in a vertical plane perpendicular to the wall. The ladder makes an angle α with the vertical, where $\tan\alpha = \frac{3}{4}$. A child of mass $2m$ stands on the ladder at C where $AC = \frac{2}{3}a$, as shown in the diagram. The ladder and the child are in equilibrium.



By modelling the ladder as a rod and the child as a particle, calculate the least possible value of the coefficient of friction between the ladder and the ground.

Solution:



Let N be the force between the ladder and the wall at B – perpendicular to the wall since no friction here. Let R be the normal reaction at A, and F the friction at A.

R(↑) $\quad R = 3mg$

M(B) $mga\sin\alpha + 2mg \times \frac{4a}{3} \times \sin\alpha + F \times 2a\cos\alpha$

$= R \times 2a\sin\alpha$

Substituting for R, $\sin\alpha$ and $\cos\alpha$:

$mga \times \frac{3}{5} + \frac{8mga}{3} \times \frac{3}{5} + F \times 2a \times \frac{4}{5} = 6mga \times \frac{3}{5}$

$F \times \frac{8a}{5} = \frac{18mga}{5} - \frac{8mga}{5} - \frac{3mga}{5} = \frac{7mga}{5}$

The ladder and the child are in equilibrium,

so $F = \frac{7mga}{5} \times \frac{5}{8a} = \frac{7mg}{8} \le \mu R$

$\Rightarrow \frac{7mg}{8} \le \mu \times 3mg, \mu \ge \frac{7}{24}$,

the least possible value for μ is $\frac{7}{24}$

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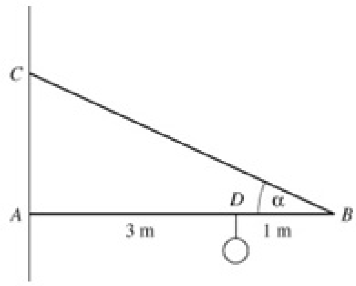
Exercise E, Question 8

Question:

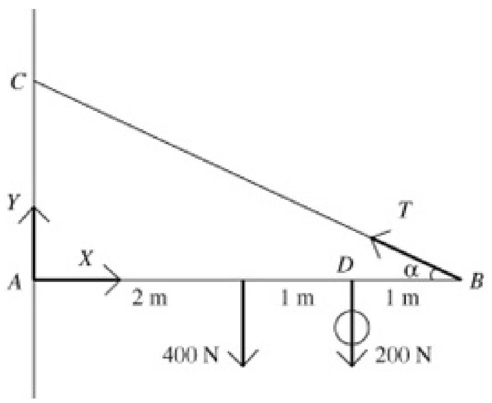
A uniform steel girder AB of weight 400 N and length 4 m, is freely hinged at A to a vertical wall. The girder is supported in a horizontal position by a steel cable attached to the girder at B. The other end of the cable is attached to the point C vertically above A on the wall, with $\angle ABC = \alpha$ where $\tan\alpha = \frac{1}{2}$. A load of weight 200 N is suspended by another cable from the girder at the point D, where $AD = 3$ m, as shown in the diagram. The girder remains horizontal and in equilibrium. The girder is modelled as a rod, and the cables as light inextensible strings.



a Show that the tension in the cable BC is $350\sqrt{5}$ N.

b Find the magnitude of the reaction on the girder at A.

Solution:



Let the tension in cable BC be T. The horizontal and vertical components of the reaction at A are X and Y respectively.

$\tan\alpha = \frac{1}{2} \Rightarrow \sin\alpha = \frac{1}{\sqrt{5}}$

a

M(A) $400\times 2 + 200\times 3 = T\sin\alpha\times 4$

$1400 = 4T\times\frac{1}{\sqrt{5}}, T = 350\sqrt{5}$ N

b

R($\rightarrow$) $X = T\cos\alpha = 350\sqrt{5}\times\frac{2}{\sqrt{5}} = 700$ N

R($\uparrow$) $Y = 600 - T\sin\alpha = 600 - 350\sqrt{5}\times\frac{1}{\sqrt{5}}$

$= 250$ N

Magnitude of the reaction

$= \sqrt{700^2 + 250^2} = \sqrt{552\,500} = 743$ N (3 s.f.)

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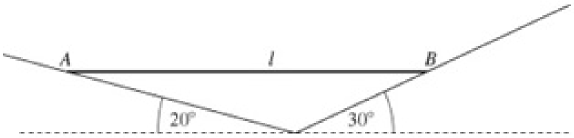
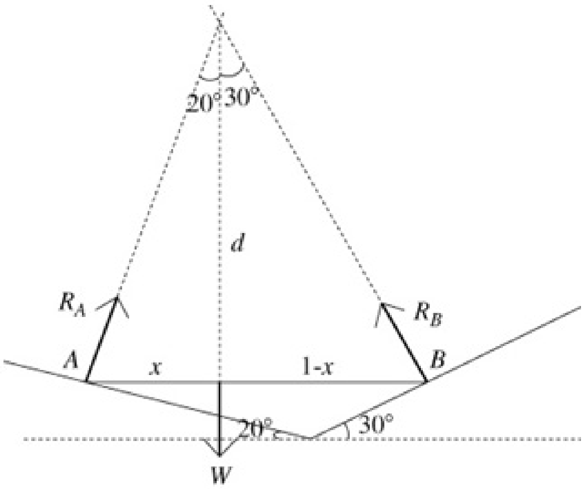
Exercise E, Question 9

Question:

A non-uniform rod AB of length l rests horizontally with its ends resting on two smooth surfaces inclined at $20°$ and $30°$ to the horizontal, as shown in the diagram. Use a geometrical method to find the distance of the centre of mass from A.



Solution:



Suppose that the centre of mass is distance x from A.
There are three forces acting, the weight and the two normal reactions, so their lines of action must be concurrent.
The length d lies in 2 separate right angled triangles:

$$\frac{x}{d} = \tan 20° \text{ and } \frac{l - x}{d} = \tan 30°$$

$$\Rightarrow \frac{x}{l - x} = \frac{\tan 20°}{\tan 30°}$$

$$x(\tan 30° + \tan 20°) = l \tan 20°$$

$$x = \frac{l \tan 20°}{\tan 30° + \tan 20°} \approx 0.39l \text{ (2 s.f.)}$$

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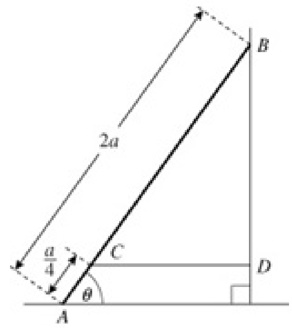
Exercise E, Question 10

Question:



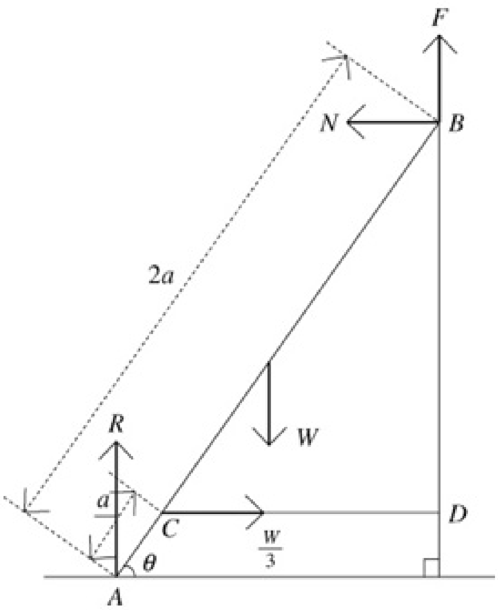
A uniform ladder, of weight W and length $2a$, rests in equilibrium with one end A on a smooth horizontal floor and the other end B against a rough vertical wall. The ladder is in a vertical plane perpendicular to the wall. The coefficient of friction between the wall and the ladder is μ. The ladder makes an angle θ with the floor, where $\tan\theta = \frac{4}{3}$. A horizontal light inextensible string CD is attached to the ladder at the point C, where $AC = \frac{1}{4}a$. The string is attached to the wall at the point D, with BD vertical, as shown in the diagram. The tension in the string is $\frac{1}{3}W$. By modelling the ladder as a rod,

a find the magnitude of the force of the floor on the ladder,

b show that $\mu \geq \frac{1}{3}$.

c State how you have used the modelling assumption that the ladder is a rod.

Solution:



Let R be the normal reaction at A, N the normal reaction at B and F the friction at B.

a

M(B) $\quad \frac{W}{3}\times\frac{7a}{4}\times\sin\theta + W\times a\times\cos\theta = R\times 2a\times\cos\theta$

Dividing by $a\cos\theta$ gives

$$2R = \frac{7W}{12}\times\tan\theta + W = \frac{7}{12}\times\frac{4}{3}W + W = \frac{16W}{9}$$

So $R = \frac{8W}{9}$

b

R($\rightarrow$) $N = \frac{W}{3}$

R($\uparrow$) $R + F = W$, $F = W - R = \frac{W}{9}$

$$F \leq \mu N \Rightarrow \frac{W}{9} \leq \mu\frac{W}{3}, \mu \geq \frac{1}{3}.$$

c

The ladder has negligible thickness / the ladder does not bend.

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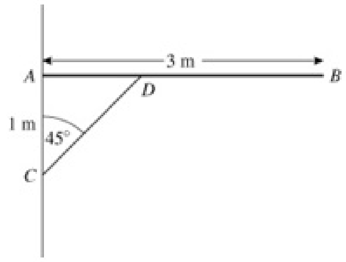
Exercise E, Question 11

Question:

A uniform pole AB of mass 40 kg and length 3 m, is smoothly hinged to a vertical wall at one end A. The pole is held in equilibrium in a horizontal position by a light rod CD. One end C of the rod is fixed to the wall vertically below A. The other end D is freely jointed to the pole so that $\angle ACD = 45°$ and $AC = 1\text{ m}$, as shown in the diagram. Find



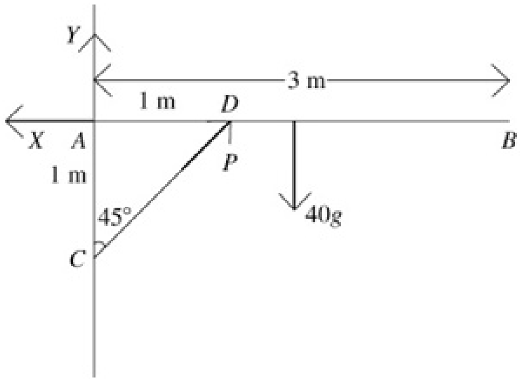
a the thrust in the rod CD,

b the magnitude of the force exerted by the wall on the pole at A.

The rod CD is removed and replaced by a longer light rod CM, where M is the midpoint of AB. The rod is freely jointed to the pole at M. The pole AB remains in equilibrium in a horizontal position.

c Show that the force exerted by the wall on the pole at A now acts horizontally.

Solution:



Let the horizontal and vertical components or the force at A be X and Y respectively.
Let the thrust in the rod be P.

a

$\text{M}(A) \quad 1 \times P \times \cos 45° = 40g \times \frac{3}{2}$

$P = \frac{60g}{\cos 45°} = 60\sqrt{2}g = 830\text{ N}(2\text{ s.f.})$

b

$\text{R}(\rightarrow) \quad X = P\cos 45° = 60g$

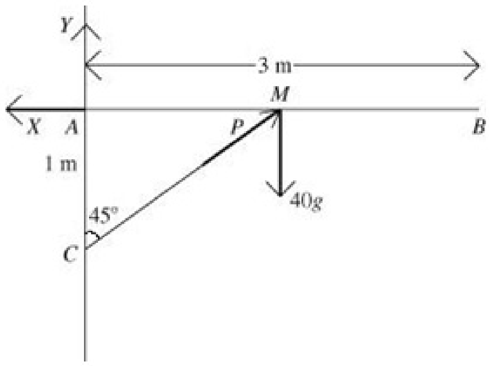
$\text{R}(\uparrow) \quad Y + P\cos 45° = 40g$

$Y = 40g - 60g = -20g$

$\text{resultant} = \sqrt{X^2 + Y^2} = 10g\sqrt{4^2 + 2^2} = 10g\sqrt{40}$

$= 620\text{ N}(2\text{ s.f.})$

c



The lines of action of P and the weight meet at M, hence the line of action of the resultant of X and Y must also pass through M (3 forces acting on a body in equilibrium). Therefore the reaction must act horizontally (i.e. no vertical component).

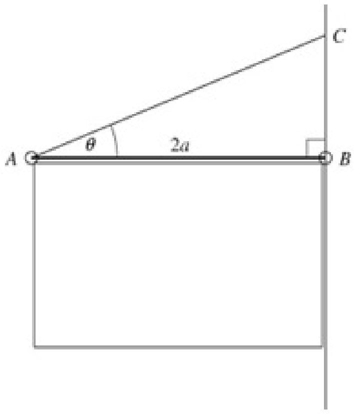
Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 12

Question:

A uniform rod AB, of weight W and length $2a$, is used to display a light banner. The rod is freely hinged to a vertical wall at point B. It is held in a horizontal position by a light wire attached to A and to a point C vertically above B on the wall. The angle CAB is θ, where $\tan\theta = \frac{1}{3}$.

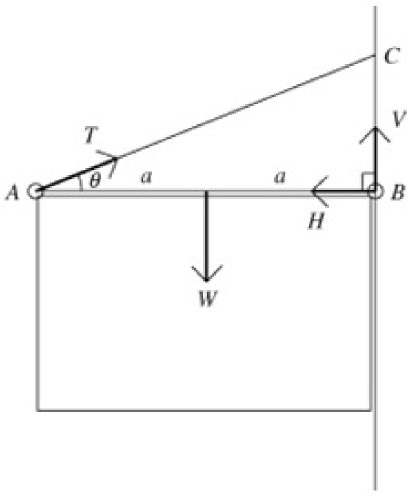


a Show that the tension in the wire is

$$\frac{W}{2\sin\theta}$$

b Find, in terms of W, the magnitude of the force exerted by the wall on the rod at B.

Solution:



T is the tension in the string, V and H are horizontal and vertical components of the force at B.

a

$$\text{M}(B) \quad a \times W = T\sin\theta \times 2a$$

$$\Rightarrow T = \frac{W}{2\sin\theta}$$

b

$$\text{R}(\rightarrow) \quad T\cos\theta = H$$

$$= \frac{W}{2\sin\theta} \times \cos\theta = \frac{W}{2\tan\theta} = \frac{3W}{2}$$

$$\text{R}(\uparrow) \quad T\sin\theta + V = W = \frac{W}{2} + V \Rightarrow V = \frac{W}{2}$$

so, using Pythagoras, the magnitude of the resultant force is

$$\frac{W}{2}\sqrt{1^2 + 3^2} = \frac{\sqrt{10}W}{2}.$$

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 13

Question:

A uniform ladder, of weight W and length 5 m, has one end on rough horizontal ground and the other touching a smooth vertical wall. The coefficient of friction between the ladder and the ground is 0.3.

Given that the top of the ladder touches the wall at a point 4 m vertically above the level of the ground,

a show that the ladder can not rest in equilibrium in this position.

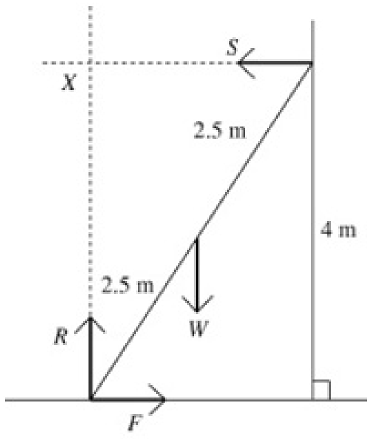
In order to enable the ladder to rest in equilibrium in the position described above, a brick is attached to the bottom of the ladder.

Assuming that this brick is at the lowest point of the ladder, but not touching the ground,

b show that the horizontal frictional force exerted by the ladder on the ground is independent of the mass of the brick,

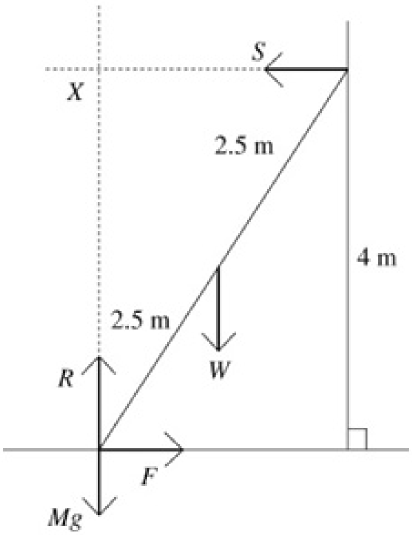
c find, in terms of W and g, the smallest mass of the brick for which the ladder will rest in equilibrium.

Solution:



a

R(↑) $F = S$, $R(\rightarrow)$ $R = W$
M(X) $1.5W = 4F$
$F \leq \mu R \Rightarrow \frac{1.5W}{4} \leq 0.3 \times R$
$\frac{3W}{8} \leq \frac{3W}{10} \Rightarrow \frac{3}{8} \leq \frac{3}{10}$,
which is false, therefore the assumption $F \leq \mu R$ must be false – the ladder can not be resting in equilibrium.



b

With the brick in place (second diagram).
M(X) $1.5W = 4F$ so
$F = \frac{1.5W}{4} = \frac{3W}{8}$, which is independent of M, the mass of the brick.

R(↑) $R = W + Mg$, R(→) $F = S$
$F \leq \mu R \Rightarrow \frac{3W}{8} \leq 0.3(W + Mg) = \frac{3(W + Mg)}{10}$
$\Rightarrow 10W \leq 8W + 8Mg$
$8Mg \geq 2W, M \geq \frac{W}{4g}$
So the smallest value for M is $\frac{W}{4g}$.

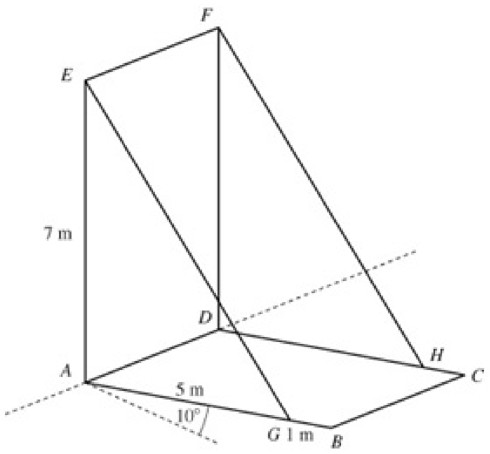
Solutionbank M2

Edexcel AS and A Level Modular Mathematics

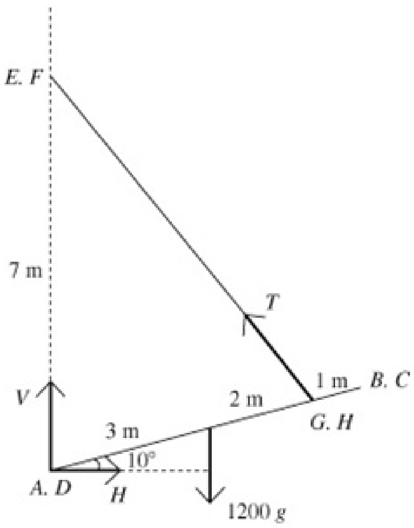
Exercise E, Question 14

Question:

A uniform drawbridge $ABCD$ is 6 m long and has mass 1200 kg. The bridge is held at $10°$ to the horizontal by two chains attached to points G and H on the bridge 5 m from the hinge and to fixed points E and F at a height of 7 m vertically above A and D. Find the force from the hinge.



Solution:



V and H are the vertical and horizontal components of the force at the hinge.
T is the tension in the cables.

We have 2 unknown forces, V and H, that we need to find. T is unknown, but we are not asked to find it. If we take moments about axes EF and GH then we can obtain 2 equations in V and H but not T:

About EF:

$$7H = 1200g \times 3\cos 10°,$$

$$H = \frac{3600g\cos 10°}{7} \approx 4960 \text{ N}$$

About GH:

$$V \times 5\cos 10° = H \times 5\sin 10° + 1200g \times 2\cos 10°$$

$$\Rightarrow V = \frac{5H\sin 10° + 2400g\cos 10°}{5\cos 10°} \approx 5580 \text{ N}$$

Combining the two components of the force at the hinge gives a force of magnitude $\sqrt{4960^2 + 5580^2} \approx 7470$ N at an angle of $\tan^{-1}\frac{5580}{4960} \approx 48°$ to the horizontal.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8\text{ m s}^{-2}$.

A stone was thrown with velocity 20 m s^{-1} at an angle of elevation of $30°$ from the top of a vertical cliff. The stone moved freely under gravity and reached the sea 5 s after it was thrown. Find

a the vertical height above the sea from which the stone was thrown,

b the horizontal distance covered by the stone from the instant when it was thrown until it reached the sea,

c the magnitude and direction of the velocity of the stone when it reached the sea.

Solution:

$R(\rightarrow) \quad u_x = 20\cos 30° = 10\sqrt{3}$
$R(\uparrow) \quad u_y = 20\sin 30° = 10$

The first step in most projectile questions is to resolve the velocity of projection horizontally and vertically.

a

$$\begin{aligned} R(\uparrow) \quad s &= ut + \tfrac{1}{2}at^2 \\ &= 10\times 5 - 4.9\times 5^2 \\ &= -72.5 \end{aligned}$$

The vertical motion is motion under constant acceleration of magnitude 9.8 m s^{-2}.

The vertical height above the sea from which the stone was thrown is 73 m (2 s.f.).

b

$$\begin{aligned} R(\rightarrow) \quad \text{distance} &= \text{speed}\times\text{time} \\ &= 10\sqrt{3}\times 5 \\ &= 50\sqrt{3} = 86.602\ldots \end{aligned}$$

The horizontal motion is motion with constant speed. In this question, the horizontal component of the velocity is $10\sqrt{3}\text{ m s}^{-1}$ throughout. It never changes.

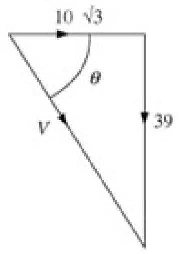
The horizontal distance covered by the stone is 87 m (2 s.f.).

c

$$\begin{aligned} R(\uparrow) \quad v^2 &= u^2 + 2as \\ v_y^2 &= 10^2 - 2\times 9.8\times(-72.5) \\ &= 1521 \\ v_y &= \sqrt{1521} = -39 \end{aligned}$$

The stone is moving downwards as it reaches the sea, so the negative square root of 1521 is the appropriate root.

As the stone reaches the sea



$$\begin{aligned} V^2 &= u_x^2 + v_y^2 \\ &= (10\sqrt{3})^2 + (-39)^2 = 1821 \\ V &= \sqrt{1821} = 42.673\ldots \\ \tan\theta &= \frac{39}{10\sqrt{3}} \Rightarrow \theta \approx 66.05° \end{aligned}$$

Velocity is a vector quantity. Finding the magnitudes and directions of vectors is found in Chapter 6 of the M1 book.

The magnitude of the velocity of the stone as it reaches the sea is 43 m s^{-1} (2 s.f.), and the direction is $66°$, (nearest degree), below the horizontal.

As the numerical value $g = 9.8$ has been used, you should give your answers to 2 significant figures. Answers cannot be more accurate than the data used to calculate them.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 2

Question:

A darts player throws darts at a dart board which hangs vertically. The motion of a dart is modelled as that of a particle moving freely under gravity. The darts move in a vertical plane which is perpendicular to the plane of the dart board. A dart is thrown horizontally with speed 12.6 m s^{-1}. It hits the board at a point which is 10 cm below the level from which it was thrown.

a Find the horizontal distance from the point where the dart was thrown to the dart board.

The darts player moves his position. He now throws a dart from a point which is at a horizontal distance of 2.5 m from the dart board. He throws the dart at an angle of elevation α to the horizontal where $\tan \alpha = \frac{7}{24}$. The dart hits the board at a point which is at the same level as the point from which it was thrown.

b Find the speed with which the dart was thrown.

Solution:

a The initial components of the velocity are

$$
\begin{aligned}
\mathrm{R}(\rightarrow) \quad u_x &= 12.6\\
\mathrm{R}(\downarrow) \quad u_y &= 0\\
\mathrm{R}(\downarrow) \quad s &= ut + \tfrac{1}{2}at^2\\
0.1 &= 0 + 4.9t^2\\
t^2 &= \frac{0.1}{4.9} = \frac{1}{49} \Rightarrow t = \frac{1}{7}\\
\mathrm{R}(\rightarrow) \quad \text{distance} &= \text{speed} \times \text{time}\\
&= 12.6 \times \tfrac{1}{7} = 1.8
\end{aligned}
$$

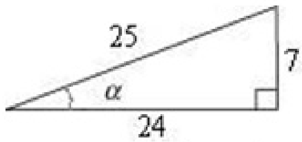
As the dart is thrown horizontally, the vertical component of the initial velocity is zero.

You usually work in metres, kilograms and seconds. Here, as the units of g are m s^{-2}, you need to convert 10 cm to 0.1 m before using the standard formula $s = ut + \frac{1}{2}at^2$ to find t.

The horizontal distance from the point where the dart was thrown to the dart board is 1.8 m.

b $\tan\alpha = \dfrac{7}{24} \Rightarrow \sin\alpha = \dfrac{7}{25}, \cos\alpha = \dfrac{24}{25}$

As $7^2 + 24^2 = 25^2$, using



you can write down $\sin\alpha$ and $\cos\alpha$.

Let U m s^{-1} be the speed of projection.

$$
\begin{aligned}
\mathrm{R}(\rightarrow) \quad u_x &= U\cos\alpha = \frac{24U}{25}\\
\mathrm{R}(\uparrow) \quad u_y &= U\sin\alpha = \frac{7U}{25}\\
\mathrm{R}(\rightarrow) \quad \text{distance} &= \text{speed} \times \text{time}\\
2.5 &= \frac{24U}{25} \times t \Rightarrow t = \frac{62.5}{24U} \quad (1)\\
\mathrm{R}(\uparrow)\ s &= ut + \tfrac{1}{2}at^2\\
0 &= \frac{7U}{25} \times t - 4.9 \times t^2
\end{aligned}
$$

As $t \neq 0$, dividing by t

$$
\begin{aligned}
0 &= \frac{7U}{25} - 4.9 \times t\\
t &= \frac{7U}{25 \times 4.9} = \frac{62.5}{24U}, \text{ from (1)}\\
U^2 &= \frac{62.5 \times 25 \times 4.9}{7 \times 24} = 45.572\ldots\\
U &= 6.750\ldots
\end{aligned}
$$

The speed with which the dart was thrown is 6.8 m s^{-1} (2 s.f.).

The separate equations for the distances travelled horizontally and vertically give simultaneous equations in U and t from which both can be found. In this question, you are not asked for t but it is quite common to be asked for t and you could substitute your answer for U into (1) to find it.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 3

Question:

A particle is projected with velocity $(8\mathbf{i}+10\mathbf{j})$ m s^{-1}, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors horizontally and vertically respectively, from a point O at the top of a cliff and moves freely under gravity.

Six seconds after projection, the particle strikes the sea at the point S. Calculate

a the horizontal distance between O and S,

b the vertical distance between O and S.

At time T seconds after projection, the particle is moving with velocity $(8\mathbf{i}-14.5\mathbf{j})$ m s^{-1}.

c Find the value of T and the position vector, relative to O, of the particle at this instant.

Solution:

The initial components of the velocity are

$R(\rightarrow)\quad u_x = 8$
$R(\uparrow)\quad u_y = 10$

When the velocity of projection is given as a vector in terms of **i** and **j**, the usual first step of resolution is simpler. The horizontal component is 8 and the vertical 10.

a

$R(\rightarrow)\quad \text{distance} = \text{speed} \times \text{time}$
$= 8 \times 6 = 48$

The horizontal distance between O and S is 48 m.

b

$R(\uparrow)\quad s = ut + \frac{1}{2}at^2$
$= 10 \times 6 - 4.9 \times 6^2 = -116.4$

The vertical distance between O and S is 120 m (2 s.f.).

c

$R(\uparrow)\quad v = u + at$
$-14.5 = 10 - 9.8T$
$T = \frac{24.5}{9.8} = \frac{245}{98} = \frac{5}{2} = 2\frac{1}{2}$

Considering the **j** components of the velocity, $v = -14.5$ and $u = 10$. Using $v = u + at$ with $a = -9.8$ then gives you the time.

$R(\rightarrow)\quad \text{distance} = \text{speed} \times \text{time}$
$= 8 \times \frac{5}{2} = 20$

The **i** component of the velocity remains 8 throughout the motion.

20 is the **i** component of the position vector.

$R(\uparrow)\quad s = ut + \frac{1}{2}at^2$
$= 10 \times \frac{5}{2} - 4.9 \times \left(\frac{5}{2}\right)^2 = -\frac{45}{8}$

$-\frac{45}{8} = -5.625$ is the **j** component of the position vector.

The position vector of the particle after $2\frac{1}{2}$ seconds is $\left(20\mathbf{i} - \frac{45}{8}\mathbf{j}\right)$ m.

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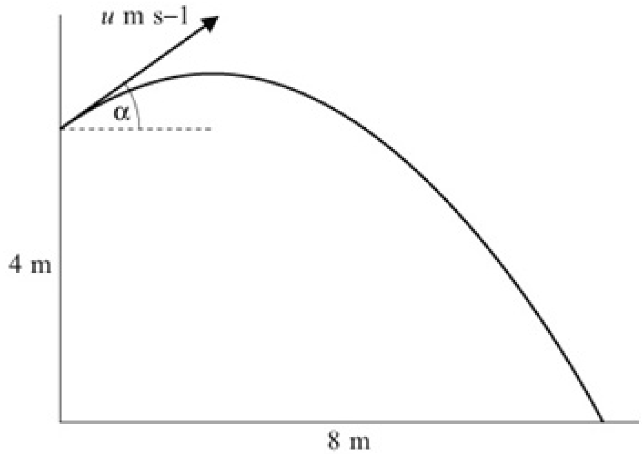
Review Exercise
Exercise A, Question 4

Question:



A ball is thrown from a point 4 m above horizontal ground. The ball is projected at an angle α above the horizontal, where $\tan\alpha = \frac{3}{4}$. The ball hits the ground at a point which is a horizontal distance 8 m from its point of projection, as shown in the figure above. The initial speed of the ball is u m s^{-1} and the time of flight is T seconds.

a Prove that $uT = 10$.

b Find the value of u.

As the ball hits the ground, its direction of motion makes an angle ϕ with the horizontal.

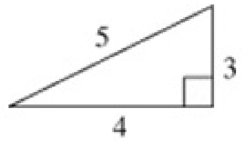
c Find $\tan\phi$.

Solution:

$\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

$R(\rightarrow) \quad u_x = u\cos\alpha = \frac{4}{5}u$

$R(\uparrow) \quad u_y = u\sin\alpha = \frac{3}{5}u$



This diagram shows that if $\tan\alpha = \frac{3}{4}$, $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$.

a

$R(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$

$8 = \frac{4}{5}u \times T$

$uT = 8 \times \frac{5}{4} = 10,$

as required

The horizontal component of the velocity remains unchanged throughout the question.

The separate equations for the distances travelled horizontally and vertically give simultaneous equations in u and T.

b

$R(\uparrow) \quad s = ut + \frac{1}{2}at^2$

$-4 = \frac{3}{5}uT - 4.9T^2$

$-4 = \frac{3}{5} \times 10 - 4.9T^2$

$4.9T^2 = 6 + 4 = 10$

$T^2 = \frac{10}{4.9} = \frac{100}{49} \Rightarrow T = \frac{10}{7}$

The ball descends 4 m before hitting the ground. So, if the upwards direction is taken as positive, $s = -4$.

From part **a**

$uT = 10$

$u \times \frac{10}{7} = 10 \Rightarrow u = 7$

c At the point where the ball hits the ground

$R(\uparrow) \quad v = u + at$

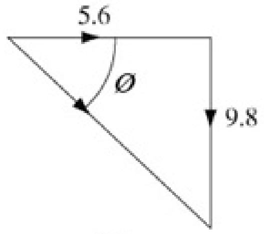
$v = v_y, u = 7\sin\alpha = 7 \times \frac{3}{5} = \frac{21}{5}, t = \frac{10}{7}, a = -9.8$

$v_y = \frac{21}{5} - 9.8 \times \frac{10}{7} = -9.8$

$u_x = 7\cos\alpha = 7 \times \frac{4}{5} = 5.6$

The vertical component of the velocity as the ball hits the ground has to be found, but you can do this in several ways. When you have a choice, $v = u + at$ is usually the simplest formula to use.

As the ball hits the ground



$\tan\varphi = \frac{9.8}{5.6} = 1.75 \ \varphi$

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Review Exercise
Exercise A, Question 5

Question:

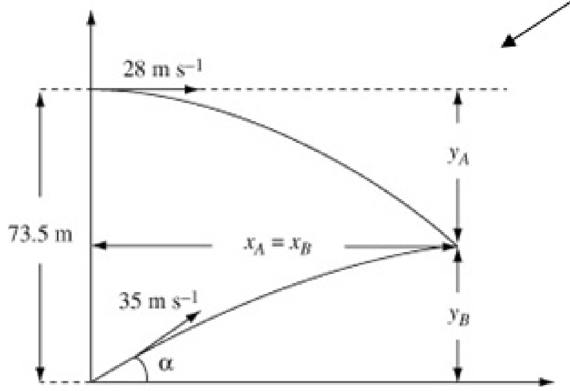
A vertical cliff is 73.5 m high. Two stones A and B are projected simultaneously.

Stone A is projected horizontally from the top of a cliff with speed 28 m s^{-1}. Stone B is projected from the bottom of the cliff with speed 35 m s^{-1} at an angle α above the horizontal. The stones move freely under gravity in the same vertical plane and collide in mid-air.

a By considering the horizontal motion of each stone, prove that $\cos\alpha = \frac{4}{5}$.

b Find the time which elapses between the instant when the stones are projected and the instant when they collide.

Solution:



It helps you to draw a sketch to illustrate the data of the question. This shows that, for A and B to collide, two conditions have to be met. Firstly, the horizontal distances moved by A and B (here labelled x_A and x_B) have to be equal. Secondly the distance fallen by $A(y_A)$ and the distance moved upwards by $B(y_B)$ must add up to the height of the cliff, 73.5 m.

a For A,

$$\begin{aligned} R(\rightarrow)\ \text{distance} &= \text{speed} \times \text{time} \\ x_A &= 28t \end{aligned}$$

For B,

$$\begin{aligned} R(\rightarrow)\quad u_x &= 35\cos\alpha \\ R(\uparrow)\quad u_y &= 35\sin\alpha \\ R(\rightarrow)\quad \text{distance} &= \text{speed} \times \text{time} \\ x_B &= 35\cos\alpha \times t \\ x_A &= x_B \\ 28\cancel{t} &= 35\cos\alpha\,\cancel{t} \\ \cos\alpha &= \frac{28}{35} = \frac{4}{5}, \text{ as required} \end{aligned}$$

For collision, the horizontal distance travelled by the stones must be the same.

As $t \neq 0$ when the stones collide, you can divide both sides of the equation by t.

b For A,

$$\begin{aligned} R(\downarrow)\quad s &= ut + \frac{1}{2}at^2 \\ y_A &= 0 + 4.9t^2 \end{aligned}$$

For B,

$$\begin{aligned} R(\uparrow)\quad s &= ut + \frac{1}{2}at^2 \\ y_B &= 35\sin\alpha \times t - 4.9t^2 \end{aligned}$$

$$\cos\alpha = \frac{4}{5} \Rightarrow \sin\alpha = \frac{3}{5}$$

$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \frac{16}{25} = \frac{9}{25}$ implies $\sin\alpha = \frac{3}{5}$ or you could use a 3, 4, 5 triangle to see this.

$$y_B = 35 \times \frac{3}{5} \times t - 4.9t^2 = 21t - 4.9t^2$$

$$y_A + y_B = 73.5$$

$$\cancel{4.9t^2} + 21t - \cancel{4.9t^2} = 73.5$$

The two distances moved vertically must add up to the height of the cliff.

$$t = \frac{73.5}{21} = 3.5$$

The time elapsed is 3.5 s.

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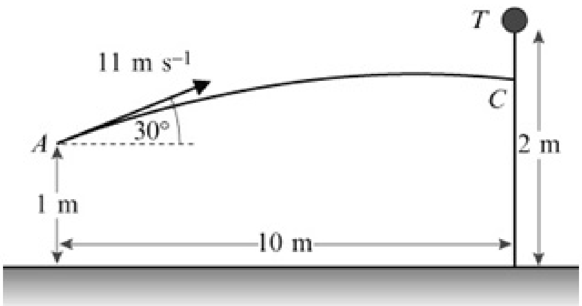
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 6

Question:



The object of a game is to throw a ball B from a point A to hit a target T which is placed at the top of a vertical pole, as shown in the figure above. The point A is 1 m above horizontal ground and the height of the pole is 2 m. The pole is a horizontal distance of 10 m from A.

The ball B is projected from A with speed 11 m s^{-1} at an angle of elevation of $30°$.

The ball hits the pole at C. The ball B and the target T are modelled as particles.

a Calculate, to 2 decimal places, the time taken for B to move from A to C.

b Show that C is approximately 0.63 m below T.

The ball is thrown again from A.

The speed of projection of B is increased to $V \text{ m s}^{-1}$, the angle of elevation remaining $30°$. This time B hits T.

c Calculate the value of V.

d Explain why, in practice, a range of values of V would result in B hitting the target.

Solution:

$$\mathrm{R}(\rightarrow) \quad u_x = 11\cos 30^\circ = 5.5\sqrt{3}$$
$$\mathrm{R}(\uparrow) \quad u_y = 11\sin 30^\circ = 5.5$$

a $\mathrm{R}(\rightarrow)$ distance = speed × time

$$10 = 5.5\sqrt{3} \times t$$
$$t = \frac{10}{5.5\sqrt{3}} = 1.049\,727\ldots$$

If the question specifies a particular accuracy, you must give your answer to that accuracy to gain full marks.

The time taken to move from A to C is 1.05 seconds (2 d.p.).

b

$$\mathrm{R}(\uparrow) \quad s = ut + \frac{1}{2}at^2$$
$$= 5.5 \times 1.05 - 4.9 \times 1.05^2 \approx 0.374$$

The distance below T is
$(1 - 0.374)$ m ≈ 0.63 m, as required.

The ball starts 1 m above the ground and hits the pole approximately 0.37 m higher than it started. That leaves another 0.63 m to reach 2 m above the ground.

c

$$\mathrm{R}(\rightarrow) \quad u_x = V\cos 30^\circ$$
$$\mathrm{R}(\uparrow) \quad u_y = V\sin 30^\circ$$

$\mathrm{R}(\rightarrow)$ distance = speed × time

$$10 = V\cos 30^\circ \times t$$
$$Vt = \frac{10}{\cos 30^\circ} = \frac{20}{\sqrt{3}} \qquad (1)$$
$$\mathrm{R}(\uparrow) \quad s = ut + \frac{1}{2}at^2$$
$$1 = V\sin 30^\circ \times t - 4.9t^2 \ldots (2)$$
$$1 = \frac{20}{\sqrt{3}} \times \frac{1}{2} - 4.9\left(\frac{20}{V\sqrt{3}}\right)^2$$
$$\frac{4.9 \times 400}{3V^2} = \frac{10}{\sqrt{3}} - 1 = 4.773\ldots$$
$$V^2 = \frac{4.9 \times 400}{3 \times 4.773\ldots} = 136.866\ldots$$
$$V = 11.699\ldots = 12 \text{ (2 s.f.)}$$

These two equations in V and t enable either to be found by elimination. In this case t has to be eliminated.

From (1), $t = \frac{20}{V\sqrt{3}}$, and this is substituted into (2).

d B and T are not particles but take up space; they have extension. This would allow a range of values of V resulting in hitting the target.

For example, the target takes up space and can be hit at its top, its bottom or anywhere in between.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 7

Question:

A particle P, projected from a point O on horizontal ground, moves freely under gravity and hits the ground again at A.

Taking O as origin, OA as the x-axis and the upward vertical at O as the y-axis, the equation of the path of P is

$$y = x - \frac{x^2}{500}$$

where x and y are measured in metres.

a By finding $\frac{dy}{dx}$, show that P was projected from O at an angle of $45°$ to the horizontal.

b Find the distance OA and the greatest vertical height attained by P above OA.

c Find the speed of projection of P.

d Find, to the nearest second, the time taken by P to move from O to A.

Solution:

a $y = x - \frac{x^2}{500}$

$\frac{dy}{dx} = 1 - \frac{x}{250}$

When $x = 0, \frac{dy}{dx} = 1$

The gradient of the direction of motion at O is 1 and the angle is given by $\tan\theta = 1$. Hence $\theta = 45^\circ$, as required.

$\frac{dy}{dx}$ is the gradient of the path of the projectile and, at any point of the path, this is the direction of motion of the particle.

b P is at A when $y = 0$

$0 = x - \frac{x^2}{500} = x\left(1 - \frac{x}{500}\right)$

At $A, x \neq 0$

$1 - \frac{x}{500} = 0 \Rightarrow x = 500$

$OA = 500$ m

The greatest height is reached when $\frac{dy}{dx} = 0$

The greatest height corresponds to the maximum point on the curve and, as this is a stationary value, using the knowledge you learnt in the C2 module, this is found by putting $\frac{dy}{dx} = 0$. The M2 module requires knowledge of the C1, C2 and C3 specifications.

$1 - \frac{x}{250} = 0 \Rightarrow x = 250$

$y = 250 - \frac{250^2}{500} = 125$

The greatest height reached is 125 m.

c Let the speed of projection be U m s^{-1}.

Initially R($\uparrow$) $u_y = U\sin 45^\circ = \frac{U}{\sqrt{2}}$

At the greatest height

R($\uparrow$) $u = \frac{U}{\sqrt{2}}, v = 0, s = 125, a = -9.8$

$$v^2 = u^2 + 2as$$

$$0^2 = \frac{U^2}{2} - 2 \times 9.8 \times 125$$

$$U^2 = 4 \times 9.8 \times 125 = 4900 \Rightarrow U = 70$$

The speed of projection is 70 m s^{-1}.

At the greatest height the vertical component of the velocity is 0. The value of s is the height found in **b**.

d R($\rightarrow$) $u_x = 70\sin 45^\circ = \frac{70}{\sqrt{2}}$

R($\rightarrow$) distance = speed $\times$ time

$$500 = \frac{70}{\sqrt{2}}t \Rightarrow t = \frac{500\sqrt{2}}{70} = 10.101\ldots$$

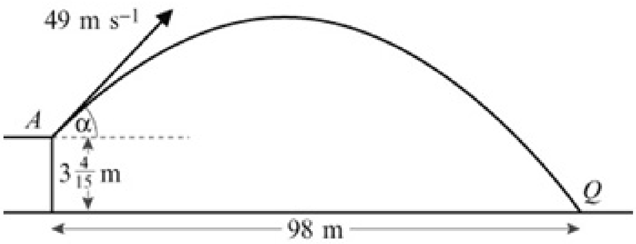
The time taken for P to move from O to A is 10 s (nearest second).

The horizontal component of the velocity is $u_x = \frac{70}{\sqrt{2}}$ throughout the motion.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 8

Question:



A golf ball is projected with speed 49 m s^{-1} at an angle of elevation α from a point A on the first floor of a driving range. Point A is at a height of $3\frac{4}{15}$ m above horizontal ground. The ball first strikes the ground at a point Q which is at a horizontal distance of 98 m from the point A, as shown in the figure above.

a Show that

$$6\tan^2\alpha - 30\tan\alpha + 5 = 0.$$

b Hence find the two possible angles of elevation.

c Find the smallest possible time of direct flight from A to Q.

Solution:

a $R(\rightarrow)\quad u_x = 49\cos\alpha$

$R(\uparrow)\quad u_y = 49\sin\alpha$

$R(\rightarrow)\quad \text{distance} = \text{speed}\times\text{time}$

$$98 = 49\cos\alpha\times t \Rightarrow t = \frac{2}{\cos\alpha}$$

These two equations are simultaneous equations in α and t. You have to eliminate t first and then use trigonometric identities to show that the printed answer is correct.

At Q

$R(\uparrow)\quad s = ut + \frac{1}{2}at^2$

$$-\frac{49}{15} = 49\sin\alpha\times t - 4.9t^2$$

From A to Q, the ball descends a vertical distance of $3\frac{4}{15}$ m $= \frac{49}{15}$ m.

Dividing by 4.9

$$-\frac{10}{15} = -\frac{2}{3} = 10\sin\alpha t - t^2$$

Multiplying by 3 and rearranging

$$3t^2 - 30\sin\alpha t - 2 = 0$$

Substituting $t = \dfrac{2}{\cos\alpha}$

$$\frac{12}{\cos^2\alpha} - 30\sin\alpha\times\frac{2}{\cos\alpha} - 2 = 0$$

Using $\frac{1}{\cos\alpha} = \sec\alpha$ and $\sec^2\alpha = 1 + \tan^2\alpha$.

$$12\sec^2\alpha - 60\tan\alpha - 2 = 0$$

$$12(\tan^2\alpha + 1) - 60\tan\alpha - 2 = 0$$

$$12\tan^2\alpha - 60\tan\alpha + 10 = 0$$

Dividing by 2

$6\tan^2\alpha - 30\tan\alpha + 5 = 0$, as required.

b $$\tan\alpha = \frac{30\pm\sqrt{(900-120)}}{12} = 4.827\ldots, 0.1726\ldots$$

$$\alpha \approx 78.3^\circ, 9.79^\circ$$

Using the formula $x = \frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$ to solve the quadratic in $\tan\alpha$.

To the nearest degree, the possible angles of elevation are 10° and 78°.

c The smallest possible time is given by

$$t = \frac{2}{\cos 9.79^\circ} \approx 2.029$$

There are 2 possible angles. As $t = \dfrac{2}{\cos\alpha}$, the smaller value of t comes from the larger value of $\cos\alpha$, which corresponds to the smaller angle. $\dfrac{2}{\cos 78.3^\circ} \approx 9.86$.

The smallest possible time of direct flight from A to Q is 2.0 s (2 s.f.).

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 9

Question:

A particle P moves on the x-axis. At time t seconds, its acceleration is $(5-2t)$ m s^{-2}, measured in the direction of x increasing. When $t=0$, its velocity is 6 m s^{-1} measured in the direction of x increasing. Find the time when P is instantaneously at rest in the subsequent motion.

Solution:

$$\begin{aligned} a &= 5-2t \\ v &= \int a \, \mathrm{d}t = \int (5-2t)\, \mathrm{d}t \\ &= 5t - t^2 + C \end{aligned}$$

When $t=0, v=6$

The initial condition, that when $t=0$ the velocity is 6 m s^{-1}, enables you to find the value of the constant of integration.

$6 = 0 - 0 + C \Rightarrow C = 6$

Hence

$v = 6 + 5t - t^2$

When P is at rest

$$\begin{aligned} 0 &= 6 + 5t - t^2 \\ t^2 - 5t - 6 &= (t-6)(t+1) = 0 \\ t &= 6, -1 \end{aligned}$$

When P is at rest, its velocity is 0.

$t > 0$

$t = 6$

The question asks you for the value of t subsequent to, that is after, $t=0$. So you must pick the positive root of the quadratic.

P is at rest at time 6 seconds.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 10

Question:

A particle P moves in a straight line in such a way that, at time t seconds, its velocity, v m s^{-1}, is given by

$$v = \begin{cases} 12t - 3t^2, & 0 \le t \le 5 \\ -\dfrac{375}{t^2}, & t > 5. \end{cases}$$

When $t = 0$, P is at the point O.

Calculate the displacement of P from O

a when $t = 5$,

b when $t = 6$.

Solution:

a For $0 \le t \le 5$

$$s = \int v\,dt = \int (12t - 3t^2)\,dt$$
$$= 6t^2 - t^3 + A$$

When $t = 0, s = 0$

$0 = 0 - 0 + A \Rightarrow A = 0$

This constant of integration is 0. Even when it seems obvious that a constant has this value, you should show sufficient working to justify the value 0.

Hence

$s = 6t^2 - t^3$

When $t = 5$

$s = 6 \times 5^2 - 5^3 = 25$

The displacement of P from O when $t = 5$ is 25 m.

b For $t > 5$

$$s = \int v\,dt = \int -375t^{-2}\,dt$$
$$= \frac{-375t^{-1}}{-1} + B = \frac{375}{t} + B$$

To integrate $-\frac{375}{t^2}$, write it as $-375t^{-2}$ and use the formula $\int t^n\,dt = \frac{t^{n+1}}{n+1} + A$.

From **a**, when $t = 5, s = 25$

The end point in **a** is the starting point in **b**.

$25 = \frac{375}{5} + B \Rightarrow B = 25 - 75 = -50$

Hence

$s = \frac{375}{t} - 50$

When $t = 6$

$s = \frac{375}{6} - 50 = 12.5$

The displacement of P from O when $t = 6$ is 12.5 m.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 11

Question:

A particle is moving in a straight line Ox.

At time t seconds the acceleration of P is a m s^{-2} and the velocity v m s^{-1} of P is given by

$$v = 2 + 8\sin kt,$$

where k is a constant.

The initial acceleration of P is 4 m s^{-2}.

a Find the value of k.

Using the value of k found in **a**,

b find, in terms of π, the values of t in the interval $0 \le t \le 4\pi$ for which $a = 0$,

c show that $4a^2 = 64 - (v-2)^2$.

Solution:

a $v = 2 + 8\sin kt$

$a = \frac{dv}{dt} = 8k\cos kt$

For any constant k, $\frac{d}{dt}(\sin kt) = k\cos kt$.

When $t = 0, a = 4$

$4 = 8k \Rightarrow k = \frac{1}{2}$

The initial condition, that the acceleration is 4 m s^{-1}, gives an equation in k which you solve.

b $a = 8 \times \frac{1}{2}\cos\frac{1}{2}t = 4\cos\frac{1}{2}t$

When $a = 0$

$\cos\frac{1}{2}t = 0 \Rightarrow \frac{1}{2}t = \frac{\pi}{2}, \frac{3\pi}{2}$

Hence
$t = \pi, 3\pi$

In all differentiation and integration of trigonometric functions, it is assumed that angles are measured in radians. $\cos\theta = 0$ when θ is an odd multiple of $\frac{\pi}{2}$.

c
$$\begin{aligned} 64 - (v-2)^2 &= 64 - \left(8\sin\tfrac{1}{2}t\right)^2 \\ &= 64 - 64\sin^2\tfrac{1}{2}t = 64\left(1 - \sin^2\tfrac{1}{2}t\right) \\ &= 64\cos^2\tfrac{1}{2}t = 4\left(4\cos\tfrac{1}{2}t\right)^2 \\ &= 4a^2\text{, as required} \end{aligned}$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 12

Question:

An aircraft is situated at rest at a point A on a runway XY which is of length 1400 m. Point A is 77 m from X. The aircraft moves along the runway towards Y with acceleration $\left(10-\frac{4}{5}t\right)$ m s^{-2}, where t seconds is the time from the instant the aircraft started to move.

a Find the speed of the aircraft when $t = 6$ and determine the distance travelled in the first 6 seconds of the aircraft's motion.

B is the point such that $AB = \frac{3}{10}AY$.

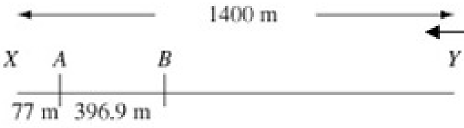
b Find the distance AB.

A safety regulation requires that the aircraft passes point B with a speed of 55 m s^{-1} or more.

c Given that $t = T$ when the aircraft passes B, form an equation for T.

d Show that $T = 10.5$ satisfies the equation, and hence determine whether or not the aircraft satisfies this safety regulation as it passes B.

Solution:



A sketch is useful to help you sort out the various distances in this question.

a $v = \int a \, dt = \int \left(10 - \frac{4}{5}t\right) dt$

$= 10t - \frac{2}{5}t^2 + C$

When $t = 0, v = 0$

$0 = 0 + 0 + C \Rightarrow C = 0$

Hence

$v = 10t - \frac{2}{5}t^2$ (1)

When $t = 6$

$v = 10 \times 6 - \frac{2}{5} \times 6^2 = 45.6$

The speed of the aircraft when $t = 6$ is 45.6 m s^{-1}.

This is an exact answer and as g, or any similar approximation, has not been used, the answer should not be rounded.

$s = \int v \, dt = \int \left(10t - \frac{2}{5}t^2\right) dt$

$= 5t^2 - \frac{2}{15}t^3 + D$

Let $s = 0$, when $t = 0$

In principle, the displacement could be measured from any point but it is sensible to measure the displacement from the starting point, which is A.

$0 = 0 - 0 + D \Rightarrow D = 0$

Hence

$s = 5t^2 - \frac{2}{15}t^3$ (2)

When $t = 6$

$s = 5 \times 6^2 - \frac{2}{15} \times 6^3 = 151.2$

The distance travelled in the first 6 s of motion is 151.2 m.

b $AY = (1400 - 77)\text{ m} = 1323\text{ m}$

$AB = \frac{3}{10}AY = 396.9\text{ m}$

c Substituting $s = 396.9$ and $t = T$ into (2)

$396.9 = 5T^2 - \frac{2}{15}T^3$

$\frac{2}{15}T^3 - 5T^2 + 396.9 = 0$

This cubic equation would be very difficult to solve directly and the question only asks you to show that 10.5 satisfies the equation. To do that, you substitute $T = 10.5$ into the left hand side of the equation and show that the calculation gives the value 0.

d Substituting $T = 10.5$ into the left hand side of the answer in **c**

$\frac{2}{15} \times 10.5^3 - 5 \times 10.5^2 + 396.9$

$= \quad 154.35 - 551.25 + 396.9$

$= \quad 551.25 - 551.25 = 0$

$T = 10.5$ satisfies the equation in **c**, as required.

Substituting $T = 10.5$ into equation (1) in **a**

$v = 10 \times 10.5 - \frac{2}{5} \times 10.5^2 = 60.9 > 55$

The aircraft satisfies the safety condition.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 13

Question:

A particle P moves along the x-axis. It passes through the origin O at time $t = 0$ with speed 7 m s^{-1} in the direction of x increasing.

At time t seconds the acceleration of P in the direction of x increasing is $(20 - 6t) \text{ m s}^{-2}$.

a Show that the velocity $v \text{ m s}^{-1}$ of P at time t seconds is given by

$$v = 7 + 20t - 3t^2.$$

b Show that $v = 0$ when $t = 7$ and find the greatest speed of P in the interval $0 \le t \le 7$.

c Find the distance travelled by P in the interval $0 \le t \le 7$.

Solution:

a

$$v = \int a \, dt = \int (20 - 6t) \, dt$$
$$= 20t - 3t^2 + A$$

When $t = 0, v = 7$

$7 = 0 - 0 + A \Rightarrow A = 7$

Hence

$v = 7 + 20t - 3t^2$, as required.

b When

$t = 7$

$v = 7 + 20 \times 7 - 3 \times 7^2$

$= 7 + 140 - 147 = 0$, as required

For the greatest speed of P

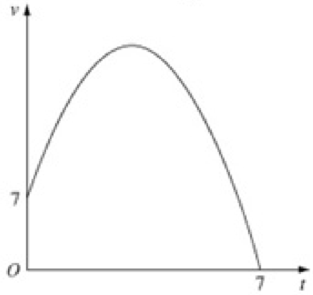
$\frac{dv}{dt} = a = 20 - 6t = 0$

$t = \frac{20}{6} = \frac{10}{3}$

When $t = \frac{10}{3}$

$v = 7 + 20 \times \frac{10}{3} - 3 \times \left(\frac{10}{3}\right)^2 = 40\frac{1}{3}$

A sketch of v against t is



This illustrates that the greatest value of the speed occurs at a maximum point.

The greatest speed of P in the interval $0 \le t \le 7$ is $40\frac{1}{3}$ m s^{-1}.

c $s = \int v \, dt = \int (7 + 20t - 3t^2) \, dt$

$= 7t + 10t^2 - t^3 + B$

When $t = 0, s = 0$

$0 = 0 + 0 - B \Rightarrow B = 0$

Hence

$s = 7t + 10t^2 - t^3$

When $t = 7$

$s = 7 \times 7 + 10 \times 7^2 - 7^3 = 196$

Finding the distance travelled is not straightforward if the particle turns round. This happens when $v = 0$. However the sketch in **b** shows that P does not turn round until $t = 7$, so the distance travelled in this interval is found by substituting $t = 7$ into the equation for s.

The distance travelled by P in the interval $0 \le t \le 7$ is 196 m.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 14

Question:

A particle P moves along a straight line. Initially, P is at rest at a point O on the line. At time t seconds (where $t \geq 0$) the acceleration of P is proportional to $(7-t^2)$ and the displacement of P from O is s metres. When $t = 3$, the speed of P is 6 m s^{-1}.

a Show that

$$s = \frac{1}{24}t^2(42-t^2).$$

b Find the total distance that P moves before returning to O.

Solution:

a

$$a = k(7-t^2) = 7k - kt^2$$
$$v = \int a\,\mathrm{d}t = \int (7k - kt^2)\,\mathrm{d}t$$
$$= 7kt - \frac{k}{3}t^3 + A$$

If $a \propto (7-t^2)$ then $a = k(7-t^2)$, where k is the constant of proportionality. You will need to use the information that the speed of P is $6\ \mathrm{m\,s^{-1}}$ when $t = 3$ to evaluate k.

You will need to integrate twice to obtain s from a.

When
$$t = 0, v = 0$$
$$0 = 0 - 0 + A \Rightarrow A = 0$$

Hence
$$v = 7kt - \frac{k}{3}t^3$$

When $t = 3, v = 6$
$$6 = 21k - 9k \Rightarrow 12k = 6 \Rightarrow k = \frac{1}{2}$$
$$v = \frac{7}{2}t - \frac{1}{6}t^3 \qquad (1)$$
$$s = \int v\,\mathrm{d}t = \int \left(\frac{7}{2}t - \frac{1}{6}t^3\right)\mathrm{d}t$$
$$= \frac{7}{4}t^2 - \frac{1}{24}t^4 + B$$

When $t = 0, s = 0$
$$0 = 0 - 0 + B \Rightarrow B = 0$$
$$s = \frac{7}{4}t^2 - \frac{1}{24}t^4 = \frac{42}{24}t^2 - \frac{1}{24}t^4$$
$$= \frac{1}{24}t^2(42 - t^2), \text{ as required}$$

b Substituting $v = 0$ into (1)
$$0 = \frac{7}{2}t - \frac{1}{6}t^3 = \frac{21}{6}t - \frac{1}{6}t^3$$
$$= \frac{1}{6}t\left(21 - t^2\right)$$

To find the total distance P moves, you will need to find the point where P reverses direction. That is where $v = 0$.

For $t > 0$
$$t^2 = 21$$

$t = \sqrt{21}$ but it is the value of t^2 you need to substitute into the expression for s. Using a decimal approximation for t would lose accuracy.

Substituting $t^2 = 21$ into the result of **a**
$$s = \frac{1}{24} \times 21 \times (42 - 21) = \frac{21^2}{24} = \frac{441}{24}$$

The total distance P moves before returning to O is $\left(2 \times \frac{441}{24}\right)\mathrm{m} = \frac{441}{12}\ \mathrm{m}$.

P moves to a point $\frac{441}{24}$ m from O and then returns to O. So the total distance moved is twice this distance.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 15

Question:

A particle P of mass 0.3 kg moves under the action of a single force $\mathbf{F}$ newtons. At time t seconds, the velocity $\mathbf{v}$ m s^{-1} of P is given by

$$\mathbf{v} = 3t^2\mathbf{i} + (6t-4)\mathbf{j}.$$

a Find the magnitude of $\mathbf{F}$ when $t = 2$.

When $t = 0$, P is at the point A. The position vector of A with respect to a fixed origin O is $(3\mathbf{j} - 4\mathbf{j})$ m. When $t = 4$, P is at the point B.

b Find the position vector of B.

Solution:

a

$$\begin{aligned}
\text{Acceleration, } \mathbf{a} &= \dot{\mathbf{v}} = 6t\mathbf{i} - 4\mathbf{j} \\
\mathbf{F} &= m\mathbf{a} \\
&= 0.3(6t\mathbf{i} + 6\mathbf{j}) \\
&= 1.8t\mathbf{i} + 1.8\mathbf{j}
\end{aligned}$$

You find $\mathbf{F}$ using Newton's Second Law $\mathbf{F} = m\mathbf{a}$, so you begin this question by differentiating the velocity to find the acceleration.

When $t = 2$

$$\begin{aligned}
\mathbf{F} &= 3.6\mathbf{i} + 1.8\mathbf{j} \\
|\mathbf{F}|^2 &= 3.6^2 + 1.8^2 = 16.2 \\
|\mathbf{F}| &= \sqrt{16.3} = 4.0249\ldots
\end{aligned}$$

The magnitude of the vector $\mathbf{F}$, often written as F, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, is given by $F^2 = |\mathbf{F}|^2 = x^2 + y^2$.

The magnitude of $\mathbf{F}$ when $t = 2$ is 4.02 (2 d.p.).

b

$$\begin{aligned}
\mathbf{r} &= \int \mathbf{v}\,\mathrm{d}t = \int \left(3t^2\mathbf{i} + (6t-4)\mathbf{j}\right)\mathrm{d}t \\
&= t^3\mathbf{i} + (3t^2 - 4t)\mathbf{j} + \mathbf{A}
\end{aligned}$$

When you integrate vectors the constant of integration is a vector.

When $t = 0, \mathbf{r} = 3\mathbf{i} - 4\mathbf{j}$

$3\mathbf{i} - 4\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{A} \Rightarrow \mathbf{A} = 3\mathbf{i} - 4\mathbf{j}$

Hence

$\mathbf{r} = \left(t^3 + 3\right)\mathbf{i} + \left(3t^2 - 4t - 4\right)\mathbf{j}$

When $t = 4$

$\mathbf{r} = \left(4^3 + 3\right)\mathbf{i} + \left(3 \times 4^2 - 4 \times 4 - 4\right)\mathbf{j} = 67\mathbf{i} + 28\mathbf{j}$

The position vector of B is $(67\mathbf{i} + 28\mathbf{j})$ m.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 16

Question:

Referred to a fixed origin O, the position vector of a particle P at time t seconds is $\mathbf{r}$ metres, where

$$\mathbf{r} = 6t^2\mathbf{i} + t^{\frac{5}{2}}\mathbf{j},\ t \geq 0.$$

At the instant when $t = 4$, find

a the speed of P,

b the acceleration of P, giving your answer as a vector.

Solution:

a

$$\mathbf{v} = \dot{\mathbf{r}} = 12t\mathbf{i} + \frac{5}{2}t^{\frac{3}{2}}\mathbf{j}$$

When $t = 4$

$$4^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = 2^3 = 8$$

$$\mathbf{v} = 48\mathbf{i} + \frac{5}{2} \times 4^{\frac{3}{2}}\mathbf{j} = 48\mathbf{i} + 20\mathbf{j}$$

$$|\mathbf{v}|^2 = 48^2 + 20^2 = 2704$$

$$|\mathbf{v}| = \sqrt{2704} = 52$$

The speed of P when $t = 4$ is 52 m s^{-1}.

You need to know that $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$.

b $\mathbf{a} = \dot{\mathbf{v}} = 12\mathbf{i} + \frac{5}{2} \times \frac{3}{2}t^{\frac{1}{2}}\mathbf{j} = 12\mathbf{i} + \frac{15}{4}t^{\frac{1}{2}}\mathbf{j}$

When $t = 4$

$$\mathbf{a} = 12\mathbf{i} + \frac{15}{4} \times 4^{\frac{1}{2}}\mathbf{j} = 12\mathbf{i} + \frac{15}{2}\mathbf{j}$$

The acceleration of P when $t = 4$ is $\left(12\mathbf{i} + \frac{15}{2}\mathbf{j}\right)\text{ m s}^{-2}$.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 17

Question:

A particle P moves in a horizontal plane. At time t seconds, the position vector of P is $\mathbf{r}$ metres relative to a fixed origin O where $\mathbf{r}$ is given by

$$\mathbf{r} = (18t - 4t^3)\mathbf{i} + ct^2\mathbf{j},$$

where c is a positive constant. When $t = 1.5$, the speed of P is 15 m s^{-1}. Find

a the value of c,

b the acceleration of P when $t = 1.5$.

Solution:

a

$\mathbf{v} = \dot{\mathbf{r}} = (18 - 12t^2)\mathbf{i} + 2ct\mathbf{j}$

When $t = 1.5$

$\mathbf{v} = (18 - 12 \times 1.5^2)\mathbf{i} + 3c\mathbf{j} = -9\mathbf{i} + 3c\mathbf{j}$

$|\mathbf{v}|^2 = (-9)^2 + (3c)^2 = 15^2$

$9c^2 = 15^2 - 9^2 = 144 \Rightarrow c^2 = \frac{144}{9} = 16$

As c is positive, $c = 4$

The speed of P is the magnitude of the velocity $\mathbf{v}$. $|\mathbf{v}|^2$ is both $(-9)^2 + (3c)^2$ and the speed squared. This gives you an equation in c.

b

$\mathbf{a} = \dot{\mathbf{v}} = -24t\mathbf{i} + 2c\mathbf{j}$

Using $c = 4$ and $t = 1.5$

$\mathbf{a} = -36\mathbf{i} + 8\mathbf{j}$

The acceleration of P when $t = 1.5$ is $(-36\mathbf{i} + 8\mathbf{j})\text{ m s}^{-2}$.

Acceleration is a vector and the answer should be given in vector form.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 18

Question:

A particle P of mass 0.4 kg moves under the action of a single force $\mathbf{F}$ newtons. At time t seconds, the velocity of P, $\mathbf{v}$ m s^{-1}, is given by

$$\mathbf{v} = (6t + 4)\mathbf{i} + (t^2 + 3t)\mathbf{j}.$$

When $t = 0$, P is at the point with position vector $(-3\mathbf{i} + 4\mathbf{j})$ m with respect to a fixed origin O. When $t = 4$, P is at the point S.

a Calculate the magnitude of $\mathbf{F}$ when $t = 4$.

b Calculate the distance OS.

Solution:

a

Acceleration, $\mathbf{a} = \dot{\mathbf{v}} = 6\mathbf{i} + (2t+3)\mathbf{j}$

You find $\mathbf{F}$ using Newton's Second Law $\mathbf{F} = m\mathbf{a}$, so you begin this part by differentiating the velocity to find the acceleration.

When $t = 4$

$$\begin{aligned} \mathbf{a} &= 6\mathbf{i} + 11\mathbf{j} \\ \mathbf{F} &= m\mathbf{a} \\ &= 0.4(6\mathbf{i} + 11\mathbf{j}) = 2.4\mathbf{i} + 4.4\mathbf{j} \\ |\mathbf{F}|^2 &= 2.4^2 + 4.4^2 = 25.12 \\ |\mathbf{F}| &= \sqrt{25.12} = 5.011\ldots \end{aligned}$$

No accuracy is specified in this question and any sensible accuracy is acceptable.

The magnitude of $\mathbf{F}$ is 5.01 (2 d.p.).

b

$$\begin{aligned} \mathbf{r} &= \int \mathbf{v}\,\mathrm{d}t = \int \left((6t+4)\mathbf{i} + (t^2+3t)\mathbf{j}\right)\mathrm{d}t \\ &= \left(3t^2+4t\right)\mathbf{i} + \left(\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2\right)\mathbf{j} + \mathbf{A} \end{aligned}$$

When you integrate vectors the constant of integration is a vector.

When $t = 0, \mathbf{r} = -3\mathbf{i} + 4\mathbf{j}$

$-3\mathbf{i} + 4\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{A} \Rightarrow \mathbf{A} = -3\mathbf{i} + 4\mathbf{j}$

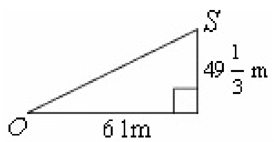
Hence

$\mathbf{r} = \left(3t^2 + 4t - 3\right)\mathbf{i} + \left(\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 + 4\right)\mathbf{j}$

When $t = 4$

$$\begin{aligned} \mathbf{r} &= \left(3\times 4^2 + 4\times 4 - 3\right)\mathbf{i} + \left(\tfrac{1}{3}\times 4^3 + \tfrac{3}{2}\times 4^2 + 4\right)\mathbf{j} \\ &= 61\mathbf{i} + 49\tfrac{1}{3}\mathbf{j} \\ |\mathbf{r}|^2 &= 61^2 + \left(49\tfrac{1}{3}\right)^2 = 6154\tfrac{7}{9} \Rightarrow |\mathbf{r}| = 78.452\ldots \\ OS &= 78.45\text{ m (2 d.p.)} \end{aligned}$$

The distance OS is found using Pythagoras' Theorem.



Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 19

Question:

Two particles P and Q move in a plane so that at time t seconds, where $t \geq 0$, P and Q have position vectors $\mathbf{r}_P$ metres and $\mathbf{r}_Q$ metres respectively, relative to a fixed origin O, where

$$\mathbf{r}_P = (3t^2 + 4)\mathbf{i} + (2t - \frac{1}{2})\mathbf{j},$$

$$\mathbf{r}_P = (t + 6)\mathbf{i} + \frac{3t^2}{2}\mathbf{j}.$$

Find

a the velocity vectors of P and Q at time t seconds,

b the speed of P when $t = 2$,

c the value of t at the instant when the particles are moving parallel to one another.

d Show that the particles collide and find the position vector of their point of collision.

Solution:

a

$$\mathbf{v}_P = \dot{\mathbf{r}}_P = 6t\mathbf{i} + 2\mathbf{j}$$
$$\mathbf{v}_Q = \dot{\mathbf{r}}_Q = \mathbf{i} + 3t\mathbf{j}$$

$\frac{\mathrm{d}}{\mathrm{d}t}((t+6)\mathbf{i}) = 1\mathbf{i} = \mathbf{i}$

The velocity of P at time t seconds is $(6t\mathbf{i} + 2\mathbf{j})$ m s^{-1} and the velocity of Q is $(\mathbf{i} + 3t\mathbf{j})$ m s^{-1}.

b When $t = 2$

$$\mathbf{v}_P = 12\mathbf{i} + 2\mathbf{j}$$
$$|\mathbf{v}_P|^2 = 12^2 + 2^2 = 148 \Rightarrow \mathbf{v}_P = \sqrt{148} = 12.165\ldots$$

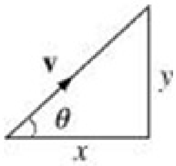
The speed of P when $t = 2$ is 12.2 m s^{-1} (3 s.f.).

c When P is moving parallel to Q

$$\frac{2}{6t} = \frac{3t}{1} \Rightarrow 18t^2 = 2 \Rightarrow t^2 = \frac{1}{9}$$
$$t \geq 0, t = \frac{1}{3}$$

When the particles are moving parallel to each other, the angle each makes with $\mathbf{i}$ is the same.

If $\mathbf{v} = x\mathbf{i} + y\mathbf{j}, \tan\theta = \frac{y}{x}$ must be the same for both velocities.



d $\mathbf{i}$ components

$$3t^2 + 4 = t + 6$$
$$3t^2 - t - 2 = (t-1)(3t+2) = 0$$
$$t = 1, -\tfrac{2}{3}$$

$\mathbf{j}$ components

$$2t - \frac{1}{2} = \frac{3t^2}{2}$$

For the particles to collide, both the $\mathbf{i}$ and $\mathbf{j}$ components of their position vectors must be the same for the same value of t. The appropriate method is to equate the $\mathbf{i}$ components and solve the resulting quadratic, and then do the same for the $\mathbf{j}$ components. If one of the roots of the quadratics is the same, then the particles collide.

Multiplying by 2 and rearranging

$$3t^2 - 4t + 1 = (t-1)(3t-1) = 0$$
$$t = 1, \tfrac{1}{3}$$

1 is a common root of the equations and, hence, P and Q collide at the point with position vector $\left(7\mathbf{i} + \frac{3}{2}\mathbf{j}\right)$m.

$t = 1$ can be substituted into either $\mathbf{r}_P$ or $\mathbf{r}_Q$ to find the position vector of the point where the particles collide.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 20

Question:

Referred to a fixed origin O, the particle R has position vector $\mathbf{r}$ metres at time t seconds, where

$$\mathbf{r} = (6\sin\omega t)\mathbf{i} + (4\cos\omega t)\mathbf{j}$$

and ω is a positive constant.

a Find $\dot{\mathbf{r}}$ and hence show that

$$v^2 = 2\omega^2(13 + 5\cos 2\omega t),$$

where v m s^{-1} is the speed of R at time t seconds.

b Deduce that

$$4\omega \le v \le 6\omega.$$

c Find $\ddot{\mathbf{r}}$.

d At the instant when $t = \frac{\pi}{3\omega}$, find the angle between $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$, giving your answer in degrees to one decimal place.

Solution:

a

$$\mathbf{v} = \dot{\mathbf{r}} = (6\omega\cos\omega t)\mathbf{i} - (4\omega\sin\omega t)\mathbf{j}$$

$$v^2 = |\mathbf{v}|^2 = 36\omega^2\cos^2\omega t + 16\omega^2\sin^2\omega t$$

$$= 36\omega^2\left(\frac{1}{2}+\frac{1}{2}\cos 2\omega t\right) + 16\omega^2\left(\frac{1}{2}-\frac{1}{2}\cos 2\omega t\right)$$

$$= 18\omega^2 + 18\omega^2\cos 2\omega t + 8\omega^2 - 8\omega^2\cos 2\omega t$$

$$= 26\omega^2 + 10\omega^2\cos 2\omega t$$

$$= 2\omega^2(13 + 5\cos 2\omega t)\text{, as required}$$

Using the double angle formulae $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 2\theta = 1 - 2\sin^2\theta$.

b As $-1 \le \cos 2\omega t \le 1$

$$2\omega^2(13-5) \le 2\omega^2(13+5\cos 2\omega t) \le 2\omega^2(13+5)$$

$$16\omega^2 \le v^2 \le 36\omega^2$$

As $v, \omega > 0$

$4\omega \le v \le 6\omega$, as required

Both the $\cos x$ and $\sin x$ have the range $-1 \le x \le 1$. In this case the limits of $\cos 2\omega t$ limit the possible values of v^2 and, hence, of v.

c

$$\ddot{\mathbf{r}} = \frac{\mathrm{d}}{\mathrm{d}t}((6\omega\cos\omega t)\mathbf{i} - (4\omega\sin\omega t)\mathbf{j})$$

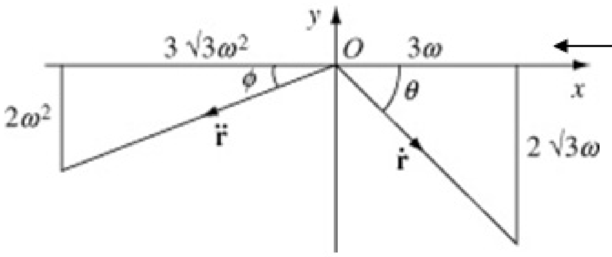
$$= -6\omega^2\sin\omega t\,\mathbf{i} - 4\omega^2\cos\omega t\,\mathbf{j}$$

d When $t = \frac{\pi}{3\omega}$

$$\dot{\mathbf{r}} = \left(6\omega\cos\frac{\pi}{3}\right)\mathbf{i} - \left(4\omega\sin\frac{\pi}{3}\right)\mathbf{j} = 3\omega\mathbf{i} - 2\sqrt{3}\omega\mathbf{j}$$

$$\ddot{\mathbf{r}} = -6\omega^2\sin\frac{\pi}{3}\mathbf{i} - 4\omega^2\cos\frac{\pi}{3}\mathbf{j} = -3\sqrt{3}\omega^2\mathbf{i} - 2\omega^2\mathbf{j}$$

Using $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.



A diagram is essential here. Once the diagram has been drawn, the problem reduces to basic trigonometry. You find the angles using the inverse tangent button on your calculator. The final angle could be given in degrees or radians (1.92^c).

$$\tan\theta = \frac{2\sqrt{3}\omega}{3\omega} = \frac{2\sqrt{3}}{3} \Rightarrow \theta = 49.106\ldots^\circ$$

$$\tan\phi = \frac{2\omega^2}{3\sqrt{3}\omega^2} = \frac{2}{3\sqrt{3}} \Rightarrow \phi = 21.051\ldots^\circ$$

The angle between $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ is

$(180 - 49.106\ldots - 21.501\ldots)^\circ = 109.8^\circ$ (1 d.p.).

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 21

Question:

Three particles of mass $3m$, $5m$ and λm are placed at the points with coordinates $(4,0)$, $(0,-3)$ and $(4, 2)$ respectively.

The centre of mass of the three particles is at $(2, k)$.

a Show that $\lambda = 2$.

b Calculate the value of k.

Solution:

The total mass is

$3m + 5m + \lambda m = (8 + \lambda)m$

a $\mathrm{M}(Oy)$

The moment of the total mass about any axis equals the sum of the moments of the individual masses about that axis. The distance of a particle from the y-axis is its x-coordinate.

$$(8+\lambda)m \times 2 = 3m \times 4 + 5m \times 0 + \lambda m \times 4$$

$$16 + 2\lambda = 12 + 4\lambda \Rightarrow 4 = 2\lambda$$

$$\lambda = 2$$

Divide throughout by m, simplify and remove the bracket.

b $\mathrm{M}(Ox)$

$$(8+\lambda)m \times k = 3m \times 0 + 5m \times -3 + \lambda m \times 2$$

$$10k = -15 + 4 \Rightarrow 10k = -11$$

$$k = -1.1$$

Divide by m and simplify using $\lambda = 2$.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 22

Question:

Particles of mass $2M$, xM and yM are placed at points whose coordinates are (2, 5), (1, 3) and (3, 1) respectively. Given that the centre of mass of the three particles is at the point (2, 4), find the values of x and y.

Solution:

The total mass is

$2M + xM + yM = (2 + x + y)M$

M(Oy)

$(2 + x + y)M \times 2 = 2M \times 2 + xM \times 1 + yM \times 3$

$4 + 2x + 2y = 4 + x + 3y$

$x - y = 0 \qquad (1)$

The moment of the total mass about any axis equals the sum of the moments of the individual masses about that axis. The distance of a particle from the y-axis is its x-coordinate.

Divide by M, simplify and collect together like terms to give the first of two simultaneous equations.

M(Ox)

$(2 + x + y)M \times 4 = 2M \times 5 + xM \times 3 + yM \times 1$

$8 + 4x + 4y = 10 + 3x + y$

$x + 3y = 2 \qquad (2)$

This is the second of the simultaneous equations. You can solve the simultaneous equations using any correct method.

$(2) - (1)$

$4y = 2 \Rightarrow y = \frac{1}{2}$

From (1)

$x = y = \frac{1}{2}$

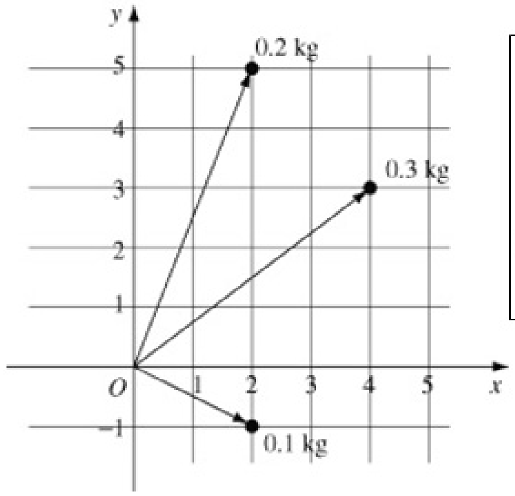
Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 23

Question:

Three particles of mass 0.1 kg, 0.2 kg and 0.3 kg are placed at the points with position vectors $(2\mathbf{i}-\mathbf{j})$ m, $(2\mathbf{i}+5\mathbf{j})$ m and $(4\mathbf{i}+2\mathbf{j})$ m respectively. Find the position vector of the centre of mass of the particles.

Solution:



This diagram illustrates the positions of the particles with respect to Cartesian axes. You take moments about the y-axis and the x-axis to find the x- and y-coordinates, respectively, of the centre of mass of the system of particles.

Let the position vector of the centre of mass be $(\bar{x}\mathbf{i}+\bar{y}\mathbf{j})$ m.

$\bar{x}$ and $\bar{y}$ are the standard symbols for the coordinates of a centre of mass.

The total mass is

$(0.1+0.2+0.3)\text{kg} = 0.6\text{ kg}$

M(Oy)

$$0.6\bar{x} = 0.1\times 2+0.2\times 2+0.3\times 4 = 1.8$$
$$\bar{x} = \frac{1.8}{0.6} = 3$$

The x-coordinates are the distances of the points from the y-axis.

M(Ox)

$$0.6\bar{y} = 0.1\times -1+0.2\times 5+0.3\times 2 = 1.5$$
$$\bar{y} = \frac{1.5}{0.6} = 2.5$$

The y-coordinates are the distances of the points from the x-axis.

The position vector of the centre of mass is $(3\mathbf{i}+2.5\mathbf{j})$ m.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 24

Question:

Three particles of mass $2M$, M and kM, where k is a constant, are placed at points with position vectors $6\mathbf{i}$ m, $4\mathbf{j}$ m and $(2\mathbf{i}-2\mathbf{j})$ m respectively. The centre of mass of the three particles has position vector $(3\mathbf{i}+c\mathbf{j})$ m, where c is a constant.

a Show that $k=3$.

b Hence find the value of c.

Solution:

The total mass is
$2M+M+kM=(3+k)M$

a $\mathrm{M}(Oy)$

$(3+k)M\times 3=2M\times 6+M\times 0+kM\times 2$

$9+3k=12+2k$

$k=3$, as required

The moment of the mass of a particle about an axis is the mass multiplied by the perpendicular distance from the particle to the axis. The particle of mass M has position vector $4\mathbf{j}$ m and so lies on Oy. So its moment about Oy is zero.

b $\mathrm{M}(Ox)$

$(3+k)M\times c=2M\times 0+M\times 4+kM\times -2$

$$\begin{aligned} \text{Using } k &= 3 \\ 6c &= 4-6=-2 \\ c &= -\tfrac{1}{3} \end{aligned}$$

Divide by M and use the result to **a**.

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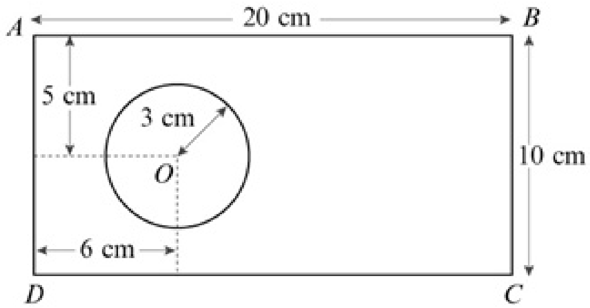
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 25

Question:



The figure shows a metal plate that is made by removing a circle of centre O and radius 3 cm from a uniform rectangular lamina $ABCD$, where $AB = 20$ cm and $BC = 10$ cm. The point O is 5 cm from both AB and CD and is 6 cm from AD.

a Calculate, to 3 significant figures, the distance of the centre of mass of the plate from AD.

The plate is freely suspended from A and hangs in equilibrium.

b Calculate, to the nearest degree, the angle between AB and the vertical.

Solution:

a

The area of the lamina is $20 \times 10 = 200 \text{ cm}^2$.

The area of the circle is $\pi \times 3^2 = 9\pi \text{ cm}^2$.

The area of the plate is $(200 - 9\pi) \text{ cm}^2$.

Let the distance of the centre of mass of the plate from AB be $\bar{x}$ cm.

	Lamina	Circle	Plate
Mass ratios	200	9π	$200 - 9\pi$
Distances	10	6	$\bar{x}$

As the lamina is uniform, masses are proportional to areas.

By symmetry, the centre of mass of the rectangular lamina is 10 cm from AB and the centre of mass of the circle is 6 cm from AB.

M(AD)

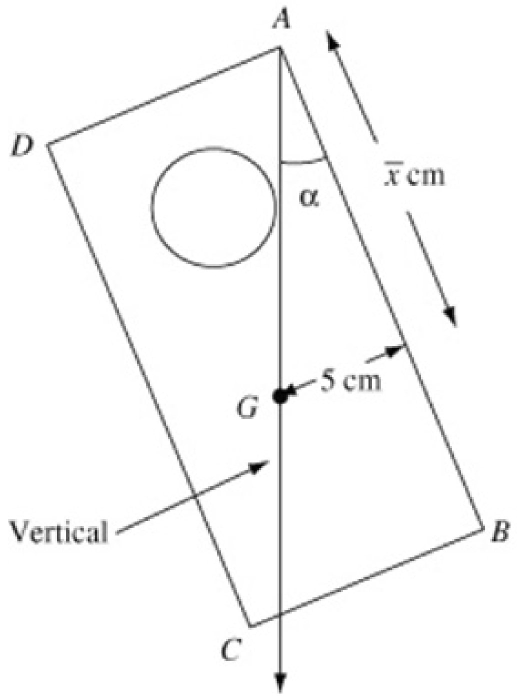
$$200 \times 100 = 9\pi \times 6 + (200 - 9\pi) \times \bar{x}$$

$$\bar{x} = \frac{2000 - 54\pi}{200 - 9\pi} = 10.658\ldots$$

The moment of the total mass (in this case the rectangle) about AD equals the sum of the moments of the circle and the plate about AD.

The distance of the centre of mass of the plate from AB is 10.7 cm (3 s.f.).

b Let the angle between AB and the vertical be α.



When the plate is suspended freely from A, its centre of mass G is vertically below the point of suspension A. The distance of G from AD was found in **a** and the distance of G from AB is 5 cm by symmetry. You calculate α using trigonometry.

$$\tan\alpha = \frac{5}{\bar{x}} = \frac{5}{10.658\ldots} = 0.4691\ldots$$

$$\alpha = 25.13\ldots^\circ$$

The angle between AB and the vertical is 25° (nearest degree).

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 26

Question:

A triangular frame ABC is made by bending a piece of wire of length 24 cm, so that AB, BC and AC are of lengths 6 cm, 8 cm and 10 cm respectively. Given that the wire is uniform, find the distance of the centre of mass of the frame from

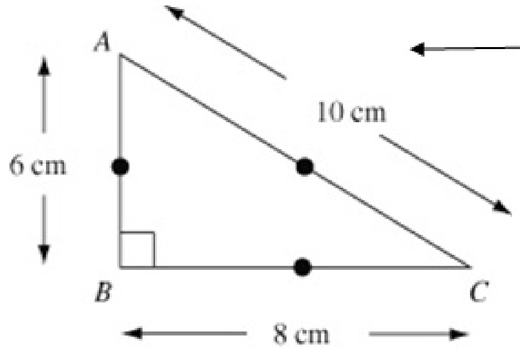
a AB,

b BC.

The frame is suspended from the corner A and hangs in equilibrium.

c Find, to the nearest degree, the acute angle made by AB with the downward vertical.

Solution:



When there is no diagram, it is a good idea to sketch one. You are expected to recognise that a triangle with sides in the ratio of 3 : 4 : 5 has a right angle.

As the wire is uniform, each side has a mass proportional to its length and the centre of mass of each side is at the middle of the side.

It is a common error to confuse a triangular frame with a solid triangle. A triangular frame consists of three straight line sections, each of which is considered separately.

Let the centre of mass G of the framework be $(\bar{x}, \bar{y})$.

Displaying the masses and coordinates in a table helps you to form correct equations.

	Framework	AB	BC	AC
Mass ratios	24	6	8	10
Distances (x)	$\bar{x}$	0	4	4
Distances (y)	$\bar{y}$	3	0	3

The centre of mass of each side is at the middle of the side.

a M(AB)

$$24\bar{x} = 6\times0+8\times4+10\times4=72$$

$$\bar{x} = \frac{72}{24}=3$$

b M(BC)

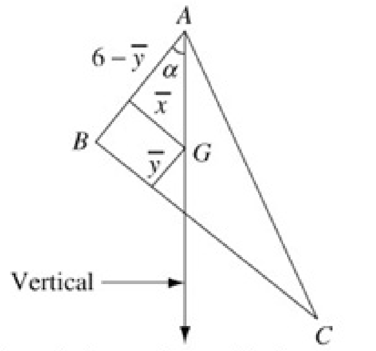
$$24\bar{y} = 6\times3+8\times0+10\times3=48$$

$$\bar{y} = \frac{48}{24}=2$$

c Let the acute angle made by AB with the downward vertical be α.

$$\tan\alpha = \frac{\bar{x}}{6-\bar{y}}=\frac{3}{6-2}=\frac{3}{4}$$

$$\alpha = 37^\circ \text{ (nearest degree).}$$



When the frame hangs freely from A, its centre of mass G is vertically below A.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 27

Question:

Three uniform rods AB, BC and CA of mass $2m$, m and $3m$ respectively have lengths l, l and $l\sqrt{2}$ respectively. The rods are rigidly joined to form a right-angled triangular framework.

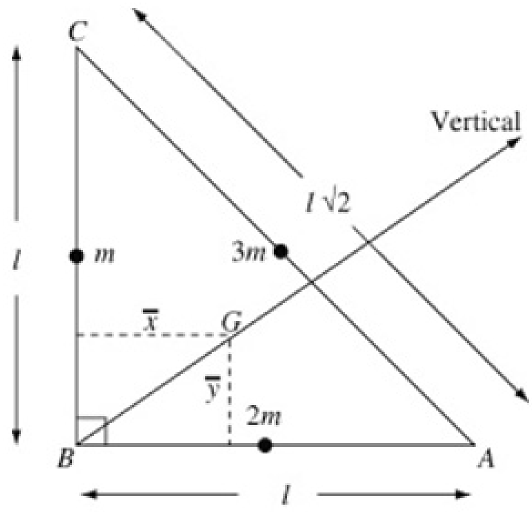
a Calculate, in terms of l, the distance of the centre of mass of the framework from

i BC,

ii AB.

b Calculate the angle, to the nearest degree, that BC makes with the vertical when the framework is freely suspended from the point B.

Solution:



As $l^2 + l^2 = (l\sqrt{2})^2$, the angle at B is a right angle.

As each rod is uniform, the centre of mass of each rod is at its mid-point. You can think of this as replacing each rod by a particle of the appropriate mass at the mid-point of the rod.

a The total mass is

$2m + m + 3m = 6m$

	Total	AB	BC	CA
Mass	$6m$	$2m$	m	$3m$
Distances (x)	$\bar{x}$	$\frac{l}{2}$	0	$\frac{l}{2}$
Distances (y)	$\bar{y}$	0	$\frac{l}{2}$	$\frac{l}{2}$

A table displays the masses and coordinates in a concise form and helps you form correct equations.

Let the centre of mass G of the framework be $(\bar{x}, \bar{y})$.

i M(AB)

$$6m \times \bar{x} = 2m \times \frac{l}{2} + m \times 0 + 3m \times \frac{l}{2} = \frac{5ml}{2}$$

$$\bar{x} = \frac{5ml}{12m} = \frac{5l}{12}$$

The moment of the total mass about any axis equals the sum of the moments of the individual masses about that axis.

ii M(BC)

$$6m \times \bar{y} = 2m \times 0 + m \times \frac{l}{2} + 3m \times \frac{l}{2} = 2ml$$

$$\bar{y} = \frac{2ml}{6m} = \frac{l}{3}$$

b Let the angle that BC makes with the vertical be α.

$$\tan\alpha = \frac{\bar{y}}{\bar{x}} = \frac{\frac{l}{3}}{\frac{5l}{12}} = \frac{4}{5}$$

$$\alpha = 39^\circ \text{ (nearest degree).}$$

When the framework hangs freely from A, its centre of mass G is vertically below A.
The vertical line has been drawn in the diagram.

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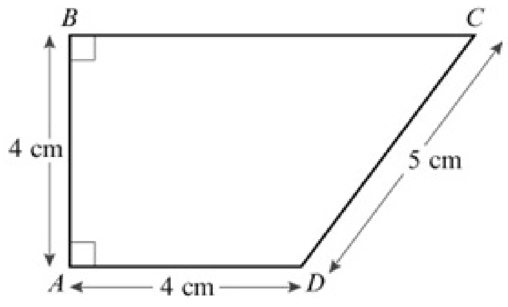
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 28

Question:



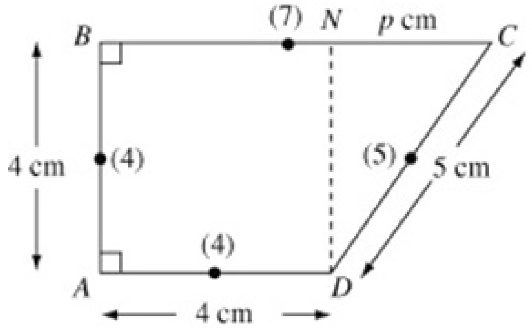
A thin uniform wire of total length 20 cm, is bent to form a frame. The frame is in the shape of a trapezium $ABCD$, where $AB = AD = 4$ cm, $CD = 5$ cm and AB is perpendicular to BC and AD, as shown in the figure.

a Find the distance of the centre of mass of the frame from AB.

The frame has mass M. A particle of mass kM is attached to the frame at C. When the frame is freely suspended from the midpoint of BC, the frame hangs in equilibrium with BC horizontal.

b Find the value of k.

Solution:



a Let N be the foot of the perpendicular from D to BC and $NC = p$ cm .

By Pythagoras' Theorem

$p^2 = 5^2 - 4^2 = 9 \Rightarrow p = 3$

Hence $BC = BN + NC = (4+3)$ cm $= 7$ cm

The total length of the frame is
$(4+7+5+4)$ cm $= 20$ cm

Let the distance of the centre of mass of the frame from AB be $\bar{x}$ cm

	Total	AB	BC	CD	DA
Mass ratios	20	4	7	5	4
Distances	$\bar{x}$	0	3.5	5.5	2

As the wire is uniform, each section of the wire has a mass proportional to its length. As the ratios of the lengths are 4 : 7 : 5 : 4, the mass ratios are also 4 : 7 : 5 : 4.

The distance of the centre of mass of CD from AB is

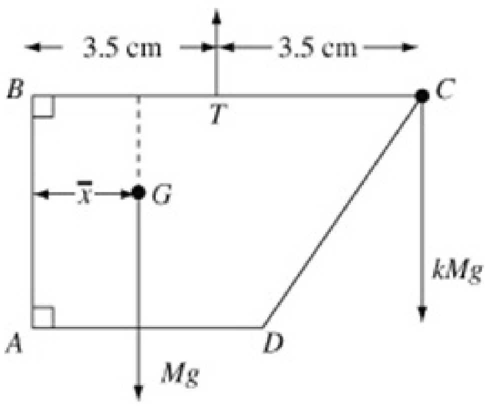
$BN + \frac{1}{2}NC = (4 + 1.5)$ cm $= 5.5$ cm.

M(AB)

$$20\bar{x} = 4\times 0 + 7\times 3.5 + 5\times 5.5 + 4\times 2 = 60$$

$$\bar{x} = \frac{60}{20} = 3$$

b



The whole weight of the framework acts at the centre of mass G. There is also a force acting upwards at the mid-point, say T, of BC, but, when you take moments about T, this force has zero moment about T.

Let T be the mid-point of BC M(T)

$$\cancel{Mg}(3.5 - \bar{x}) = k\,\cancel{Mg} \times 3.5$$

$$3.5 - 3 = 3.5k$$

$$k = \frac{0.5}{3.5} = \frac{1}{7}$$

The perpendicular distance from T to the line of action of the weight of the framework is $3.5 - \bar{x} = 0.5$ cm .

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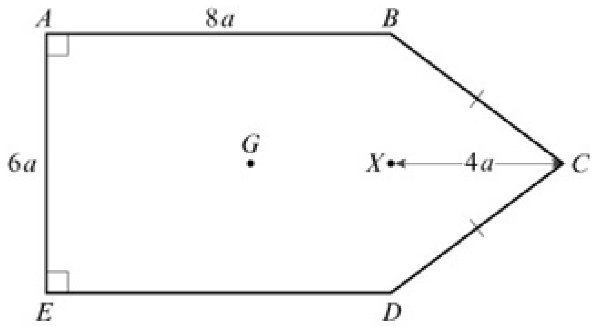
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 29

Question:



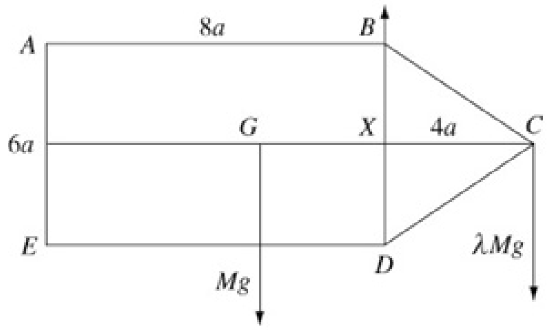
The figure shows a uniform lamina $ABCDE$ such that $ABDE$ is a rectangle, $BC = CD$, $AB = 8a$ and $AE = 6a$. The point X is the mid-point of BD and $XC = 4a$. The centre of mass of the lamina is at G.

a Show that $GX = \frac{44}{15}a$.

The mass of the lamina is M. A particle of mass λM is attached to the lamina at C. The lamina is suspended from B and hangs freely under gravity with AB horizontal.

b Find the value of λ.

Solution:



a The area of rectangle $ABDE$ is $6a \times 8a = 48a^2$

The area of ΔBCD is $\frac{1}{2} \times 6a \times 4a = 12a^2$

The area of lamina $ABCDE$ is $48a^2 + 12a^2 = 60a^2$

You break the lamina up into two parts. The rectangle $ABDE$, with dimensions $8a \times 6a$, and the triangle BCD, with base $6a$ and height $4a$.

	Lamina	Rectangle	Triangle
Mass ratios	$60a^2$	$48a^2$	$12a^2$
Displace-ments from X	GX	$4a$	$-\frac{4}{3}a$

The centre of mass of the triangle is $\frac{1}{3}h = \frac{1}{3} \times 4a$ from the base BD of the triangle.

M(X)

$$60a^2 \times GX = 48a^2 \times 4a + 12a^2 \times \left(-\frac{4}{3}a\right)$$

$$= 192a^3 - 16a^3 = 176a^3$$

$$GX = \frac{176a^3}{60a^2} = \frac{44}{15}a, \text{ as required}$$

Taking moments about X, the rectangle is on one side of X, here taken as positive, and the triangle is on the other side of X, here taken as negative.

b With the particle at C

M(X)

$$Mg \times GX = \lambda Mg \times 4a$$

$$\cancel{Mg} \times \frac{44}{15}\cancel{a} = \lambda \cancel{Mg} \times 4\cancel{a}$$

$$\lambda k = \frac{44}{15 \times 4} = \frac{11}{15}$$

There is a vertical force at B but, as the line of action of this force passes through X, its moment about X is zero.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 30

Question:

A uniform square plate $ABCD$ has mass $10M$ and the length of a side of the plate is $2l$. Particles of mass M, $2M$, $3M$ and $4M$ are attached at A, B, C and D respectively. Calculate, in terms of l, the distance of the centre of mass of the loaded plate from

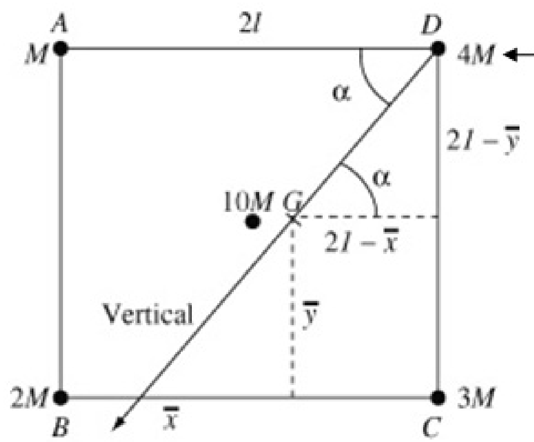
a AB,

b BC.

The loaded plate is freely suspended from the vertex D and hangs in equilibrium.

c Calculate, to the nearest degree, the angle made by DA with the downward vertical.

Solution:



The angle required in **c** is marked α in the diagram. It has been brought into a right angled triangle, in which you can use trigonometry, using alternate (Z) angles.

The total mass is

$$M + 2M + 3M + 4M + 10M = 20M$$

Let the distance of the centre of mass, G say, of the loaded plate from AB and BC be $\bar{x}$ cm and $\bar{y}$ cm respectively.

Total	Plate	A	B	C	D	
Mass	$20M$	$10M$	M	$2M$	$3M$	$4M$
Distances (x)	$\bar{x}$	l	0	0	$2l$	$2l$
Distances (y)	$\bar{y}$	l	$2l$	0	0	$2l$

A table displays the masses and coordinates in a concise form and helps you form correct equations.

a M(AB)

$$20M \times \bar{x} = 10M \times l + 3M \times 2l + 4M \times 2l = 24Ml$$

$$\bar{x} = \frac{24Ml}{20M} = \frac{6l}{5}$$

b M(BC)

$$20M \times \bar{y} = 10M \times l + M \times 2l + 4M \times 2l = 20Ml$$

$$\bar{y} = \frac{20Ml}{20M} = l$$

c Let the angle made by DA with the downward vertical be α.

$$\tan\alpha = \frac{2l - \bar{y}}{2l - \bar{x}} = \frac{2l - l}{2l - \frac{6l}{5}} = \frac{l}{\frac{4l}{5}} = \frac{5}{4}$$

$\alpha = 51°$ (nearest degree).

When the framework hangs freely from D, its centre of mass G is vertically below D. The vertical line has been drawn in the diagram.

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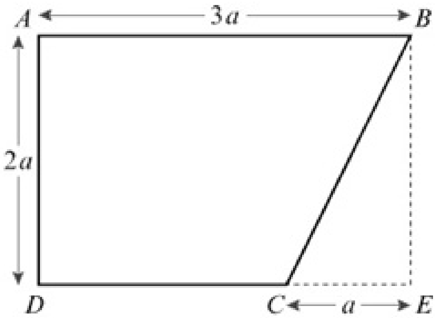
Review Exercise
Exercise A, Question 31

Question:



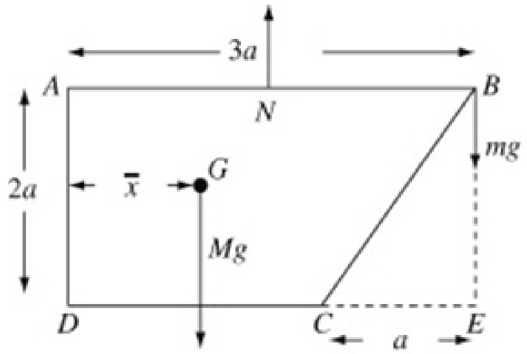
A uniform lamina $ABCD$ is made by taking a uniform sheet of metal in the form of a rectangle $ABED$, with $AB = 3a$ and $AD = 2a$, and removing the triangle BCE, where C lies on DE and $CE = a$, as shown in the figure.

a Find the distance of the centre of mass of the lamina from AD.

The lamina has mass M. A particle of mass m is attached to the lamina at B. When the loaded lamina is freely suspended from the midpoint of AB, it hangs in equilibrium with AB horizontal.

b Find m in terms of M.

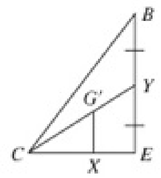
Solution:



a The area of rectangle $ABCD$ is $3a \times 2a = 6a^2$.
The area of triangle BCE is $\frac{1}{2}a \times 2a = a^2$.
The area of lamina $ABCD$ is $6a^2 - a^2 = 5a^2$.

Let the distance of the centre of mass of the lamina, say G, from AD be $\bar{x}$ cm.

	Rectangle	Lamina	Triangle
Mass ratio	$6a^2$	$5a^2$	a^2
Distances	$\frac{3}{2}a$	$\bar{x}$	$\frac{8}{3}a$



If G' is the centre of mass of the triangle and Y is the mid-point of BE then $CG' : G'Y = 2:1$.
Using similar triangles,
$\frac{CX}{XE} = \frac{CG'}{G'Y} = \frac{2}{1}$ and, as $CE = a$, $XE = \frac{1}{3}a$. So G' is $\frac{1}{3}a$ from BE and $\left(3a - \frac{1}{3}a\right) = \frac{8}{3}a$ from AD.

M(AD)

$$6a^2 \times \frac{3}{2}a = 5a^2 \times \bar{x} + a^2 \times \frac{8}{3}a$$
$$5a^2\bar{x} = 9a^3 - \frac{8}{3}a^3 = \frac{19}{3}a^3$$
$$\bar{x} = \frac{19}{15}a$$

The moment of the complete rectangle about AD is the sum of the moments of the lamina and the triangle which has been removed from the rectangle.

b Let N be the mid-point of AB
M (N)

$$M\cancel{g}\left(\frac{3}{2}a - \bar{x}\right) = m\cancel{g} \times \frac{3}{2}a$$
$$\frac{3}{2}M\cancel{a} - \frac{19}{15}M\cancel{a} = \frac{3}{2}m\cancel{a}$$
$$\frac{3}{2}m = \left(\frac{3}{2} - \frac{19}{15}\right)M = \frac{7}{30}M$$
$$m = \frac{2}{3} \times \frac{7}{30}M = \frac{7}{45}M$$

There is a vertical force at N holding the loaded lamina in equilibrium but this has no moment about N.

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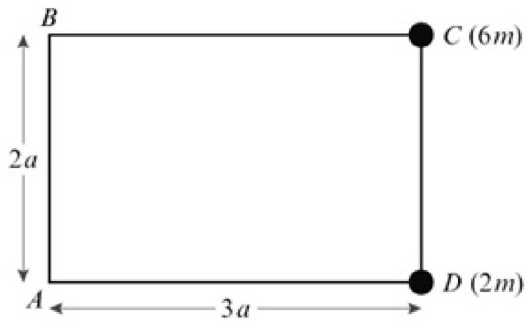
Review Exercise
Exercise A, Question 32

Question:



The figure shows four uniform rods joined to form a rectangular framework $ABCD$, where $AB = CD = 2a$ and $BC = AD = 3a$. Each rod has mass m. Particles of mass $6m$ and $2m$ are attached to the framework at points C and D respectively.

a Find the distance of the centre of mass of the loaded framework from

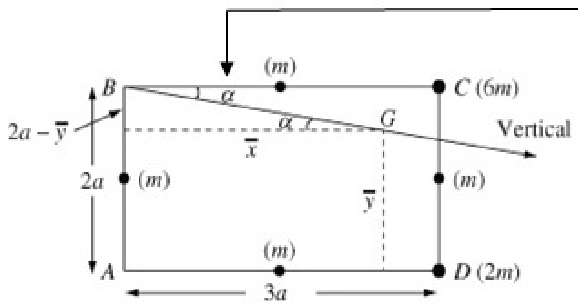
i AB,

ii AD.

The loaded framework is freely suspended from B and hangs in equilibrium.

b Find the angle which BC makes with the vertical.

Solution:



The angle required in **b** is marked α in the diagram. It has been brought into a right angled triangle, in which you can use trigonometry, using alternate (Z) angles.

a The total mass is

$m+m+m+m+2m+6m=12m$

Let the distance of the centre of mass of the loaded framework, say G, from AB and AD be $\bar{x}$ and $\bar{y}$ respectively.

It is a common error to confuse a rectangular frame with a solid rectangle.
The frame consists of four straight sections, each of which is considered separately.

		Rods				Particles	
	Total	AB	BC	CD	DA	C	D
Masses	$12m$	m	m	m	m	$6m$	$2m$
Distances (x)	$\bar{x}$	0	$\frac{3}{2}a$	$3a$	$\frac{3}{2}a$	$3a$	$3a$
Distances (y)	$\bar{y}$	a	$2a$	a	0	$2a$	0

There is a lot of information in this question and summarising it in a table helps you both to check that you have not overlooked anything and to obtain correct moments equations.

i M (AB)

$$12m\times\bar{x} = m\times\frac{3}{2}a+m\times 3a+m\times\frac{3}{2}a+6m\times 3a+2m\times 3a=30ma$$

$$\bar{x} = \frac{30ma}{12m}=\frac{5}{2}a$$

ii M (AD)

$$12m\times\bar{y} = m\times a + m\times 2a + m\times a + 6m\times 2a = 16ma$$

$$\bar{y} = \frac{16ma}{12m} = \frac{4}{3}a$$

b Let α be the angle BC makes with the vertical.

When the loaded framework hangs freely from B, its centre of mass G is vertically below B. The downward vertical has been drawn in the diagram.

$$\tan\alpha = \frac{2a-\bar{y}}{\bar{x}}$$

$$= \frac{2a-\frac{4}{3}a}{\frac{5}{2}a} = \frac{2}{3}\times\frac{2}{5} = \frac{4}{15}$$

$\alpha = 15°$ (nearest degree)

It is a good idea to write down an expression for the angle in terms of $\bar{x}$ and $\bar{y}$ and not to immediately use the expressions obtained in **a** in terms of a. Everyone makes mistakes from time to time and writing the expression using the general terms $\bar{x}$ and $\bar{y}$ makes your method clear.

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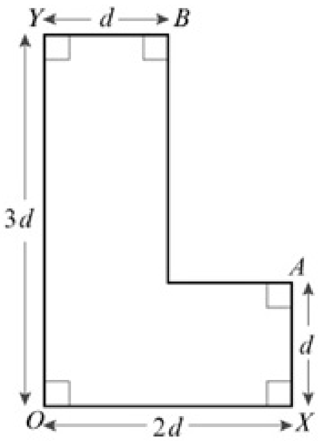
Review Exercise
Exercise A, Question 33

Question:



The figure shows a uniform L-shaped lamina with $OX = 2d, OY = 3d$ and $OX = YB = d$. The angles at O, A, B, X and Y are all right angles.

Find, in terms of d, the distance of the centre of mass of the lamina

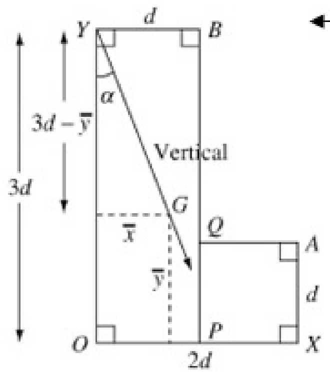
a from OX,

b from OY.

The lamina is suspended from the point Y and hangs freely in equilibrium.

c Find, to the nearest degree, the angle that OY makes with the vertical.

Solution:



You divide the L-shaped lamina into parts, each with a known centre of mass. This can be done in different ways. Here the lamina has been divided up into a rectangle $OXBP$ of dimensions $3d \times d$ and a square $AXPQ$ of side d.

The area of rectangle $OXBP$ is $3d \times d = 3d^2$.
The area of square $AXQP$ is $d \times d = d^2$.
The area of the L-shaped lamina is $3d^2 + d^2 = 4d^2$.
Let the distances of the centre of mass of the lamina, say G, from OX and OY be $\bar{x}$ and $\bar{y}$ respectively.

	Lamina	$OYBP$	$AXPQ$
Mass ratios	$4d^2$	$3d^2$	d^2
Distances (x)	$\bar{x}$	$\frac{1}{2}d$	$\frac{3}{2}d$
Distances (y)	$\bar{y}$	$\frac{3}{2}d$	$\frac{1}{2}d$

As the lamina is uniform, the masses of the lamina, rectangle and square are proportional to their areas. You could 'cancel' the d^2 here and just use 4 : 3 : 1. This would shorten the working a little.

a M (OX)

The distance of the centre of mass from OX is $\bar{y}$, not $\bar{x}$.

$$4d^2 \times \bar{y} = 3d^2 \times \frac{3}{2}d + d^2 \times \frac{1}{2}d = 5d^3$$

$$\bar{y} = \frac{5d^3}{4d^2} = \frac{5}{4}d$$

b M(OY)

$$4d^2 \times \bar{x} = 3d^2 \times \frac{1}{2}d + d^2 \times \frac{3}{2}d = 3d^3$$

$$\bar{x} = \frac{3d^3}{4d^2} = \frac{3}{4}d$$

c Let α be the angle OY makes with the vertical.

When the L-shaped lamina hangs freely from Y, its centre of mass G is vertically below Y.
The vertical has been drawn in the diagram.
The arrow is pointing downwards.

$$\tan\alpha = \frac{\bar{x}}{3d - \bar{y}} = \frac{\frac{3}{4}d}{3d - \frac{5}{4}d} = \frac{3}{4} \times \frac{4}{7} = \frac{3}{7}$$

$$\alpha = 23° \text{ (nearest degree)}$$

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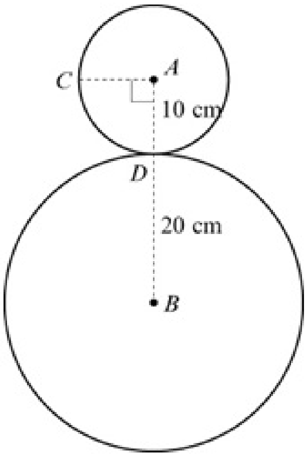
Review Exercise
Exercise A, Question 34

Question:



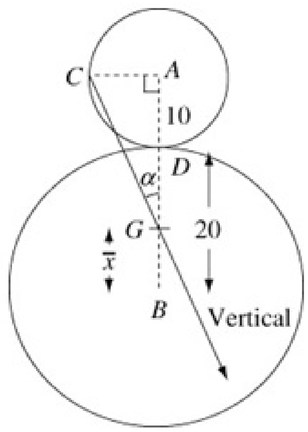
The figure shows a decoration which is made by cutting 2 circular discs from a sheet of uniform card. The discs are joined so that they touch at a point D on the circumference of both discs. The discs are coplanar and have centres A and B with radii 10 cm and 20 cm respectively.

a Find the distance of the centre of mass of the decoration from B.

The point C lies on the circumference of the smaller disc and $\angle CAB$ is a right angle. The decoration is freely suspended from C and hangs in equilibrium.

b Find, in degrees to one decimal place, the angle between AB and the vertical.

Solution:



a The area of the smaller circle is $\pi 10^2 = 100\pi \text{ cm}^2$.

The area of the larger circle is $\pi 20^2 = 400\pi \text{ cm}^2$.

The area of the decoration is $100\pi + 400\pi = 500\pi \text{ cm}^2$.

Let the distance of the centre of mass, say G, from B be $\bar{x}$ cm.

	Decoration	Small circle	Large circle
Mass ratios	500π	100π	400π
Distances from B (cm)	$\bar{x}$	30	0

As the card is uniform, the masses of the decoration and circles are proportional to their areas. You could simplify the ratio $500\pi : 100\pi : 400\pi$ to 5 : 1 : 4 and this would simplify your working.

The centre of the larger circle is at B.

M(B)

$$500\pi \times \bar{x} = 100\pi \times 30 + 400\pi \times 0$$

$$\bar{x} = \frac{3000\cancel{\pi}}{500\cancel{\pi}} = 6$$

The distance of the centre of mass of the decoration from B is 6 cm.

b Let the angle between AB and the vertical be α.

$$AC = 10 \text{ cm}$$

$$AG = (10 + (20 - 6)) \text{ cm} = 24 \text{ cm}$$

$$\tan\alpha = \frac{10}{24} = \frac{5}{12}$$

$$\alpha = 22.6° \text{ (1 d.p.)}$$

When the decoration is freely suspended from C, its centre of mass G is vertically below C.
The vertical has been drawn in the diagram with the arrow pointing directly down. You use the result of **a** and trigonometry in the right angled triangle GAC to complete the question.

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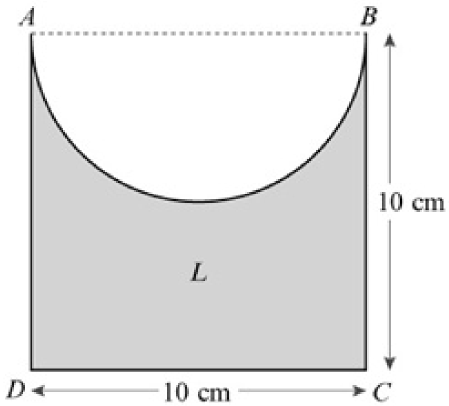
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 35

Question:



A uniform lamina L is formed by taking a uniform square sheet of material $ABCD$ of side 10 cm and removing a semicircle with diameter AB from the square, as shown in the figure.

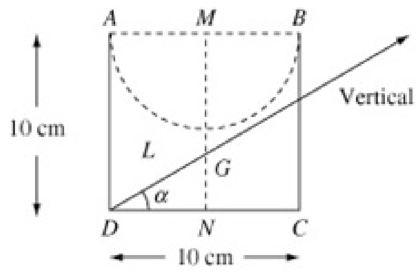
a Find, in cm to 2 decimal places, the distance of the centre of mass of the lamina from the midpoint of AB.

[The centre of mass of a uniform semicircular lamina, radius a, is at a distance $\frac{4a}{3\pi}$ from the centre of the bounding diameter.]

The lamina is freely suspended from D and hangs at rest.

b Find, in degrees to one decimal place, the angle between CD and the vertical.

Solution:



The area radius of the semi-circle is $\frac{1}{2}\pi r^2$ with $r = 5$.

a The area of the semi-circle is $\frac{1}{2}\pi \times 5^2 = \frac{25}{2}\pi \text{ cm}^2$.

The area of the square is $10 \times 10 = 100 \text{ cm}^2$.

The area of L is $\left(100 - \frac{25}{2}\right)\pi \text{ cm}^2$.

Let M be the mid-point of AB and N the mid-point of DC.
Let G be the centre of mass of L and $MG = \bar{x}$ cm

	Square	Semi-circle	L
Mass ratios	100	$\frac{25}{2}\pi$	$100 - \frac{25}{2}\pi$
Distances from M (cm)	5	$\frac{20}{3\pi}$	$\bar{x}$

The question gives the expression $\frac{4a}{3\pi}$ for the centre of mass of semi-circle and $a = 5$.

M (M)

$$100 \times 5 = \frac{25}{2}\cancel{\pi} \times \frac{20}{3\cancel{\pi}} + \left(100 - \frac{25}{2}\pi\right)\bar{x}$$

$$\left(100 - \frac{25}{2}\pi\right)\bar{x} = 500 - \frac{250}{3}$$

$$\bar{x} = \frac{500 - \frac{250}{3}}{100 - \frac{25}{2}\pi} = 6.860\,958\ldots$$

The question, by asking for 2 decimal places, implies there are no neat answers to this question and you must use your calculator to find the answers.

The distance of the centre of mass of L from the mid-point of AB is 6.86 cm (2 d.p.).

b Let the angle between CD and the vertical be α.

$$GN = (10 - \bar{x}) \text{ cm}, DN = 5 \text{ cm}$$

$$\tan\alpha = \frac{GN}{DN} = \frac{10 - 6.860}{5} = 0.6278\ldots$$

$$\alpha = 32.1^\circ \text{ (1 d.p.)}$$

When L is suspended freely from D, the centre of mass G hangs vertically below D. The downward vertical is drawn in the diagram.

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Review Exercise
Exercise A, Question 36

Question:

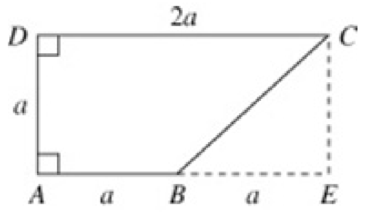
A uniform lamina $ABCD$ is in the form of a trapezium in which $AB = AD = a$, $CD = 2a$ and $\angle BAD = \angle ADC = 90°$

a Find the distance of the centre of mass of the lamina from AD and from AB.

The lamina stands with the edge AB on a plane inclined at an angle α to the horizontal with A higher than B. The lamina is in a vertical plane through a line of greatest slope of the plane.

b Given that the lamina is on the point of overturning about B, find the value of $\tan\alpha$.

Solution:



When there is no diagram in the question, sketch the diagram yourself. This makes the layout of the question clearer.

a Let the perpendicular from C to AB produced meet AB produced at E.

The area of rectangle $AECD = 2a \times a = 2a^2$.

The area of triangle $BEC = \frac{1}{2}a \times a = \frac{1}{2}a^2$.

The area of the lamina $= 2a^2 - \frac{1}{2}a^2 = \frac{3}{2}a^2$.

This solution treats the lamina as if it was made from a uniform rectangle, of dimensions a by $2a$, by removing the right angled triangle BEC. There are other equally valid alternatives.

Let the distance of the centre of mass of the lamina, say G, from AD and AB be $\bar{x}$ and $\bar{y}$ respectively.

	Rectangle	Triangle	Lamina
Mass ratios	$2a^2$	$\frac{1}{2}a^2$	$\frac{3}{2}a^2$
Distances (x)	a	$\frac{5}{3}a$	$\bar{x}$
Distances (y)	$\frac{1}{2}a$	$\frac{1}{3}a$	$\bar{y}$

G is $\frac{1}{3}a$ from CE and so it is $2a - \frac{1}{3}a = \frac{5}{3}a$ from AD.

M (AD)

$$2a^2 \times a = \frac{1}{2}a^2 \times \frac{5}{3}a + \frac{3}{2}a^2 \times \bar{x}$$

$$2a^3 = \frac{5}{6}a^3 + \frac{3}{2}a^2\bar{x} \Rightarrow \frac{3}{2}a^2\bar{x} = 2a^3 - \frac{5}{6}a^3 = \frac{7}{6}a^3$$

$$\bar{x} = \frac{7}{6} \times \frac{2}{3}a = \frac{7}{9}a$$

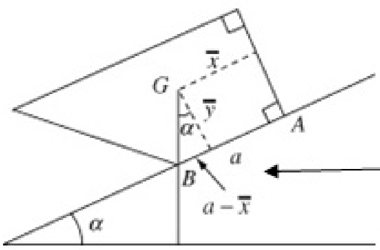
M (AD)

$$2a^2 \times \frac{1}{2}a = \frac{1}{2}a^2 \times \frac{1}{3}a + \frac{3}{2}a^2 \times \bar{y}$$

$$a^3 = \frac{1}{6}a^3 + \frac{3}{2}a^2\bar{y} \Rightarrow \frac{3}{2}a^2\bar{y} = a^3 - \frac{1}{6}a^3 = \frac{5}{6}a^3$$

$$\bar{y} = \frac{5}{6} \times \frac{2}{3}a = \frac{5}{9}a$$

b



When the lamina is on the point of overturning about B, the centre of mass G of the lamina is vertically above B.

$$\tan \alpha = \frac{a - \bar{x}}{\bar{y}} = \frac{\frac{2}{9}a}{\frac{5}{9}a} = \frac{2}{5}$$

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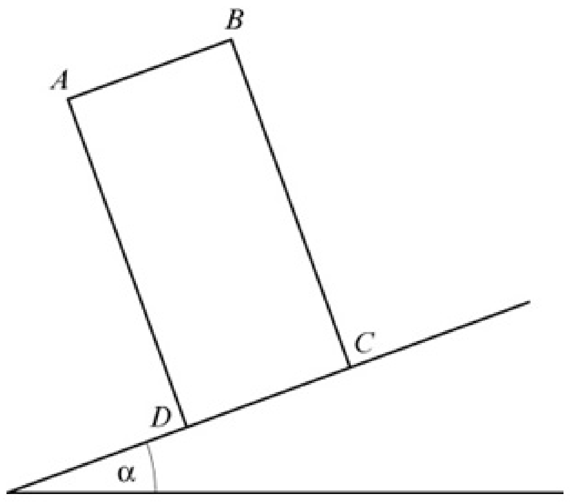
Review Exercise
Exercise A, Question 37

Question:



A thin uniform rectangular metal plate $ABCD$ of mass M rests on a rough plane inclined at an angle α to the horizontal. The plate lies in a vertical plane containing a line of greatest slope of the inclined plane, with the edge CD in contact with the plane and C further up the plane than D, as shown in the figure. The lengths of AB and BC are 10 cm and 30 cm respectively. The plane is sufficiently rough to prevent the plate from slipping.

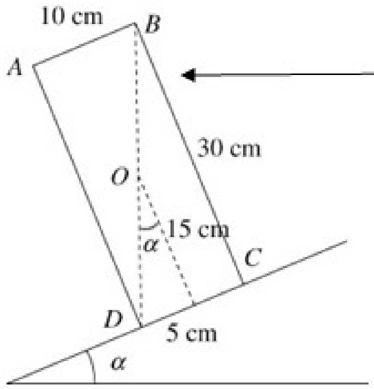
a Find, to the nearest degree, the greatest value which α can have if the plate does not topple.

A small stud of mass m is fixed to the plate at the point C.

b Given that $\tan \alpha = \frac{1}{2}$, find, in terms of M, the smallest value of m which will enable the plate to stay in equilibrium without toppling.

Solution:

a



The greatest value which α can take without toppling about D corresponds to the situation where the centre of plate, here labelled O, is vertically above D. As drawn here, if O was to the left of D, the weight of the plate would have an anticlockwise moment about D and the plate would rotate about D.

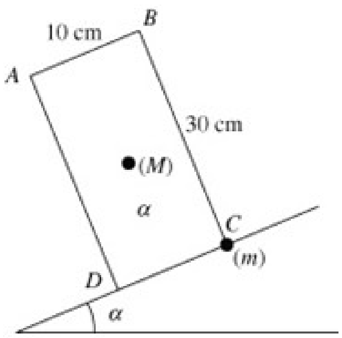
$$\tan\alpha = \frac{5}{15} = \frac{1}{3}$$

$$\alpha = 18^\circ \text{ (nearest degree)}$$

The point O is 5 cm from AD and 15 cm from CD.

The greatest value α can have if the plate does not topple is 18° (nearest degree).

b



The greatest value of α occurs when the centre of mass, say G, of the plate combined with the stud is vertically above D. The first step in this part must, therefore, be finding the position of G.

Let the centre of mass, say G, of the plate and stud combined be $\bar{x}$ cm and $\bar{y}$ cm from AD and DC respectively.

	Total	Plate	Stud
Mass	$M+m$	M	m
Distances (x)	$\bar{x}$	5	10
Distances (y)	$\bar{y}$	15	0

M (AD)

$$(M+m)\bar{x} = M\times 5 + m\times 10$$

$$\bar{x} = \frac{5M+10m}{M+m}$$

M (DC)

$$(M+m)\bar{y} = M\times 15$$

$$\bar{y} = \frac{15M}{M+m}$$

Given that $\tan\alpha = \frac{1}{2}$

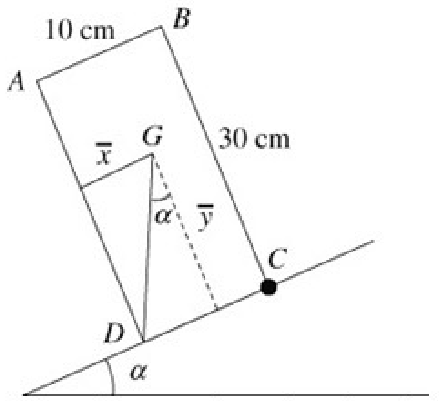
$$\tan\alpha = \frac{\bar{x}}{\bar{y}} = \frac{1}{2} \Rightarrow 2\bar{x} = \bar{y}$$

$$\frac{2(5M+10m)}{\cancel{M+m}} = \frac{15M}{\cancel{M+m}}$$

$15M$

$$10M + 20m = 15M \Rightarrow 20m = 5M \Rightarrow m = \frac{1}{4}M$$

The smallest value of m which will enable the plate to stay in equilibrium without toppling is $\frac{1}{4}M$.



When the system is on the point of toppling about D, the centre of mass of the plate and stud combined, G, is vertically above O.
The diagram above illustrates that $\tan\alpha = \frac{\bar{x}}{\bar{y}}$ and, as you are given that $\tan\alpha = \frac{1}{2}$, you can form an equation relating m and M.

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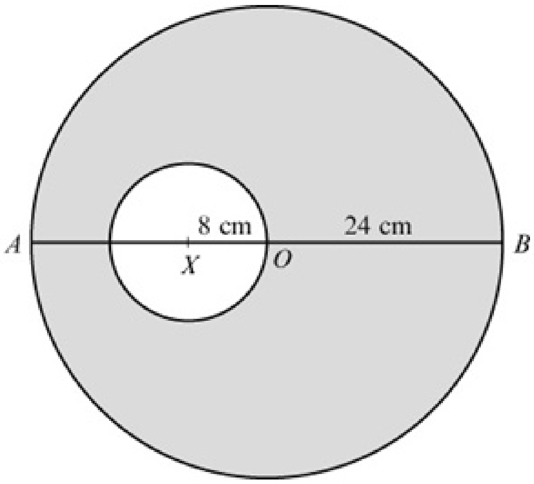
Review Exercise
Exercise A, Question 38

Question:



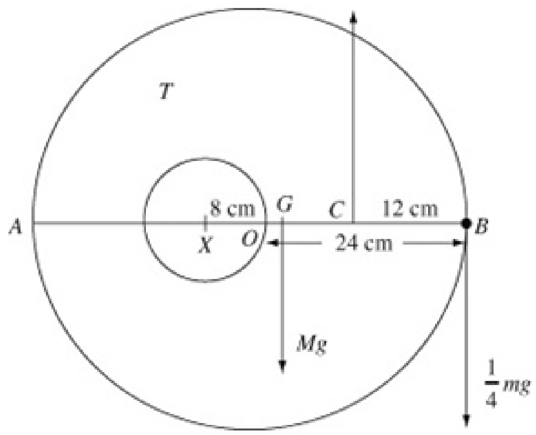
The figure shows a template T made by removing a circular disc, of centre X and radius 8 cm, from a uniform circular lamina, of centre O and radius 24 cm. The point X lies on the diameter AOB of the lamina and $AX = 16$ cm . The centre of mass of T lies at the point G.

a Find AG.

The template T is free to rotate about a smooth fixed horizontal axis, perpendicular to the plane of T, which passes through the midpoint of OB. A small stud of mass $\frac{1}{4}m$ is fixed at B, and T and the stud are in equilibrium with AB horizontal.

b Modelling the stud as a particle, find the mass of T in terms of m.

Solution:



a Let $AG = \bar{x}$ cm .

The area of the larger circle is $\pi \times 24^2 = 576\pi$ cm^2.

The area of the smallest circle is $\pi \times 8^2 = 64\pi$ cm^2.

The area of T is $576\pi - 64\pi = 64\pi$ cm^2.

$AX = (24 - 8)$ cm $= 16$ cm

	Larger circle	Smaller circle	T
Mass ratios	576π	64π	512π
Distance (cm)	24	16	$\bar{x}$

The large circle is uniform so the masses of the large circle, small circle and T are proportional to their areas.

The mass ratios $576\pi : 64\pi : 512\pi$ can be simplified to 9 : 1 : 8 and using these reduced ratios would ease the working.

M (A)

$576\cancel{\pi} \times 24 = 64\cancel{\pi} \times 16 + 512\cancel{\pi} \times \bar{x}$

$\bar{x} = \dfrac{576 \times 24 - 64 \times 16}{512} = 25$

$AG = 25$ cm

b Let C be the mid-point of OB and the mass of T be M.
$BC = 12$ cm, $CG = 11$ cm

M (C)

$M\cancel{g} \times 11 \quad = \quad \frac{1}{4} m\cancel{g} \times 12$

$11M \quad = \quad 3m \Rightarrow M = \frac{3}{11} m$

There is a force acting vertically upwards on T at C but, when moments are taken about C, this has zero moment.

The mass of T is $\frac{3}{11} m$.

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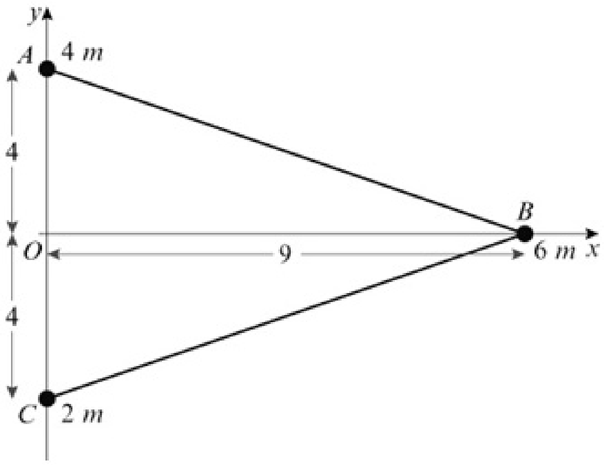
Review Exercise
Exercise A, Question 39

Question:



The figure shows a triangular lamina ABC. The coordinates of A, B and C are (0, 4), (9, 0) and (0, −4) respectively. Particles of mass $4m$, $6m$ and $2m$ are attached at A, B and C respectively.

a Calculate the coordinates of the centre of mass of the three particles, *without the lamina*.

The lamina ABC is uniform and of mass km. The centre of mass of the combined system consisting of the three particles and the lamina has coordinates $(4, \lambda)$.

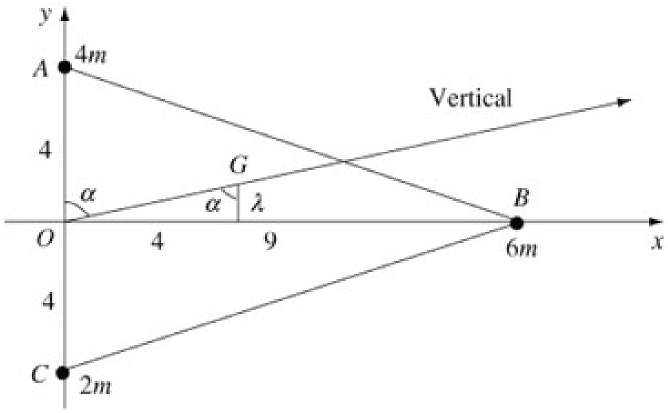
b Show that $k = 6$.

c Calculate the value of λ.

The combined system is freely suspended from O and hangs at rest.

d Calculate, in degrees to one decimal place, the angle between AC and the vertical.

Solution:



a Let the centre of mass of the three particles, without the lamina, be $(\bar{x}, \bar{y})$.
The total mass of the three particles is
$4m + 6m + 2m = 12m$

	Total	A	B	C
Mass	$12m$	$4m$	$6m$	$2m$
Distances (x)	$\bar{x}$	0	9	0
Distances (y)	$\bar{y}$	4	0	-4

You must take into account the signs of the coordinates.

M (Oy)

$$12m \times \bar{x} = 6m \times 9$$

$$\bar{x} = \frac{54m}{12m} = \frac{9}{2}$$

M (Ox)

$$12m \times \bar{y} = 4m \times 4 + 2m \times (-4) = 8m$$

$$\bar{y} = \frac{8m}{12m} = \frac{2}{3}$$

The centre of mass of the three particles, without the lamina, has coordinates $\left(\frac{9}{2}, \frac{2}{3}\right)$.

b The centre of mass of the lamina is (3, 0).
Let the centre of mass of the combined system of the lamina and the three particles be at the point G.

The question gives you that the coordinates of G are $(4, \lambda)$.

The total mass of the system is

$$12m + km = (12 + k)m$$

	Combined system	Particles	Lamina
Mass	$(12+k)m$	$12m$	km
Distances (x)	4	$\frac{9}{2}$	3
Distances (y)	λ	$\frac{2}{3}$	0

Although you could start again, considering the three particles separately, it is sensible to use the result from **a** for the centre of mass of the three particles.

M (Oy)

$(12+k)\cancel{m}\times 4 = 12\cancel{m}\times\frac{9}{2}+k\cancel{m}\times 3$

$48+4k = 54+3k$

$k=6$, as required

c M (Ox)

$(12+k)m\times\lambda = 12m\times\frac{2}{3}+km\times 0$

Using $k=6$

$18\lambda = 8$

$\lambda = \frac{8}{18}=\frac{4}{9}$

d Let the angle between AC and the vertical be α.

$\tan\alpha = \frac{4}{\lambda} = 4\div\frac{4}{9} = 9$

$\alpha = 83.7°$ (1 d.p.)

When the particle is freely suspended from O, the centre of mass of the combined system G hangs vertically below O.

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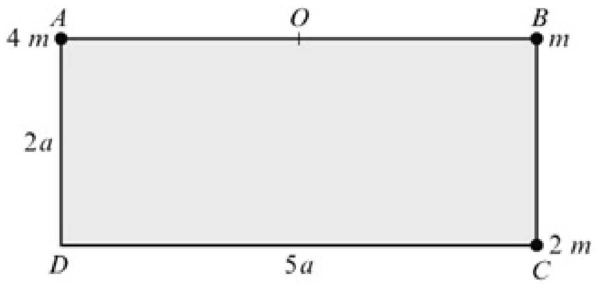
Review Exercise
Exercise A, Question 40

Question:



A loaded plate L is modelled as a uniform rectangular lamina $ABCD$ and three particles. The sides CD and AD of the lamina have length $5a$ and $2a$ respectively and the mass of the lamina is $3m$. The three particles have mass $4m$, m and $2m$ and are attached at the points A, B and C respectively, as shown in the figure.

a Show that the distance of the centre of mass of L from AD is $2.25a$.

b Find the distance of the centre of mass of L from AB.

The point O is the midpoint of AB. The loaded plate L is freely suspended from O and hangs at rest under gravity.

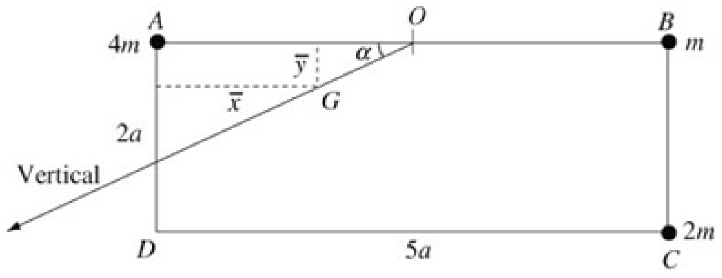
c Find, to the nearest degree, the size of the angle that AB makes with the horizontal.

A horizontal force of magnitude P is applied at C in the direction CD. The loaded plate L remains suspended from O and rests in equilibrium with AB horizontal and C vertically below B.

d Show that $P = \frac{5}{4}mg$.

e Find the magnitude of the force on L at O.

Solution:



Let the distances of the centre of mass of L, say G, from AD and AB be $\bar{x}$ and $\bar{y}$ respectively.

The distance $\bar{y}$ is measured from AB, not DC.

The mass of L is $3m+4m+m+2m=10m$.

		Rectangle	Particles		
	L	$ABCD$	A	B	C
Mass	$10m$	$3m$	$4m$	m	$2m$
Distances (x)	$\bar{x}$	$2.5a$	0	$5a$	$5a$
Distances (y)	$\bar{y}$	a	0	0	$2a$

a M (AD)

$$10m\times\bar{x} = 3m\times2.5a+m\times5a+2m\times5a=22.5ma$$

$$\bar{x} = \frac{22.5ma}{10m}=2.25a, \text{ as required}$$

b M (AB)

$$10m\times\bar{y} = 3m\times a+2m\times2a=7ma$$

$$\bar{y} = \frac{7ma}{10m}=0.7a$$

c Let α be the angle between OA and the vertical

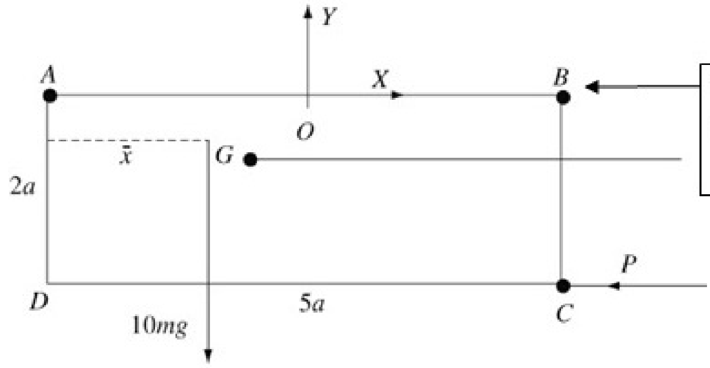
$$\tan\alpha = \frac{\bar{y}}{2.5a-\bar{x}}=\frac{0.7a}{0.25a}=2.8$$

$$\alpha = 70^\circ \text{ (nearest degree)}$$

When L is freely suspended from O, the centre of mass G of the complete system hangs vertically below O.

The angle that AB makes with the horizontal is $(90-70)^\circ=20^\circ$ (nearest degree).

d



The total weight $10mg$ of the loaded plate L acts at the centre of mass G of the plate.

M (O)

$$P \times 2a = 10mg \times (2.5a - \bar{x})$$
$$= 10mg \times 0.25a$$
$$P = \frac{2.5mg\cancel{a}}{2\cancel{a}} = \frac{5}{4}mg, \text{ as required}$$

Using $\bar{x} = 2.25a$ from **a**.

e Let the horizontal and vertical components of the force acting on L at O be X and Y respectively.

R($\rightarrow$) $X = P = \frac{5}{4}mg$

R($\uparrow$) $Y = 10mg$

Let the magnitude of the force acting on L at O be R.

$$R^2 = X^2 + Y^2$$
$$= \left(\frac{5}{4}mg\right)^2 + (10mg)^2 = \frac{1625}{4}m^2g^2$$
$$R = \frac{\sqrt{1625}}{4}mg = \frac{5\sqrt{65}}{4}mg$$

This exact answer or approximate answers, such as $10.1mg$, would be accepted.

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Review Exercise
Exercise A, Question 41

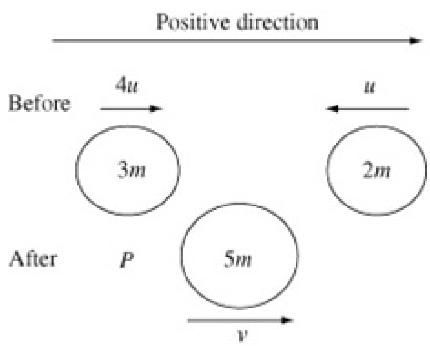
Question:

Two particles, of mass $3m$ and $2m$, are moving in opposite directions in a straight horizontal line with speeds $4u$ and u respectively. The particles collide and coalesce to form a single particle P. Calculate

a the speed of P in terms of u,

b the loss in kinetic energy, in terms of m and u, due to the collision.

Solution:



a Let the speed of P be v.

Conservation of linear momentum

$$4m\times 3m + 2m\times(-u) = 5mv$$
$$10mu = 5mv$$
$$v = \frac{10u}{5} = 2u$$

Part **a** of this question is in the M1 specification. The prerequisites in the M2 specification includes 'A knowledge of the specification for M1 … is assumed and may be tested'.

b The kinetic energy before impact is

$$\frac{1}{2}3m\times(4u)^2 + \frac{1}{2}2m\times u^2 = 24mu^2 + mu^2 = 25mu^2$$

The kinetic energy after impact is

$$\frac{1}{2}5mv^2 = \frac{1}{2}5m\times(2u)^2 = 10mu^2$$

The loss in kinetic energy is

$$25mu^2 - 10mu^2 = 15mu^2$$

The direction of motion of the particle of mass $2m$ is not relevant when working out its kinetic energy. In **a**, the direction was needed for calculating the linear momentum. Linear momentum is a vector quantity; kinetic energy is not.

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Review Exercise
Exercise A, Question 42

Question:

A particle of mass 4 kg is moving in a straight horizontal line. There is a constant resistive force of magnitude R newtons. The speed of the particle is reduced from 25 m s^{-1} to rest over a distance of 200 m.

Use the work–energy principle to calculate the value of R.

Solution:

The loss in kinetic energy is

$$\frac{1}{2}mu^2 = \frac{1}{2}4 \times 25^2 = 1250 \text{ J}$$

The initial kinetic energy of the particle is 1250 J. Its final kinetic energy is 0, as the particle has been brought to rest.

The work done by the resistive force is given by

$$\begin{aligned} \text{work done} &= \text{force} \times \text{distance moved} \\ W &= R \times 200 \end{aligned}$$

Using the work–energy principle

$$\begin{aligned} 200R &= 1250 \\ R &= \frac{1250}{200} = 6.25 \end{aligned}$$

The work done by the resistive force is equal to the loss of energy of the particle.

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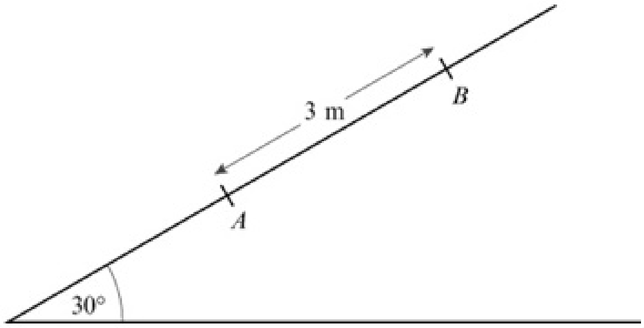
Review Exercise
Exercise A, Question 43

Question:



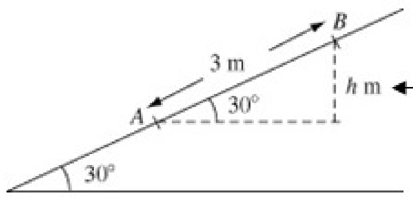
A particle P of mass 2 kg is projected from a point A up a line of greatest slope AB of a fixed plane. The plane is inclined at an angle of $30°$ to the horizontal and $AB = 3$ m with B above A, as shown in the figure. The speed of P at A is 10 m s^{-1}.

a Assuming the plane is smooth, find the speed of P at B.

The plane is now assumed to be rough. At A the speed of P is 10 m s^{-1} and at B the speed of P is 7 m s^{-1}.

b By using the work–energy principle, or otherwise, find the coefficient of friction between P and the plane.

Solution:



The change in the potential energy of P depends on the vertical distance it has moved. You find this using trigonometry.

a Let the vertical distance moved by P be h m.

$\frac{h}{3} = \sin 30° \Rightarrow h = 3\sin 30° = 1.5$

The potential energy gained by P is given by
$\text{PE} = mgh = 2 \times 9.8 \times 1.5 = 29.4$

Let the speed of P at B be v m s^{-1}.

The kinetic energy lost by P is given by

$$\begin{aligned} \text{K.E.} &= \tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2 \\ &= \tfrac{1}{2}2\times 10^2 - \tfrac{1}{2}2v^2 = 100 - v^2 \end{aligned}$$

Using the principle of conservation of energy

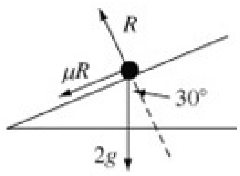
$$\begin{aligned} 100 - v^2 &= 29.4 \\ v^2 &= 100 - 29.4 = 70.6 \\ v &= \sqrt{70.6} = 8.402\ldots \end{aligned}$$

If no forces other than gravity are acting on the particle, as mechanical energy is conserved, the loss of kinetic energy must equal the gain in potential energy.

The speed of P at B is 8.4 m s^{-1} (2 s.f.).

As a numerical value of g has been used, you should round your final answer to 2 significant figures. Three significant figures are also acceptable.

b



Let the normal reaction between the particle and the plane have magnitude R N.

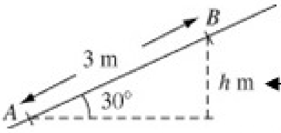
R(↖) $R = 2g\cos 30°$

The frictional force is given by

$F = \mu R = \mu 2g \cos 30°$

The kinetic energy lost by P is given by

$$\begin{aligned} \text{K.E.} &= \tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2 \\ &= \tfrac{1}{2}2\times 10^2 - \tfrac{1}{2}2\times 7^2 = 51 \end{aligned}$$



The change in the potential energy of P depends on the vertical distance it has moved. You find this using

The potential energy gained by P is the same as in **a**.

The vertical height moved is the same as in **a**.

The total loss of mechanical energy, in J, is
$51 - 29.4 = 21.6$

The work done by friction is given by

$$\begin{aligned} \text{work done} &= \text{force} \times \text{distance moved} \\ W &= \mu R \times 3 = \mu 2g \cos 30° \times 3 = \mu \times 50.922\ldots \end{aligned}$$

Using the work–energy principle

$\mu \times 50.922\ldots = 21.6 \Rightarrow \mu = 0.424\ldots$

The work done by the friction is equal to the total loss of energy of the particle.

The coefficient of friction is 0.42 (2 s.f.).

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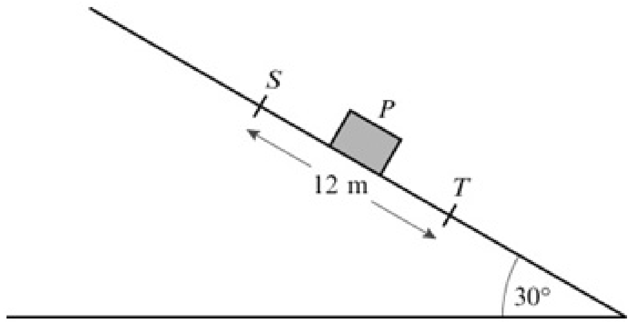
Review Exercise
Exercise A, Question 44

Question:



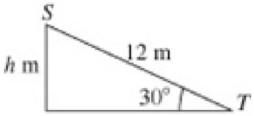
A small package is modelled as a particle P of mass 0.6 kg. The package slides down a rough plane from a point S to a point T, where $ST = 12$ m. The plane is inclined at $30°$ to the horizontal and ST is a line of greatest slope of the plane, as shown in the figure. The speed of P at S is 10 m s^{-1} and the speed of P at T is 9 m s^{-1}. Calculate

a the total loss of energy of P in moving from S to T,

b the coefficient of friction between P and the plane.

Solution:

a



In moving from S to T, P descends a vertical distance of h m, where

$\frac{h}{12} = \sin 30° \Rightarrow h = 12\sin 30° = 6$

The potential energy, in J, lost by P is given by

$mgh = 0.6 \times 9.8 \times 6 = 35.28$

The change in the potential energy of P depends on the vertical distance it has moved. You find this using trigonometry.

The kinetic energy, in J, lost by P is given by

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}m(u^2 - v^2)$$

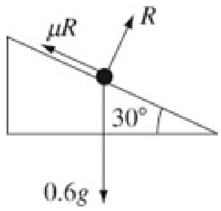
$$= \frac{1}{2} \times 0.6 \times (10^2 - 9^2) = 5.7$$

The total loss of energy of P is

$(35.28 + 5.7)$ J $= 40.98$ J $= 41$ J (2 s.f.).

As P moves from S to T both kinetic and potential energy are lost.

b



Let the normal reaction between the particle and the plane have magnitude R N.

R($\nearrow$) $R = 0.6g\cos 30°$

The frictional force is given by

$F = \mu R = \mu 0.6g\cos 30° = \mu \times 5.092\,29\ldots$

The work done by friction is given by

$$\text{work done} = \text{force} \times \text{distance moved}$$

$$W = F \times 12 = \mu \times 61.106\ldots$$

Using the work–energy principle

$\mu \times 61.106\ldots = 40.98 \Rightarrow \mu = 0.6706$

The coefficient of friction is 0.67 (2 s.f.).

Friction opposes motion and acts up the plane. The work done by friction against the motion of the particle equals the total loss of energy of the particle. You should use the unrounded answer from **a** for the total energy loss.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 45

Question:

A particle P has mass 4 kg. It is projected from a point A up a line of greatest slope of a rough plane inclined at an angle α to the horizontal, where $\tan \alpha = \frac{3}{4}$. The coefficient of friction between P and the plane is $\frac{2}{7}$. The particle comes to rest instantaneously at the point B on the plane, where $AB = 2.5$ m . It then moves back down the plane to A.

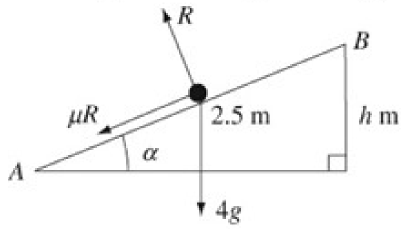
a Find the work done by friction as P moves from A to B.

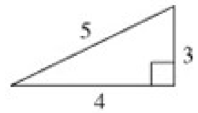
b Using the work–energy principle, find the speed with which P is projected from A.

c Find the speed of P when it returns to A.

Solution:

a $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$





This diagram illustrates that if $\tan\alpha = \frac{3}{4}, \sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$.

Let the normal reaction between the particle and the plane have magnitude R N.

R(↖) $R = 4g\cos\alpha = 4 \times 9.8 \times \frac{4}{5} = 31.36$

The work done by friction is given by

$$\begin{aligned} \text{work done} &= \text{force} \times \text{distance moved} \\ W &= \mu R \times AB \\ &= \frac{2}{7} \times 31.36 \times 2.5 = 22.4 \end{aligned}$$

The magnitude of the frictional force is given by $F = \mu R$ for the particle's motion both up and down the plane. The direction of the frictional force changes but its magnitude does not.

The work done by friction in moving from A to B is 22.4 J.

b Let the vertical distance moved by P in moving from A to B be h m.

$\frac{h}{2.5} = \sin\alpha \Rightarrow h = 2.5 \times \frac{3}{5} = 1.5$

The potential energy, in J, gained by P in moving from A to B is given by

$mgh = 4 \times 9.8 \times 1.5 = 58.8$

Let the speed of P at A be u m s^{-1}.

The kinetic energy, in J, lost by P in moving from A to B is given by

$\frac{1}{2}mu^2 = 2u^2$

At B the particle is instantaneously at rest and has no kinetic energy. So all of the initial kinetic energy has been lost.

The mechanical energy, in J, lost by P in moving from A to B is given by

$2u^2 - 58.8$

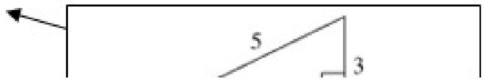
In moving from A to B, kinetic energy has been lost and potential energy gained. The difference between the values is the net loss.

Using the work–energy principle

$$\begin{aligned} 22.4 &= 2u^2 - 58.8 \\ u^2 &= \frac{58.8 + 22.4}{2} = 40.6 \\ u &= \sqrt{40.6} = 6.371\ldots \end{aligned}$$

The speed of P at A is 6.4 m s^{-1} (2 s.f.).

a $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$



R

c Let the speed of P when it returns to A be v m s^{-1}.

The work done by friction as P moves from B to A is the same that it does as P moves from A to B. Hence the total work done by friction is

$2 \times 22.4 = 44.8$ J

The work done depends on the normal reaction and the distance moved. Both the reaction and the distance are the same as the particle moves from A to B and from B to A. So you can find the total work done by friction by doubling your answer to **a**.

By the work–energy principle

$$\begin{aligned} 44.8 &= \tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2 \\ &= 2 \times 40.6 - 2v^2 \\ v^2 &= \frac{81.2 - 44.8}{2} = 18.2 \\ v &= \sqrt{18.2} = 4.266\ldots \end{aligned}$$

As the particle is now at the same level as it started, there is no change in its potential energy. So the change in mechanical energy is just the loss in kinetic energy.

The speed of P when it returns to A is 4.3 m s^{-1} (2 s.f.).

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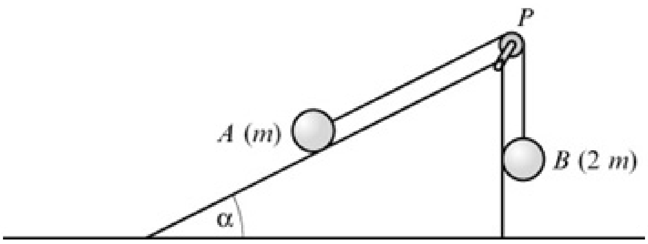
Review Exercise
Exercise A, Question 46

Question:



Two particles A and B of mass m and $2m$ respectively are attached to the ends of a light inextensible string. The particle A lies on a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The string passes over a small light pulley P fixed at the top of the plane. The particle B hangs freely below P, as shown in the figure. The particles are released from rest with the string taut and the section of the string from A to P parallel to a line of greatest slope of the plane. The coefficient of friction between A and the plane is $\frac{5}{8}$. When each particle has moved a distance h, B has not reached the ground and A has not reached P.

a Find an expression for the potential energy lost by the system when each particle has moved a distance h.

When each particle has moved a distance h, they are moving with speed v.

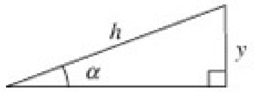
b Using the work–energy principle, find an expression for v^2, giving your answer in the form kgh where k is a number.

Solution:

a $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

You can sketch a 3, 4, 5 triangle to check these relations.

As B descends a distance h, A moves a distance h up the plane. Let the vertical displacement of A be y.



$\frac{y}{h} = \sin\alpha = \frac{3}{5} \Rightarrow y = \frac{3}{5}h$

The potential energy lost by B is $2mgh$.

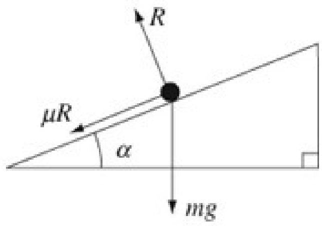
B has mass $2m$ and descends a distance h.

The potential energy gained by A is
$mgy = mg \times \frac{3}{5}h = \frac{3}{5}mgh$

A has mass m and ascends a vertical distance $\frac{3}{5}h$.

The net loss in potential energy of the system is
$2mgh - \frac{3}{5}mgh = \frac{7}{5}mgh$

b For A



Let the normal reaction between the particle and the plane have magnitude R N.
R(↖) $R = mg\cos\alpha = mg \times \frac{4}{5} = \frac{4}{5}mg$

The work done by friction is given by

$$\begin{aligned} \text{work done} &= \text{force} \times \text{distance moved} \\ W &= \mu R \times h \\ &= \frac{5}{8} \times \frac{4}{5}mg \times h = \frac{1}{2}mgh \end{aligned}$$

The gain in kinetic energy is

$\frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 = \frac{3}{2}mv^2$

Both particles start from rest, so the system has no initial kinetic energy.

The net loss of mechanical energy is

$\frac{7}{5}mgh - \frac{3}{2}mv^2$

The total loss in mechanic energy is the difference between the loss in potential energy you worked out in **a** less the kinetic energy gained.

Using the work-energy principle

$$\frac{1}{2}mgh = \frac{7}{5}mgh - \frac{3}{2}mv^2$$

$$\frac{3}{2}mv^2 = \left(\frac{7}{5} - \frac{1}{2}\right)mgh = \frac{9}{10}mgh$$

$$v^2 = \frac{2}{3} \times \frac{9}{10}gh = \frac{3}{5}gh$$

The work done by friction against the motion of the particle A equals the total loss of energy of the system. Other than gravity, there is no force acting on B. The only force causing the loss of mechanical energy is the friction acting on A.

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Review Exercise
Exercise A, Question 47

Question:

The engine of a car is working at a constant rate of 6 kW in driving a car along a straight horizontal road at 54 km h^{-1}. Find, in N, the magnitude of the resistance to motion of the car.

Solution:

Let F N be the magnitude of the driving force produced by the engine and R N be the magnitude of the resistance to motion.

$\text{R}(\rightarrow)\quad F - R = 0 \Rightarrow R = F$

This is Newton's Second Law $\mathbf{F} = m\mathbf{a}$. As the car is travelling at a constant speed the acceleration is zero.

$6 \text{ kW} = 6000 \text{ W}$

Before you use the formula $P = Fv$, you have to convert the data in kilowatts and kilometres an hour into base SI units.

$$54 \text{ km h}^{-1} = \frac{54 \times 1000}{3600} \text{ m s}^{-1} = 15 \text{ m s}^{-1}$$

$$P = Fv$$

Power = force × speed

$$6000 = F \times 15 \Rightarrow F = \frac{6000}{15} = 400$$

Hence $R = F = 400$

The resistance to motion has magnitude 400 N.

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Edexcel AS and A Level Modular Mathematics

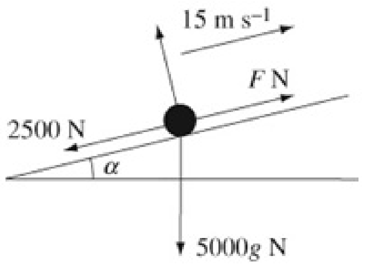
Review Exercise
Exercise A, Question 48

Question:

A lorry of mass 5000 kg moves at a constant speed of 15 m s^{-1} up a hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{15}$. The resistance experienced by the lorry is constant and has magnitude 2500 N.

Find, in kW, the rate of working of the lorry's engine.

Solution:



Let FN be the magnitude of the driving force produced by the engine.

R(↗) $\mathbf{F} = m\mathbf{a}$

$F - 2500 - 5000g\sin\alpha = 0$

$F = 2500 + 5000 \times 9.8 \times \frac{1}{15} = 2826.\dot{6}$

$P = Fv$

$= 2826.\dot{6} \times 15 = 42\,400$

The rate of working of the lorry's engine is 42 kW (2 s.f.).

As the lorry is travelling at a constant speed, its acceleration is zero.

The component of the weight down the plane is $5000g\cos(90-\alpha) = 5000g\sin\alpha$.

$P = Fv$ gives the answer in watts. To give the answer in kilowatts, as the question requires, you divide by 1000. Either 42 kW or 42.4 kW would be an acceptable answer.

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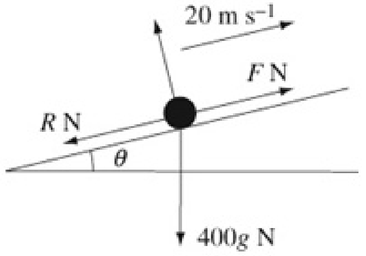
Review Exercise
Exercise A, Question 49

Question:

A car of mass 400 kg is moving up a straight road inclined at an angle θ to the horizontal where $\sin\theta = \frac{1}{14}$. The resistance to motion of the car from non-gravitational forces is modelled as a constant force of magnitude R newtons. When the car is moving at a constant speed of 20 m s^{-1}, the power developed by the car's engine is 10 kW.

Find the value of R.

Solution:



Let FN be the magnitude of the driving force produced by the engine.

$$10\text{ kW} = 10\,000\text{ W}$$
$$P = Fv$$
$$10\,000 = F \times 20 \Rightarrow F = 500$$

Before you use the formula $P = Fv$, you have to convert kilowatts to watts.

$$\text{R}(\nearrow)\quad \mathbf{F} = m\mathbf{a}$$
$$F - R - 400g\sin\theta = 0$$
$$R = F - 400g\sin\theta$$
$$= 500 - 400 \times 9.8 \times \frac{1}{14} = 220$$

As the car is travelling at a constant speed, its acceleration is zero.

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Review Exercise
Exercise A, Question 50

Question:

A lorry of mass 1500 kg moves along a straight horizontal road. The resistance to motion of the lorry has magnitude 750 N and the lorry's engine is working at a rate of 36 kW.

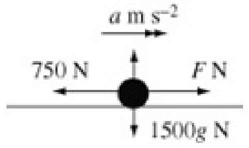
a Find the acceleration of the lorry when its speed is 20 m s^{-1}.

The lorry comes to a hill inclined at an angle α to the horizontal, where $\sin \alpha = \frac{1}{10}$.
The magnitude of the resistance to motion from non-gravitational forces remains 750 N. The lorry moves up the hill at a constant speed of 20 m s^{-1}.

b Find the rate at which the lorry is now working.

Solution:

a



Let the acceleration of the lorry be a m s^{-2} and the driving force of the engine have magnitude FN.

$$36\text{ kW} = 36\,000\text{ W}$$

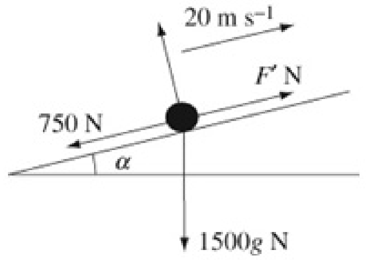
kW must be converted to W.

$$P = Fv$$
$$36\,000 = F \times 20 \Rightarrow F = 1800$$
$$\text{R}(\rightarrow)\quad \mathbf{F} = m\mathbf{a}$$
$$F - 750 = 1500a$$
$$1800 - 750 = 1500a$$
$$a = \frac{1800-750}{1500} = 0.7$$

The acceleration of the lorry when the speed is 20 m s^{-1} is 0.7 m s^{-2}.

This result is only true at one instant in time. The speed would now increase and the driving force and acceleration decrease.

b



Let the driving force of the engine have magnitude F' N.

The driving forces in **a** and **b** are different and it is a good idea to avoid confusion by using different symbols for the forces.

In this part of the question the lorry is moving at a constant speed and the acceleration is zero.

$$\text{R}(\nearrow)\quad \mathbf{F} = m\mathbf{a}$$
$$F' - 750 - 1500g\sin\alpha = 0$$
$$F' = 750 + 1500 \times 9.8 \times \frac{1}{10} = 2220$$
$$P = Fv$$
$$= 2220 \times 20 = 44\,400$$

The rate at which the lorry is now working is 44.4 kW.

This question asks for no particular form of the answer, so you could give your answer in either W or kW. Two or three significant figures are acceptable.

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Review Exercise
Exercise A, Question 51

Question:

A car of mass 1200 kg moves along a straight horizontal road. The resistance to motion of the car from non-gravitational forces is of constant magnitude 600 N. The car moves with constant speed and the engine of the car is working at a rate of 21 kW.

a Find the speed of the car.

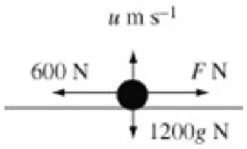
The car moves up a hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{14}$.

The car's engine continues to work at 21 kW and the resistance to motion from non-gravitational forces remains of magnitude 600 N.

b Find the constant speed at which the car moves up the hill.

Solution:

a



Let the speed of the car be u m s^{-1} and the driving force of the engine have magnitude F N.

$21\text{ kW} = 21\,000\text{ W}$

$$R(\rightarrow)\quad F - 600 = 0 \Rightarrow F = 600$$

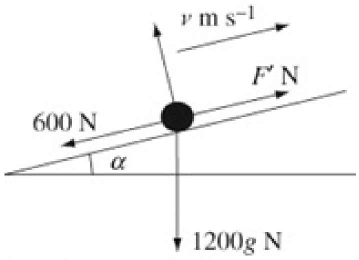
$$P = Fv$$

$$21\,000 = 600u \Rightarrow u = \frac{21\,000}{600} = 35$$

The speed of the car is 35 m s^{-1}.

In both parts of this question, as the car is moving with constant speed, the acceleration is zero. So the vector sum of the forces acting on the car is zero.

b



Let the speed of the car be u m s^{-1} and the driving force of the engine have magnitude F' N.

The driving forces in **a** and **b** are different and it is a good idea to avoid confusion by using different symbols for the forces.

$$R(\nearrow)\quad F' - 1200g\sin\alpha - 600 = 0$$

$$F' = 1200 \times 9.8 \times \frac{1}{14} + 600 = 1440$$

$$P = Fv$$

$$21\,000 = 1440v \Rightarrow v = 14.58\dot{3}$$

The constant speed of the car as it moves up the hill is 15 m s^{-1} (2 s.f.).

Although there is an exact answer, $14\frac{7}{12}$, a numerical value for g has been used in the question and the answer should be rounded to 2 significant figures. Three significant figures (14.6) is also acceptable.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 52

Question:

A car of mass 800 kg tows a caravan of mass 480 kg along a straight level road. The tow-bar connecting the car and the caravan is horizontal and of negligible mass. With the car's engine working at a rate of 24 kW, the car and caravan are travelling at a constant speed of 25 m s^{-1}.

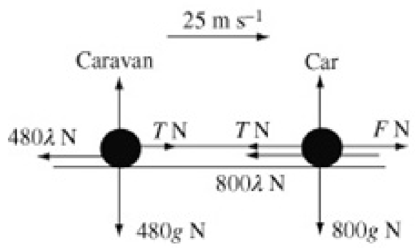
a Calculate the magnitude of the total resistance to the motion of the car and the caravan.

The resistance to the motion of the car has magnitude 800λ newtons and the resistance to the motion of the caravan has magnitude 480λ newtons, where λ is a constant. Find

b the value of λ,

c the tension in the tow-bar.

Solution:



a Let F N be the magnitude of the driving force produced by the engine of the car and R N be the total magnitude of the resistance to the motion of both the car and the caravan.

For the combined car and caravan
$R(\rightarrow)\ F - R = 0 \Rightarrow F = R$

When you consider the car and caravan combined, the tensions at the ends the tow-bar cancel one another out and can be ignored. When you consider the caravan and car separately, the tensions have to be included in your equations.

$$24\text{ kW} = 24\,000\text{ W}$$
$$P = Fv$$
$$24\,000 = F \times 25 \Rightarrow F = \frac{24\,000}{25} = 960$$
$$R = F = 960$$

The magnitude of the total resistance to the motion of the car and caravan is 960 N.

b

$$480\lambda + 800\lambda = 960$$
$$1280\lambda = 960 \Rightarrow \lambda = \frac{960}{1280} = 0.75$$

The sum of the resistances to the caravan and the car must equal the total resistance, 960 N.

c Let the magnitude of the tension in the tow-bar be T N.
For the caravan alone

You could consider the car alone. In that case your working would be
$F - T - 800\lambda = 0$
$T = F - 800\lambda = 960 - 800 \times 0.75 = 360$.

$$R(\rightarrow)\quad T - 480\lambda = 0$$
$$T = 480\lambda = 480 \times 0.75 = 360$$

The tension in the tow-bar is 360 N.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 53

Question:

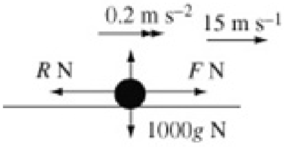
A car of mass 1000 kg is moving along a straight horizontal road. The resistance to motion is modelled as a constant force of magnitude R newtons. The engine of the car is working at a constant rate of 12 kW. When the car is moving with speed 15 m s^{-1}, the acceleration of the car is 0.2 m s^{-2}.

a Show that $R = 600$.

The car now moves with constant speed $U \text{ m s}^{-1}$ downhill on a straight road inclined at θ to the horizontal, where $\sin\theta = \frac{1}{40}$. The engine of the car is now working at a rate of 7 kW. The resistance to motion from non-gravitational forces remains of magnitude R newtons.

b Calculate the value of U.

Solution:

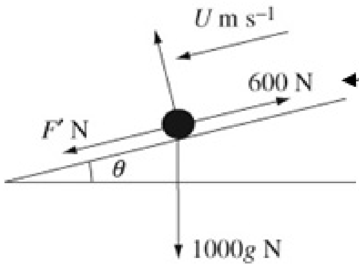


a Let F N be the magnitude of the driving force produced by the engine of the car.

$$\begin{aligned}
12\,\text{kW} &= 12\,000\ \text{W} \\
P &= Fv \\
12\,000 &= F \times 15 \\
F &= \frac{12\,000}{15} = 800 \\
\text{R}(\rightarrow)\quad \mathbf{F} &= m\mathbf{a} \\
F - R &= 1000 \times 0.2 \\
R &= F - 1000 \times 0.2 \\
&= 800 - 200 = 600, \text{ as required}
\end{aligned}$$

Using Newton's second law, the vector sum of the forces on the car equals the mass times acceleration.

b



Resistance acts against motion. As the car is travelling down the hill, the resistance of 600 N acts up the hill.

Let the driving force of the engine have magnitude F' N.

$$\text{R}(\swarrow)\ F' + 1000g\sin\theta - 600 = 0$$

$$F' = 600 - 1000 \times 9.8 \times \frac{1}{40} = 355$$

As the car is travelling at a constant speed, there is no acceleration.

$$\begin{aligned}
7\,\text{kW} &= 7000\ \text{W} \\
P &= Fv \\
7000 &= 355U \\
U &= \frac{7000}{355} = 19.718\ldots = 20\ (2\ \text{s.f.})
\end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 54

Question:

A car of mass 1000 kg is moving along a straight road with constant acceleration f m s^{-2}. The resistance to motion is modelled as a constant force of magnitude 1200 N. When the car is travelling at 12 m s^{-1}, the power generated by the engine of the car is 24 kW.

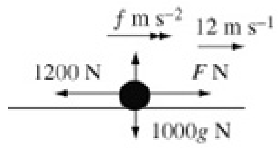
a Calculate the value of f.

When the car is travelling at 14 m s^{-1}, the engine is switched off and the car comes to rest without braking in a distance d metres.

b Assuming the same model for resistance, use the work–energy principle to calculate the value of d.

c Give a reason why the model used for resistance may not be realistic.

Solution:



a Let F N be the magnitude of the driving force produced by the engine of the car.

$$24\text{ kW} = 24\,000\text{ W}$$
$$P = Fv$$
$$24\,000 = F \times 12 \Rightarrow F = 2000$$

$$\text{R}(\rightarrow)\quad \mathbf{F} = m\mathbf{a}$$
$$F - 1200 = 1000f$$
$$2000 - 1200 = 1000f \Rightarrow f = \frac{800}{1000} = 0.8$$

b The kinetic energy, in J, lost as the car is brought to rest is

$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}1000 \times 14^2 = 98\,000$$

The final kinetic energy is zero.

$$\text{Work done by resistance} = \text{Energy lost}$$
$$\text{Resistance} \times \text{distance} = \text{Energy lost}$$
$$1200d = 98\,000$$
$$d = \frac{98\,000}{1200} = 81\tfrac{2}{3}$$

You use the work-energy principle. The work done by the resistance (1200 N) in bringing the car to rest is equal to the kinetic energy lost.

c Resistance usually varies with speed.

As the speed slows down, the resistance to motion usually decreases. In this case, this might mean that the car would travel further.

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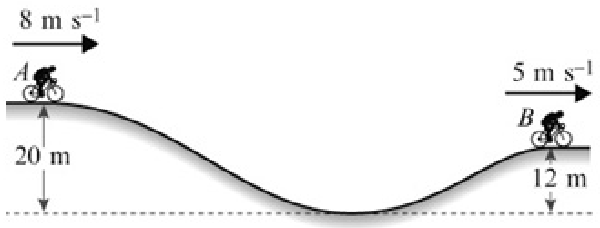
Review Exercise
Exercise A, Question 55

Question:



The figure shows the path taken by a cyclist in travelling on a section of a road. When the cyclist comes to the point A on the top of the hill she is travelling at 8 m s^{-1}. She descends a vertical distance of 20 m to the bottom of the hill. The road then rises to the point B through a vertical distance of 12 m. When she reaches B her speed is 5 m s^{-1}. The total mass of the cyclist and the cycle is 80 kg and the total distance along the road from A to B is 500 m. By modelling the resistance to the motion of the cyclist as of constant magnitude 20 N,

a find the work done by the cyclist in moving from A to B.

At B the road is horizontal.

b Given that at B the cyclist is accelerating at 0.5 m s^{-2}, find the power generated by the cyclist at B.

Solution:

a From A to B, the cyclist descends

$$(20-12)\text{ m} = 8\text{ m}$$

The potential energy, in J, lost in travelling from A to B is given by

$$mgh = 80 \times 9.8 \times 8 = 6272$$

The kinetic energy, in J, lost in travelling from A to B is given by

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}m\left(u^2 - v^2\right)$$
$$= 40\left(8^2 - 5^2\right) = 1560$$

The total mechanical energy lost is $(6272 + 1560)\text{ J} = 7832\text{ J}$

The work done by resistance due to non-gravitational forces is given by

$$W = \text{force} \times \text{distance moved}$$
$$= 20 \times 500 = 10\,000$$

$$(10\,000 - 7832)\text{ J} = 2168\text{ J}$$

The work done by the cyclist in moving from A to B is 2200 J (2 s.f.).

Whatever the path taken, the potential energy lost in travelling from A to B depends solely on the difference in levels between A and B.

The non-gravitational resistances to motion have worked 10 000 J against the motion. However, the mechanical energy lost is only 7832 J. The difference between these values is the work that has been done by the cyclist.

b At B, let the force generated by the cyclist be FN.

$$\text{R}(\rightarrow) \quad \mathbf{F} = m\mathbf{a}$$
$$F - 20 = 80 \times 0.8$$
$$P = Fv$$
$$= 60 \times 5 = 300$$

The power generated by the cyclist is 300 W.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 56

Question:

A van of mass 1500 kg is driving up a straight road inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{12}$. The resistance to motion due to non-gravitational forces is modelled as a constant force of magnitude 1000 N.

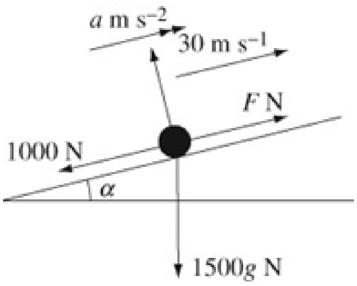
a Given that initially the speed of the van is 30 m s^{-1} and that the van's engine is working at a rate of 60 kW, calculate the magnitude of the initial deceleration of the van.

When travelling up the same hill, the rate of working of the van's engine is increased to 80 kW.

b Using the same model for the resistance due to non-gravitational forces, calculate in m s^{-1} the constant speed which can be sustained by the van at this rate of working.

c Give one reason why the use of this model for resistance may mean your answer to part **b** is too high.

Solution:



a Let F N be the magnitude of the driving force produced by the engine of the van and a m s^{-2} be the acceleration of the van.

From the wording of the question, you would expect a to be negative and this gives you a check on your working.

$$60\text{ kW} = 60\,000\text{ W}$$
$$P = Fv$$
$$60\,000 = F \times 30 \Rightarrow F = 2000$$

R(↗) $\mathbf{F} = m\mathbf{a}$

$$F - 1000 - 1500g\sin\alpha = 1500 \times a$$
$$1500a = 2000 - 1000 - 15 \times 9.8 \times \frac{1}{12} = -225$$
$$a = -\frac{225}{1500} = -0.15$$

The deceleration of the van is 0.15 m s^{-2}.

A common error is to leave the answer as -0.15. That is the acceleration and you have been asked for the deceleration.

b Let F' N be the magnitude of the driving force produced by the engine of the van at the increased power and u m s^{-1} be the constant speed.

R(↗) $F - 1000 - 1500g\sin\alpha = 0$

In **b**, the van is travelling at a constant speed and the acceleration is zero.

$$F' = 1000 + 1500 \times 9.8 \times \frac{1}{12} = 2225 *$$
$$80\text{ kW} = 80\,000\text{ W}$$
$$P = Fv$$
$$80\,000 = 2225u \Rightarrow u = \frac{80\,000}{2225} = 35.955\ldots$$

The constant speed is 36 m s^{-1} (2 s.f.).

c The resistance usually increases with speed. The equation $*$ in **b** would then give an increased value of F'.

Unless the speed is very high, resistances to motion are usually proportional to speed.

As $v = \frac{80\,000}{F'}$, an increased value of F' gives a lower value of v, and the answer in **b** would be too high.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 57

Question:

A model car has weight 200 N. It undergoes tests on a straight hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{10}$. The engine of the car works at a constant rate of P watts.

When the car goes up the hill it is observed to travel at a constant speed of 8 m s^{-1}. Given that the total resistance to the motion of the car from forces other than gravity is R newtons,

a express P in terms of R.

When the car runs down the same hill with the engine running at the same rate, it is observed to travel at a constant speed of 24 m s^{-1}.

In an initial model of the situation the resistance to motion due to non-gravitational forces is assumed to be constant whatever the speed of the car.

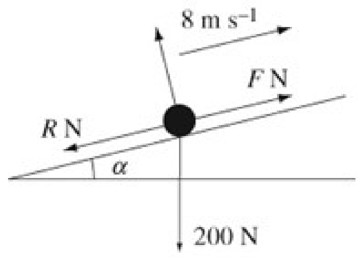
b Using this model, find an estimate for the value of P.

In a refined model the resistance to motion due to non-gravitational forces is assumed to be proportional to the speed of the car.

c Using this model, find a revised estimate for P.

Solution:

a



Let F N be the magnitude of the driving force produced by the engine of the model car.

$$P = Fv$$

$$P = F \times 8 \Rightarrow F = \frac{P}{8} \qquad (1)$$

$$\mathrm{R}(\nearrow) F - R - 200 \sin\alpha = 0$$

To obtain an expression for P in terms of R, you will have to eliminate F from the two equations you get from $P = Fv$ and resolving up the hill.

You have been given the weight of the model car in newtons and not the mass in kilograms. g is not needed. It is a common error, when not reading the question carefully, to add g when weights are given and not masses.

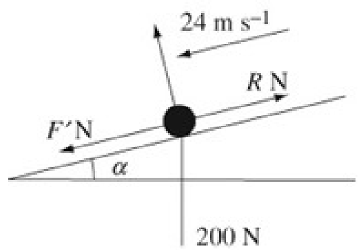
$$F = R + 200 \times \frac{1}{10} = R + 20 \qquad (2)$$

Eliminating F from (1) and (2)

$$\frac{P}{8} = R + 20$$

$$P = 8(R+20) \qquad (3)$$

b



Let F' N be the magnitude of the driving force produced by the engine of the model car as it moves downhill.

$$P = Fv$$

$$P = F' \times 24 \Rightarrow F' = \frac{P}{24} \qquad (4)$$

$$\mathrm{R}(\swarrow) F' - R + 200 \sin\alpha = 0$$

$$F' = R - 200 \times \frac{1}{10} = R - 20 \qquad (5)$$

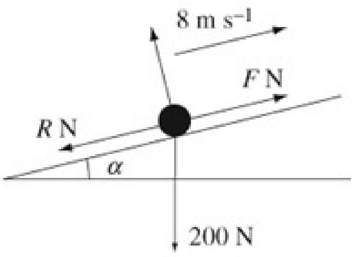
Eliminating F' from (4) and (5)

$$\frac{P}{24} = R - 20$$

$$P = 24(R-20) \qquad (6)$$

You start **b** by repeating **a**, with the directions of the driving force and resistance reversed. You can then find R and P by solving simultaneous equations.

a



Let F N be the magnitude of the driving
Eliminating P from (3) and (6)

$$
\begin{aligned}
8(R+20) &= 24(R-20) \\
8R+160 &= 24R-480 \\
16R &= 640 \Rightarrow R = 40
\end{aligned}
$$

Substituting $R = 40$ into (6)

$$P = 24(40-20) = 480$$

c When the resistance is proportional to the speed, equation (4) is still correct.

$$\text{R}(\swarrow)F' - 3R + 200\sin\alpha = 0$$

$$F' = 3R - 200 \times \frac{1}{10} = 3R - 20 \ldots 1.4 \text{ m s}^{-2} \qquad (5)'$$

The speed in **c**, 24 m s^{-1}, is three times the speed in **a** 8 m s^{-1}. If the resistance is proportional to the speed, the resistance in **c** must be three times the resistance in **a**. In **a** the resistance is R, so here it is $3R$. Solving **c**, is essentially a repeat of **b**, replacing R by $3R$ where appropriate.

Eliminating F' from (4) and (5)$'$

$$
\begin{aligned}
\frac{P}{24} &= 3R-20 \\
P &= 24(3R-20) \qquad (6)'
\end{aligned}
$$

Eliminating P from (3) and (6)$'$

$$
\begin{aligned}
8(R+20) &= 24(3R-20) \\
8R+160 &= 72R-480 \\
64R &= 640 \Rightarrow R = 10
\end{aligned}
$$

Substituting $R = 10$ into (6)$'$

$$P = 24(30-20) = 240$$

The answer here is half the answer in **b**. In many questions, it is assumed that resistance is a constant and you are sometimes asked to comment on this. This question shows the error which can follow from such an assumption.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 58

Question:

A car of mass 1000 kg is towing a trailer of mass 1500 kg along a straight horizontal road. The tow-bar joining the car to the trailer is modelled as a light rod parallel to the road. The total resistance to motion of the car is modelled as having constant magnitude 750 N. The total resistance to motion of the trailer is modelled as a force of magnitude R newtons, where R is a constant. When the engine is working at a rate of 50 kW, the car and the trailer travel at a constant speed of 25 m s^{-1}.

a Show that $R = 1250$.

When travelling at 25 m s^{-1} the driver of the car disengages the engine and applies the brakes. The brakes provide a constant braking force of magnitude 1500 N to the car. The resisting forces of magnitude 750 N and 1250 N are assumed to remain unchanged. Calculate

b the deceleration of the car while braking,

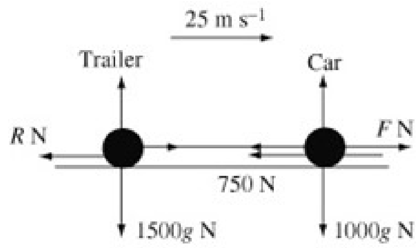
c the thrust in the tow-bar while braking,

d the work done, in kJ, by the braking force in bringing the car and the trailer to rest.

e Suggest how the modelling assumption that the resistances to motion are constant could be refined to be more realistic.

Solution:

a



Let F N be the magnitude of the driving force produced by the engine of the car.

$$
\begin{aligned}
50\text{ kW} &= 50\,000\text{ W} \\
P &= Fv \\
50\,000 &= F\times 25 \Rightarrow F = 2000
\end{aligned}
$$

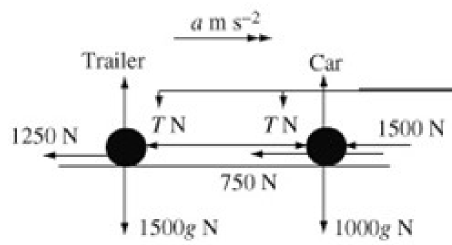
For the car and trailer combined

$\text{R}(\rightarrow)\ F - 750 - R = 0$

$R = F - 750 = 2000 - 750 = 1250$, as required

When you consider the car and trailer combined, the tensions at the ends the tow-bar cancel one another out and can be ignored.

b



As the car brakes, the forces in the tow-bar are thrusts and act in the directions shown in this diagram. The forces in the tow-bar in **a** are tensions and act in the opposite directions to thrusts.

Let the acceleration of the car while braking be a m s^{-2}.

For the car and trailer combined

$$
\begin{aligned}
\text{R}(\rightarrow)\quad \mathbf{F} &= m\mathbf{a} \\
-1500 - 750 - 1250 &= 2500a \\
2500a = -3500 \Rightarrow a &= -\frac{3500}{2500} = -1.4
\end{aligned}
$$

The deceleration of the car while braking is 1.4 m s^{-2}.

c Let the magnitude of the thrust in the tow-bar while braking be T N.

For the trailer alone

$$\begin{aligned} \text{R}(\rightarrow) \quad \mathbf{F} &= m\mathbf{a} \\ -1250 - T &= 1500a = 1500 \times (-1.4) \\ T &= 1500 \times 1.4 - 1250 = 850 \end{aligned}$$

The magnitude of the thrust in the tow-bar while braking is 850 N.

d To find the distance travelled in coming to rest

$v^2 = u^2 + 2as$

The work done by a force is its magnitude multiplied by the distance moved by its point of application. This formula you learnt for the M1 module gives you a straightforward way of finding this distance.

$$0^2 = 25^2 + 2 \times (-1.4)s$$

$$s = \frac{25^2}{2.8}$$

The work done, in J, by the braking force of 1500 N is given by

Work done = force × distance moved

$$W = 1500 \times s = 1500 \times \frac{25^2}{2.8} = 334\,821$$

The work done by the braking force in bringing the car and the trailer to rest is 335 kJ (3 s.f.).

e The resistance could be modelled as varying with speed.

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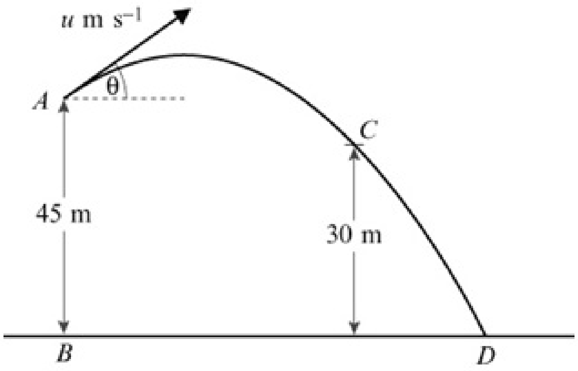
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 59

Question:



A particle P is projected from a point A with speed u m s^{-1} at an angle of elevation θ, where $\cos\theta = \frac{4}{5}$. The point B, on horizontal ground, is vertically below A and $AB = 45$ m. After projection, P moves freely under gravity passing through a point C, 30 m above the horizontal ground, before striking the ground at the point D, as shown in the figure above.

Given that P passes through C with speed 24.5 m s^{-1},

a using conservation of energy, or otherwise, show that $u = 17.5$,

b find the size of the angle which the velocity of P makes with the horizontal as P passes through C,

c find the distance BD.

Solution:

a The vertical distance fallen by P in moving from A to C is $(45-30)\text{ m} = 15\text{ m}$.

Using the principle of conservation of energy,
Kinetic energy gained = Potential energy lost

$$\frac{1}{2}\cancel{m}v^2 - \frac{1}{2}\cancel{m}u^2 = \cancel{m}gh$$

$$\frac{1}{2}\times 24.5^2 - \frac{1}{2}u^2 = 9.8\times 15$$

$$u^2 = 24.5^2 - 2\times 9.8\times 15 = 306.25$$

$$u = \sqrt{306.25} = 17.5\text{, as required}$$

The mass of the particle cancels throughout this equation. The calculations in this question are independent of the mass of P.

This equation has a similar form to $v^2 = u^2 + 2as$. However, it would be an error to use this formula, which is a formula for motion in a straight line, as P is not moving in a straight line.

b $\mathrm{R}(\rightarrow)$ $u_x = u\cos\theta = 17.5\times\frac{4}{5} = 14$

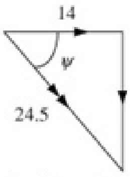
Let the required angle be ψ

$$\cos\psi = \frac{14}{24.5} = \frac{4}{7}$$

$$\psi = 55.15\ldots^\circ = 55^\circ \text{ (nearest degree)}$$

The horizontal component of the velocity is constant throughout the motion.

At C, the velocity of P and its components are illustrated in this diagram.



ψ can now be found using trigonometry. There is no need to find the vertical component of the velocity at C.

c $\mathrm{R}(\uparrow)$ $u_y = u\sin\theta = 17.5\times\frac{3}{5} = 10.5$

To find the time taken for P to move from A to D

$$\mathrm{R}(\uparrow)\quad s = ut + \frac{1}{2}at^2$$

$$-45 = 10.5t - 4.9t^2$$

$$4.9t^2 - 10.5t - 45 = 0$$

$$49t^2 - 105t - 450 = 0$$

$$(7t-30)(7t+15) = 0$$

$$t = \frac{30}{7}\text{, as } t > 0$$

These factors are difficult to spot and you can use the formula for a quadratic. You should, however, obtain an exact answer.

$$\mathrm{R}(\rightarrow)\quad \text{distance} = \text{speed}\times\text{time}$$

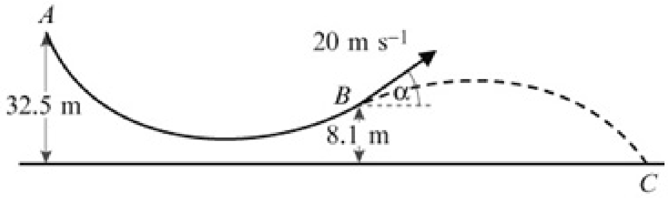
$$= 14\times\frac{30}{7} = 60$$

$$BD = 60\text{ m}$$

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 60

Question:



In a ski-jump competition, a skier of mass 80 kg moves from rest at a point A on a ski-slope. The skier's path is an arc AB. The starting point A of the slope is 32.5 m above horizontal ground. The end B of the slope is 8.1 m above the ground. When the skier reaches B she is travelling at 20 m s^{-1} and moving upwards at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$, as shown in the figure. The distance along the slope from A to B is 60 m. The resistance to motion while she is on the slope is modelled as a force of constant magnitude R newtons.

a By using the work–energy principle, find the value of R.

On reaching B, the skier then moves through the air and reaches the ground at the point C. The motion of the skier in moving from B to C is modelled as that of a particle moving freely under gravity.

b Find the time the skier takes to move from B to C.

c Find the horizontal distance from B to C.

d Find the speed of the skier immediately before she reaches C.

Solution:

a The kinetic energy, in J, gained in moving from A to B is

$$\frac{1}{2}mv^2 = \frac{1}{2}80 \times 20^2 = 16\,000$$

The potential energy, in J, lost in moving from A to B is

$$mgh = 40 \times 9.8 \times (32.5 - 8.1) = 19\,129.6$$

The net loss of mechanical energy is

$$(19\,129.6 - 16\,000)\text{ J} = 3129.6\text{ J}$$

The net loss in mechanical energy is the work done by the resistance to motion.

The work done by the resisting force of R newtons, in J, is given by

$$\begin{aligned} \text{Work} &= \text{force} \times \text{distance} \\ &= R \times 60 \end{aligned}$$

By the work–energy principle

$$\begin{aligned} 60R &= 3129.6 \\ R &= \frac{3129.6}{60} = 52.15 = 52 \text{ (2 s.f.)} \end{aligned}$$

b $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

$$\text{R}(\rightarrow) \quad u_x = 20\cos\alpha = 20 \times \frac{4}{5} = 16$$

$$\text{R}(\uparrow) \quad u_y = 20\sin\alpha = 20 \times \frac{3}{5} = 12$$

You can sketch a 3, 4, 5 triangle to check these relations.

To find the time taken to move from B to C

$$\begin{aligned} \text{R}(\uparrow) \quad s &= ut + \tfrac{1}{2}at^2 \\ -8.1 &= 12t - 4.9t^2 \\ 4.9t^2 - 12t - 8.1 &= 0 \\ 49t^2 - 120t - 81 &= 0 \\ (t-3)(49t+27) &= 0 \end{aligned}$$

$t = 3$, as $t > 0$

Rearranging the quadratic and multiplying by 10.

The time taken to move from B to C is 3 s.

c $\rightarrow$ distance $=$ speed $\times$ time $= 16 \times 3 = 48$

The horizontal distance from B to C is 48 m.

d Let the speed of the skier immediately before reaching C be w m s^{-1}.

Using conservation of energy

$$\frac{1}{2}\cancel{m}w^2 - \frac{1}{2}\cancel{m}v^2 = \cancel{m}gh$$

$$\begin{aligned} w^2 &= v^2 + 2gh \\ &= 20^2 + 2 \times 9.8 \times 8.1 = 558.76 \\ w &= \sqrt{558.76} = 23.638\ldots \end{aligned}$$

Cancelling the m and rearranging the formula. This result is similar to $v^2 = u^2 + 2as$. However, it would be an error to use this formula, which is a formula for motion in a straight line, as the skier is not moving in a straight line. You must establish the result using the principle of conservation of energy.

The speed of the skier immediately before reaching C is 24 m s^{-1} (2 s.f.).

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8 \text{ m s}^{-2}$.

[*In this question* **i** *and* **j** *are perpendicular unit vectors in a horizontal plane.*]

A ball has a mass 0.2 kg. It is moving with velocity $(30\mathbf{i})$ m s^{-1} when it is struck by a bat. The bat exerts an impulse of $(-4\mathbf{i}+4\mathbf{j})$ Ns on the ball. Find

a the velocity of the ball immediately after the impact,

b the angle through which the ball is deflected as a result of the impact,

c the kinetic energy lost by the ball in the impact.

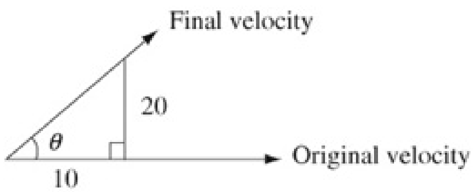
Solution:

a $-4\mathbf{i}+4\mathbf{j} = 0.2\mathbf{v} - 0.2\times 30\mathbf{i}$

$\mathbf{v} = 10\mathbf{i} + 20\mathbf{j} \text{ m s}^{-1}$

Impulse = change of momentum

b



$\tan\theta = \frac{20}{10}$

$\theta = 63.4°$

c K. E. lost

$= \frac{1}{2}\times 0.2\times 30^2 - \frac{1}{2}\times 0.2(10^2 + 20^2)$

$= 40 \text{ J}$

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

2 Review Exercise

Exercise A, Question 2

Question:

A particle P of mass 0.75 kg is moving under the action of a single force $\mathbf{F}$ newtons. At time t seconds, the velocity $\mathbf{v}$ m s^{-1} of P is given by

$$\mathbf{v} = (t^2+2)\mathbf{i} - 6t\mathbf{j}.$$

a Find the magnitude of $\mathbf{F}$ when $t = 4$.

When $t = 5$, the particle P receives an impulse of magnitude $9\sqrt{2}$ Ns in the direction of the vector $\mathbf{i}-\mathbf{j}$.

b Find the velocity of P immediately after the impulse.

Solution:

a $\mathbf{v} = (t^2+2)\mathbf{i} - 6t\mathbf{j}$

$\mathbf{a} = 2t\mathbf{i} - 6\mathbf{j}$ — Differentiate to find $\mathbf{a}$.

$\mathbf{F} = 0.75(2t\mathbf{i} - 6\mathbf{j})$ — Use $\mathbf{F} = m\mathbf{a}$ to find $\mathbf{F}$.

$t = 4$ — Make $t = 4$.

$\mathbf{F} = 0.75(8\mathbf{i} - 6\mathbf{j})$

$= 6\mathbf{i} - 4.5\mathbf{j}$

$|\mathbf{F}| = \sqrt{(6^2+4.5^2)} = 7.5\text{ N}$ — The magnitude of $\mathbf{F}$ is the modulus of the vector.

b

$\mathbf{I} = p\mathbf{i} - p\mathbf{j}$ — Impulse is parallel to $\mathbf{i}-\mathbf{j}$.

$|\mathbf{I}| = \sqrt{(p^2+p^2)} = p\sqrt{2}$

But $|\mathbf{I}| = 9\sqrt{2}\ \therefore p = 9$

$9\mathbf{i} - 9\mathbf{j} = 0.75(\mathbf{v} - (27\mathbf{i} - 30\mathbf{j}))$ — Impulse = change of momentum

$36\mathbf{i} - 36\mathbf{j} = 3\mathbf{v} - 81\mathbf{i} + 90\mathbf{j}$ — For the initial velocity, $t = 5$.

$3\mathbf{v} = 117\mathbf{i} - 126\mathbf{j}$

$\mathbf{v} = 39\mathbf{i} - 42\mathbf{j}$ m s^{-1}

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 3

Question:

A tennis ball of mass 0.2 kg is moving with velocity $(-10\mathbf{i})$ m s^{-1} when it is struck by a tennis racket. Immediately after being struck, the ball has velocity $(15\mathbf{i}+15\mathbf{j})$ m s^{-1}. Find

a the magnitude of the impulse exerted by the racket on the ball,

b the angle, to the nearest degree, between the vector $\mathbf{i}$ and the impulse exerted by the racket,

c the kinetic energy gained by the ball as a result of being struck.

Solution:

a

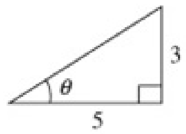
$$\mathbf{I} = 0.2((15\mathbf{i}+15\mathbf{j})-(-10\mathbf{i}))$$
$$= 5\mathbf{i}+3\mathbf{j}$$

Use impulse = change of momentum.

$$|\mathbf{I}| = \sqrt{(5^2+3^2)} = \sqrt{34}$$
$$= 5.83 \text{ Ns}$$

The magnitude of the impulse is the modulus of the vector.

b



$$\tan\theta = \frac{3}{5}$$
$$\theta = 31^\circ \text{ (nearest degree)}$$

c K. E. gained

$$= \frac{1}{2}\times 0.2\times(15^2+15^2)-\frac{1}{2}\times 0.2\times 10^2$$
$$= 35 \text{ J}$$

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 4

Question:

At time t seconds the acceleration, $\mathbf{a}$ m s^{-2}, of a particle P relative to a fixed origin O, is given by $\mathbf{a} = 2\mathbf{i} + 6t\mathbf{j}$. Initially the velocity of P is $(2\mathbf{i} - 4\mathbf{j})$ m s^{-1}.

a Find the velocity of P at time t seconds.

At time $t = 2$ seconds the particle P is given an impulse $(3\mathbf{i} - 1.5\mathbf{j})$ Ns.

Given that the particle P has mass 0.5 kg,

b find the speed of P immediately after the impulse has been applied.

Solution:

a

$$\mathbf{v} = \int (2\mathbf{i} + 6t\mathbf{j})\,dt$$

$$= 2t\mathbf{i} + 3t^2\mathbf{j} + \mathbf{c}$$

Don't forget the (vector) constant of integration.

$$t = 0 \quad \mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$$

$$\therefore \mathbf{c} = 2\mathbf{i} - 4\mathbf{j}$$

$$\mathbf{v} = (2t + 2)\mathbf{i} + (3t^2 - 4)\mathbf{j}$$

b

$$t = 2 \quad \mathbf{v} = 6\mathbf{i} + 8\mathbf{j}$$

Find the velocity immediately before the impact.

$$3\mathbf{i} - 1.5\mathbf{j} = 0.5(\mathbf{V} - (6\mathbf{i} + 8\mathbf{j}))$$

Impulse = change of momentum.

$$6\mathbf{i} - 3\mathbf{j} = \mathbf{V} - 6\mathbf{i} - 8\mathbf{j}$$

$$\mathbf{V} = 12\mathbf{i} + 5\mathbf{j}$$

$$|\mathbf{V}| = \sqrt{(12^2 + 5^2)} = 13$$

Speed is the modulus of the velocity.

$$\text{Speed} = 13 \text{ m s}^{-1}$$

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2 Review Exercise
Exercise A, Question 5

Question:

The unit vectors **i** and **j** lie in a vertical plane, **i** being horizontal and **j** vertical.

A ball of mass 0.1 kg is hit by a bat which gives it an impulse of $(3.5\mathbf{i} + 3\mathbf{j})$ Ns. The velocity of the ball immediately after being hit is $(10\mathbf{i} + 25\mathbf{j})$ m s^{-1}.

a Find the velocity of the ball immediately before it is hit.

In the subsequent motion the ball is modelled as a particle moving freely under gravity. When it is hit the ball is 1 m above horizontal ground.

b Find the greatest height of the ball above the ground in the subsequent motion.

The ball is caught when it is again 1 m above the ground.

c Find the distance from the point where the ball is hit to the point where it is caught.

Solution:

a

$$3.5\mathbf{i}+3\mathbf{j} = 0.1\left[(10\mathbf{i}+25\mathbf{j})-(u\mathbf{i}+v\mathbf{j})\right]$$
$$u\mathbf{i}+v\mathbf{j} = (-25\mathbf{i}-5\mathbf{j})\text{ m s}^{-1}$$

Impulse = change of momentum .

b Vertical motion $\uparrow$ +ve :

$$u = 25\text{ m s}^{-1}$$
$$v = 0\text{ m s}^{-1}$$

The **j** component is the vertical velocity.

$$a = -9.8\text{ m s}^{-2}$$
$$v^2 = u^2 + 2as$$
$$0 = 625 - 2\times 9.8\,s$$
$$s = \frac{625}{19.6}$$

Height above ground

$$= \frac{625}{19.6} + 1 = 32.9\text{ m}$$

c Vertical motion $\uparrow$ +ve :

Use the vertical motion to find the time taken.

$$s = 0$$
$$a = -9.8\text{ m s}^{-2}$$
$$u = 25\text{ m s}^{-1}$$
$$s = ut + \frac{1}{2}at^2$$
$$0 = 25t - \frac{1}{2}\times 9.8\,t^2$$
$$0 = t(25 - 4.9t)$$
$$t = \frac{25}{4.9}\ (\text{or } t = 0)$$

At start.

Horizontal motion:

This will give you the distance.

$$s = 10\times\frac{25}{4.9}$$
$$= 51\text{ m}$$

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2 Review Exercise

Exercise A, Question 6

Question:

Two particles, A and B, of mass m and $3m$ respectively, lie at rest on a smooth horizontal table. The coefficient of restitution between the particles is 0.25.

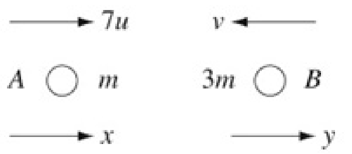
The particles A and B are given speeds of $7u$ and u respectively towards each other so that they collide directly. Find

a the speeds of A and B after the collision,

b the loss in kinetic energy due to the collision.

Solution:

a



$e = 0.25$

Draw a diagram!

$$7mu - 3mu = mx + 3my$$

Conservation of momentum.

$$4u = x + 3y \quad (1)$$

$$0.25 \times (7u + u) = y - x$$

Newton's Law of Restitution.

$$2u = y - x \quad (2)$$

$$(1)+(2) \quad 6u = 4y$$

$$y = \frac{3u}{2}$$

Solve (1) and (2) simultaneously.

$$\text{In (2)} \quad 2u = \frac{3u}{2} - x$$

$$x = -\frac{u}{2}$$

The minus sign shows the arrow in the diagram is pointing in the wrong direction.

A has speed $\frac{u}{2}$.

B has speed $\frac{3u}{2}$.

Speed is always positive.

b K.E. lost

$$= \frac{1}{2} \times m \times (7u)^2 + \frac{1}{2} \times 3m \times u^2$$

$$- \left(\frac{1}{2} m \times \left(\frac{u}{2}\right)^2 + \frac{1}{2} \times 3m \times \left(\frac{3u}{2}\right)^2 \right)$$

$$= \frac{1}{2} m \times 49u^2 + \frac{3}{2} mu^2 - \left(\frac{mu^2}{8} + \frac{27mu^2}{8} \right)$$

$$= \frac{45}{2} mu^2$$

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2 Review Exercise
Exercise A, Question 7

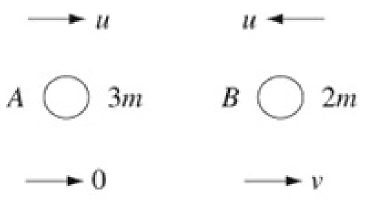
Question:

Two uniform smooth spheres A and B are of equal size and have masses $3m$ and $2m$ respectively. They are both moving in the same straight line with speed u, but in opposite directions, when they are in direct collision with each other. Given that A is brought to rest by the collision, find

a the coefficient of restitution between the spheres,

b the kinetic energy lost in the impact.

Solution:



a
$$3mu - 2mu = 2mv$$
$$mu = 2mv$$

Conservation of momentum.

$$e(u+u) = v$$
$$2ue = v$$

Newton's Law of Restitution.

Eliminating v:
$$mu = 2m(2ue)$$
$$e = \frac{1}{4}$$

b K.E. lost
$$= \frac{1}{2}\times 3mu^2 + \frac{1}{2}\times 2mu^2 - \left(0+\frac{1}{2}\times 2m\left(\frac{1}{2}u\right)^2\right)$$

$v = 2ue = 2u\times\frac{1}{4} = \frac{1}{2}u$

$$= \frac{5}{2}mu^2 - \frac{1}{4}mu^2$$
$$= \frac{9}{4}mu^2$$

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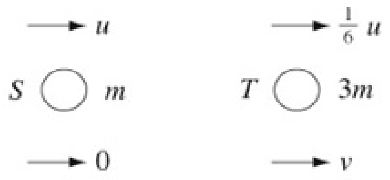
2 Review Exercise
Exercise A, Question 8

Question:

A smooth sphere S of mass m is moving on a smooth horizontal plane with speed u. It collides directly with another smooth sphere T, of mass $3m$, whose radius is the same as S. The sphere T is moving in the same direction as S with speed $\frac{1}{6}u$. The sphere S is brought to rest by the impact.

Find the coefficient of restitution between the spheres.

Solution:



$$mu + 3m \times \frac{1}{6}u = 3mv$$

Conservation of momentum.

$$\frac{3}{2}u = 3v$$

$$u = 2v$$

$$e(u - \frac{1}{6}u) = v$$

Newton's Law of Restitution.

$$\therefore \frac{5}{6}eu = \frac{1}{2}u$$

$$e = \frac{6}{5} \times \frac{1}{2}$$

$$e = \frac{3}{5}$$

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2 Review Exercise
Exercise A, Question 9

Question:

A smooth sphere S of mass m is moving with speed u on a smooth horizontal plane. The sphere S collides with another smooth sphere T, of equal radius to S but of mass km, moving in the same straight line and in the same direction with speed λu, $0 < \lambda < \frac{1}{2}$. The coefficient of restitution between S and T is e.

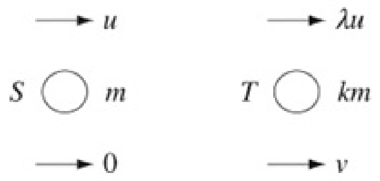
Given that S is brought to rest by the impact,

a show that $e = \dfrac{1+k\lambda}{k(1-\lambda)}$.

b Deduce that $k > 1$.

Solution:

a



$mu + km\lambda u = kmv$ — Conservation of momentum.

$u(1+k\lambda) = kv \quad (1)$

$v = e(u - \lambda u) = eu(1-\lambda) (2)$ — Newton's Law of Restitution.

Eliminate v from (1) and (2)

$$u(1+k\lambda) = keu(1-\lambda)$$

$$e = \frac{1+k\lambda}{k(1-\lambda)}$$

b

$e \le 1$ — Any coefficient of restitution satisfies $0 \le e \le 1$.

$\Rightarrow \quad 1+k\lambda \le k - k\lambda$

$$\frac{1}{1-2\lambda} \le k$$

but $0 < \lambda < \frac{1}{2} \Rightarrow 0 < 1-2\lambda < 1$ and $k > 1$.

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2 Review Exercise
Exercise A, Question 10

Question:

A particle A, of mass $2m$, is moving with speed u on a horizontal table when it collides directly with a particle B, of mass $3m$, which is at rest. The coefficient of restitution between the particles is e.

a Find, in terms of e and u, the velocities of A and B immediately after the collision.

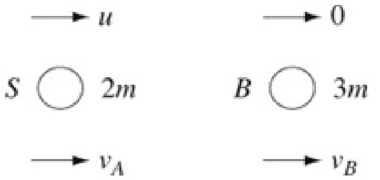
b Show that, for all possible values of e, the speed of A immediately after the collision is not greater than $\frac{2}{5}u$.

Given that the magnitude of the impulse exerted by B on A is $\frac{11}{5}mu$,

c find the value of e.

Solution:

a



$$2mu = 2mv_A + 3mv_B$$

$$2u = 2u_A + 3v_B \quad (1)$$

$$eu = v_B - v_A \quad (2)$$

Conservation of momentum.

Newton's Law of Restitution.

$(1) + 2\times(2)$

$$2u + 2eu = 3v_B + 2v_B$$

$$v_B = \frac{2u}{5}(1+e)$$

In (1)

$$2u = 2v_A + \frac{6u}{5}(1+e)$$

$$v_A = u - \frac{3u}{5}(1+e)$$

$$v_A = \frac{u}{5}(2-3e)$$

Solve (1) and (2) simultaneously.

b

$e = 0 \; v_A = \frac{2u}{5}$

Using the smallest possible value for e gives the maximum velocity of A.

$e = 1 \; v_A = -\frac{u}{5}$

The maximum speed of A is $\frac{2u}{5}$.

Using the largest possible value for e gives the minimum velocity of A. This minimum must be considered as it may give a greater *speed*.

c For B:

$$\frac{11mu}{5} = 3m \times \frac{2u}{5}(1+e)$$

Impulse = change of momentum

$$11 = 6 + 6e$$

$$e = \frac{5}{6}$$

Work with *one* of A and B. It is easier to work with B as B's initial momentum is zero.

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2 Review Exercise
Exercise A, Question 11

Question:

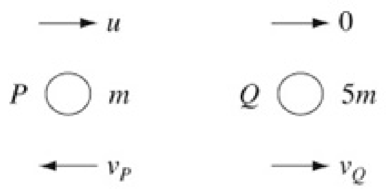
A sphere P, of mass m, is moving in a straight line with speed u on the surface of a smooth horizontal table. Another sphere Q, of mass $5m$ and having the same radius as P, is initially at rest on the table. The sphere P strikes the sphere Q directly, and the direction of motion of P is reversed by the impact. The coefficient of restitution between P and Q is e.

a Find an expression, in terms of u and e, for the speed of P after the impact.

b Find the set of possible values of e.

Solution:

a



The direction of motion of P is reversed by the impact.

$mu = 5mv_Q - mv_P$ ← Conservation of momentum

$u = 5v_Q - v_P$ (1)

$eu = v_P + v_Q$ (2) ← Newton's Law of Restitution

Eliminate v_Q ← v_Q is not needed!

$u = 5(eu - v_P) - v_P$

$6v_P = 5eu - u$

$v_P = \frac{u}{6}(5e - 1)$

b

$v_P > 0$ ← You know the direction for v_P is correct.

$\therefore 5e - 1 > 0 \quad e > \frac{1}{5}$

But

$0 \le e \le 1$ ← This is *always* true.

$\therefore \frac{1}{5} < e \le 1$

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2 Review Exercise
Exercise A, Question 12

Question:

A smooth sphere A of mass m is moving with speed u on a smooth horizontal table when it collides directly with another smooth sphere B of mass $3m$, which is at rest on the table. The coefficient of restitution between A and B is e. The spheres have the same radius and are modelled as particles.

a Show that the speed of B immediately after the collision is $\frac{1}{4}(1+e)u$.

b Find the speed of A immediately after the collision.

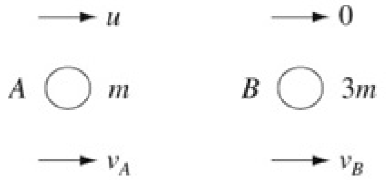
Immediately after the collision the total kinetic energy of the spheres is $\frac{1}{6}mu^2$.

c Find the value of e.

d Hence show that A is at rest after the collision.

Solution:

a



$$mu = mv_A + 3mv_B$$

Conservation of momentum.

$$u = v_A + 3v_B \quad (1)$$

$$eu = v_B - v_A \quad (2)$$

Newton's Law of Restitution.

$$(1)+(2) \quad u + eu = 4v_B$$

$$v_B = \frac{1}{4}(1+e)u$$

b Using (2):

$$v_A = v_B - eu$$

$$= \frac{1}{4}(1+e)u - eu$$

$$= \frac{1}{4}(1-3e)u$$

c K.E. after impact

$$= \frac{1}{2}mv_A^2 + \frac{1}{2}\times 3mv_B^2$$

$$= \frac{1}{2}m\left(\frac{1}{4}(1-3e)u\right)^2 + \frac{3}{2}m\left(\frac{1}{4}(1+e)u\right)^2$$

$$= \frac{1}{2}m\frac{u^2}{16}(1-6e+9e^2) + \frac{3}{2}m\frac{u^2}{16}\left(1+2e+e^2\right)$$

$$= \frac{mu^2}{32}\left(1-6e+9e^2+3+6e+3e^2\right)$$

$$= \frac{mu^2}{32}\left(4+12e^2\right)$$

$$= \frac{mu^2}{8}(1+3e^2)$$

$$\text{K.E. after impact} = \frac{1}{6}mu^2$$

From question.

$$\therefore \frac{1}{8}(1+3e^2) = \frac{1}{6}$$

$$6+18e^2 = 8$$

$$18e^2 = 2$$

$$e^2 = \frac{1}{9}$$

$$e = \frac{1}{3} \quad (e > 0)$$

d $v_A = \frac{u}{4}(1-3e)$

$$= \frac{u}{4}(1-3\times\frac{1}{3})$$

$$= 0$$

$\therefore$ A is at rest.

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2 Review Exercise
Exercise A, Question 13

Question:

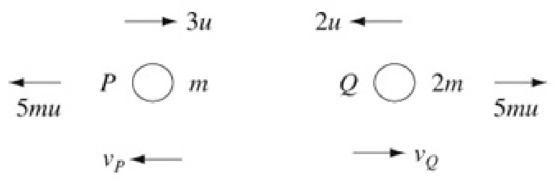
A particle P of mass m is moving with speed $3u$ in a straight line on a smooth horizontal plane. It collides with another particle Q of mass $2m$ which is moving with speed $2u$ along the same straight line but in the opposite direction. The coefficient of restitution between P and Q is e. The magnitude of the impulse given to each particle during the collision is $5mu$, and both P and Q have their directions of motion reversed by the collision.

a Show that $e = \frac{1}{2}$.

b Calculate the loss of kinetic energy due to the collision.

Solution:

a



For P:

$$5mu = mv_P - m(-3u)$$
$$v_P = 2u$$

Impulse = change of momentum, but watch signs.

For Q:

$$5mu = 2mv_Q - 2m(-2u)$$
$$v_Q = \frac{1}{2}u$$
$$e(3u + 2u) = v_P + v_Q$$
$$5eu = 2u + \frac{1}{2}u$$

Newton's Law of Restitution.

$$e = \frac{2\frac{1}{2}}{5} = \frac{1}{2}$$

b Loss of K.E.

$$= \frac{1}{2}m(3u)^2 + \frac{1}{2} \times 2m \times (2u)^2 - \left(\frac{1}{2}m(2u)^2 + \frac{1}{2} \times 2m \times \left(\frac{u}{2}\right)^2\right)$$
$$= \frac{9}{2}mu^2 + 4mu^2 - \left(2mu^2 + \frac{mu^2}{4}\right)$$
$$= \frac{25}{4}mu^2$$

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2 Review Exercise
Exercise A, Question 14

Question:

A smooth uniform sphere S of mass m is moving on a smooth horizontal plane with speed u. The sphere collides directly with another smooth uniform sphere T, of the same radius as S and a mass $2m$, which is at rest on the plane. The coefficient of restitution between the spheres is e.

a Show that the speed of T after the collision is $\frac{1}{3}u(1+e)$.

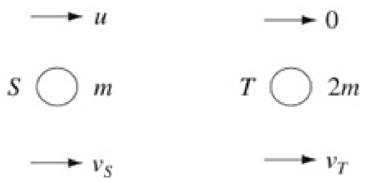
Given that $e > \frac{1}{2}$,

b **i** find the speed of S after the collision,

ii determine whether the direction of motion of S is reversed by the collision.

Solution:

a



$mu = mv_S + 2mv_T$

$u = v_S + 2v_T$ (1) Conservation of momentum.

$eu = v_T - v_S$ (2) Newton's Law of Restitution.

(1) + (2) $u + eu = 3v_T$

$v_T = \frac{1}{3}u(1+e)$

b **i** from (2)

$eu = \frac{1}{3}u(1+e) - v_S$

$v_S = \frac{1}{3}u(1+e) - eu$

$v_S = \frac{1}{3}u(1-2e)$

but $e > \frac{1}{2} \Rightarrow 1 - 2e < 0$

$\therefore$ Speed of S is $\frac{1}{3}u(2e-1)$. Speed must be positive.

ii The arrow in the diagram was the wrong way round, as shown in **b (i)**, so the direction of motion was reversed.

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2 Review Exercise
Exercise A, Question 15

Question:

A particle P of mass $3m$ is moving with speed $2u$ in a straight line on a smooth horizontal table. The particle P collides with a particle Q of mass $2m$ moving with speed u in the opposite direction to P. The coefficient of restitution between P and Q is e.

a Show that the speed of Q after the collision is $\frac{1}{5}u(9e+4)$.

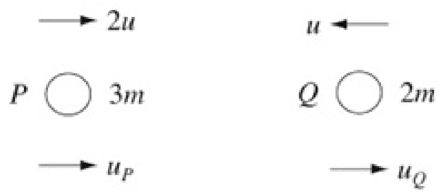
As a result of the collision, the direction of motion of P is reversed.

b Find the range of possible values of e.

c Given that the magnitude of the impulse of P on Q is $\frac{32}{5}mu$, find the value of e.

Solution:

a



Speed must be positive.

$$3m \times 2u - 2mu = 3mu_P + 2mu_Q$$

Conservation of momentum.

$$\therefore 4u = 3u_P + 2u_Q \qquad (1)$$

$$e(2u + u) = u_Q - u_P$$

Newton's Law of Restitution.

$$3eu = u_Q - u_P \qquad (2)$$

Eliminate u_P between (1) and (2):

$$4u = 3(u_Q - 3eu) + 2u_Q$$
$$4u = 5u_Q - 9eu$$
$$u_Q = \frac{1}{5}u(9e + 4)$$

b Using (2)

$$u_P = u_Q - 3eu$$
$$= \frac{1}{5}u(9e + 4) - 3eu$$
$$= \frac{2}{5}u(2 - 3e)$$

But

$$u_P < 0$$

Direction of motion of P is reversed.

$$\therefore 2 - 3e < 0$$

$$e > \frac{2}{3}$$

$$\therefore \frac{2}{3} < e \le 1$$

Use the general condition $0 \le e \le 1$.

c For P:

$$\frac{32}{5}mu = 2m \times \frac{1}{5}u(9e + 4) + 2mu$$

Impulse = change of momentum.

$$32 = 2(9e + 4) + 10$$
$$18e = 14$$
$$e = \frac{7}{9}$$

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2 Review Exercise
Exercise A, Question 16

Question:

A small smooth ball A of mass m is moving on a horizontal table with speed u when it collides directly with another small smooth ball B of mass $3m$ which is at rest on the table. The balls have the same radius and the coefficient of restitution between the balls is e. The direction of motion of A is reversed as a result of the collision.

a Find, in terms of e and u the speeds of A and B immediately after the collision.

In the subsequent motion B strikes a vertical wall, which is perpendicular to the direction of motion of B, and rebounds.

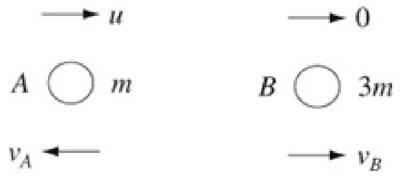
The coefficient of restitution between B and the wall is $\frac{3}{4}$.

Given that there is a second collision between A and B,

b find the range of values of e for which the motion described is possible.

Solution:

a



The direction of motion of A is reversed.

$$mu = -mv_A + 3mv_B$$

Conservation of momentum.

$$u = 3v_B - v_A \qquad (1)$$

$$eu = v_A + v_B \qquad (2)$$

Newton's Law of Restitution.

$$(1)+(2) \quad u(1+e) = 4v_B$$

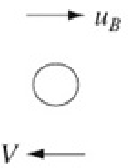
$$v_B = \frac{1}{4}u(1+e)$$

Using (2)

$$eu = v_A + \frac{1}{4}u(1+e)$$

$$v_A = \frac{1}{4}u(3e-1)$$

b



$$V = \frac{3}{4}\times\frac{1}{4}u(1+e)$$

$$= \frac{3}{16}u(1+e)$$

Newton's Law of Restitution with $e = \frac{3}{4}$.

For a second collision

$V > v_A$

B must move faster than A.

$$\frac{3u}{16}(1+e) > \frac{1}{4}u(3e-1)$$

$$3+3e > 12e-4$$

$$7 > 9e$$

$$e < \frac{7}{9}$$

and

$$v_A > 0$$

The direction of motion of A is correct in the diagram.

$$\therefore \quad e > \frac{1}{3}$$

So $\frac{1}{3} < e < \frac{7}{9}$

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2 Review Exercise
Exercise A, Question 13

Question:

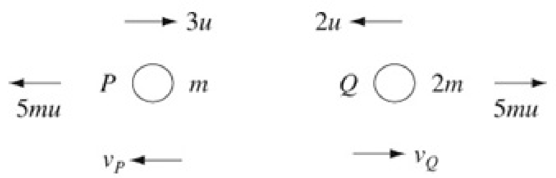
A particle P of mass m is moving with speed $3u$ in a straight line on a smooth horizontal plane. It collides with another particle Q of mass $2m$ which is moving with speed $2u$ along the same straight line but in the opposite direction. The coefficient of restitution between P and Q is e. The magnitude of the impulse given to each particle during the collision is $5mu$, and both P and Q have their directions of motion reversed by the collision.

a Show that $e = \frac{1}{2}$.

b Calculate the loss of kinetic energy due to the collision.

Solution:

a



For P:

$$5mu = mv_P - m(-3u)$$
$$v_P = 2u$$

Impulse = change of momentum, but watch signs.

For Q:

$$5mu = 2mv_Q - 2m(-2u)$$
$$v_Q = \frac{1}{2}u$$
$$e(3u + 2u) = v_P + v_Q$$
$$5eu = 2u + \frac{1}{2}u$$
$$e = \frac{2\frac{1}{2}}{5} = \frac{1}{2}$$

Newton's Law of Restitution.

b Loss of K.E.

$$= \frac{1}{2}m(3u)^2 + \frac{1}{2} \times 2m \times (2u)^2 - \left(\frac{1}{2}m(2u)^2 + \frac{1}{2} \times 2m \times \left(\frac{u}{2}\right)^2\right)$$
$$= \frac{9}{2}mu^2 + 4mu^2 - \left(2mu^2 + \frac{mu^2}{4}\right)$$
$$= \frac{25}{4}mu^2$$

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2 Review Exercise
Exercise A, Question 18

Question:

A smooth sphere P of mass $2m$ is moving in a straight line with speed u on a smooth horizontal table. Another smooth sphere Q of mass m is at rest on the table. The sphere P collides directly with Q. The coefficient of restitution between P and Q is $\frac{1}{3}$. The spheres are modelled as particles.

a Show that, immediately after the collision, the speeds of P and Q are $\frac{5}{9}u$ and $\frac{8}{9}u$ respectively.

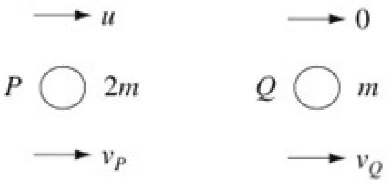
After the collision, Q strikes a fixed vertical wall which is perpendicular to the direction of motion of P and Q. The coefficient of restitution between Q and the wall is e. When P and Q collide again, P is brought to rest.

b Find the value of e.

c Explain why there must be a third collision between P and Q.

Solution:

a



$2mu = 2mv_P + mv_Q$ — Conservation of momentum.

$2u = 2v_P + v_Q$ (1)

$\frac{1}{3}u = v_Q - v_P$ (2) — Newton's Law of Restitution.

$(1) + 2 \times (2)\ \frac{8u}{3} = v_Q + 2v_Q$

$3v_Q = \frac{8u}{3}$

$v_Q = \frac{8u}{9}$

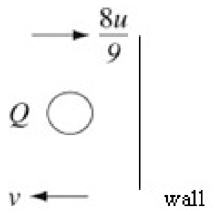
Using (2)

$v_P = v_Q - \frac{1}{3}u$

$v_P = \frac{8u}{9} - \frac{1}{3}u$

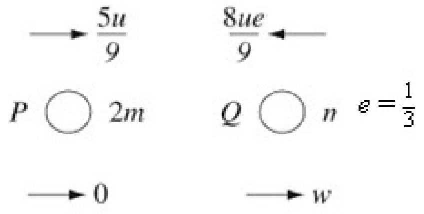
$v_P = \frac{5u}{9}$

b



$v = \frac{8ue}{9}$

Newton's Law of Restitution.



Here, $e = \frac{1}{3}$ again.

$$2m \times \frac{5u}{9} - m \times \frac{8ue}{9} = mw$$

Conservation of momentum.

$$10u - 8ue = 9w \qquad (3)$$

$$\frac{1}{3}\left(\frac{5u}{9} + \frac{8ue}{9}\right) = w$$

Newton's Law of Restitution.

$$5u + 8ue = 27w \qquad (4)$$

Eliminating w between (3) and (4):

$$3(10u - 8ue) = 5u + 8ue$$
$$30u - 24ue = 5u + 8ue$$
$$32ue = 25u$$
$$e = \frac{25}{32}$$

c Q is now moving towards the wall once more.
After Q hits the wall, it will return to collide with P once more.

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Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 19

Question:

Two small smooth spheres, P and Q of equal radius, have masses $2m$ and $3m$ respectively. The sphere P is moving with speed $5u$ on a smooth horizontal table when it collides directly with Q which is at rest on the table. The coefficient of restitution between P and Q is e.

a Show that the speed of Q immediately after the collision is $2(1+e)u$.

After the collision, Q hits a smooth vertical wall which is at the edge of the table and perpendicular to the direction of motion of Q. The coefficient of restitution between Q and the wall is f, $0 < f \leq 1$.

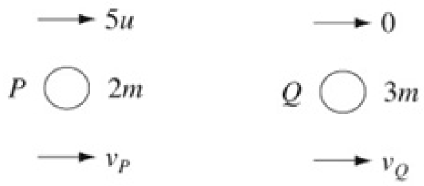
b Show that, when $e = 0.4$, there is a second collision between P and Q.

Given that $e = 0.8$ and there is a second collision between P and Q,

c find the range of possible values of f.

Solution:

a



$2m \times 5u = 2mv_P + 3mv_Q$

Conservation of momentum.

$10u = 2v_P + 3v_Q \quad (1)$

$e \times 5u = v_Q - v_P \quad (2)$

Newton's Law of Restitution.

$(1) + 3 \times (2)$:

$10u + 15eu = 3v_Q + 2v_Q$

$10u + 15eu = 5v_Q$

$v_Q = 2u + 3eu = 2(1+e)u$

b From (2)

Find direction of motion for P, as if P is moving towards the wall there must be a second collision between P and Q.

$v_P = v_Q - 5eu$

$= 2(1+e)u - 5eu$

$v_P = 2 \times 1.4u - 5 \times 0.4u$

$= 0.8u$

$e = 0.4$ in **b**.

$v_P > 0 \therefore P$ moves towards the wall and will collide with Q after Q rebounds from the wall.

c

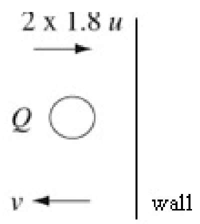
$e = 0.8$

$v_P = 2 \times 1.8u - 5 \times 0.8u$

$= -0.4u$

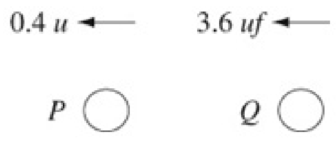
Q hits the wall:

$e = 0.8$



Newton's Law of Restitution.

$v = 3.6uf$



For a second collision

$3.6uf > 0.4u$

$f > \frac{0.4}{3.6} = \frac{1}{9}$

Range for f is

$\frac{1}{9} < f \leq 1$

All coefficients of restitution are less than or equal to 1.

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2 Review Exercise

Exercise A, Question 20

Question:

A particle A of mass $2m$, moving with speed $2u$ in a straight line on a smooth horizontal table, collides with a particle B of mass $3m$, moving with speed u in the same direction as A. The coefficient of restitution between A and B is e.

a Show that the speed of B after the collision is

$$\frac{1}{5}u(7+2e).$$

b Find the speed of A after the collision, in terms of u and e.

The speed of A after the collision is $\frac{11}{10}u$.

c Show that $e = \frac{1}{2}$.

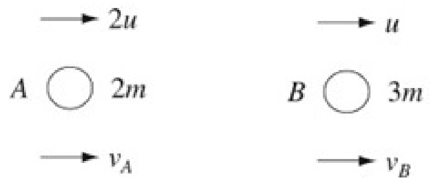
At the instant of collision, A and B are at a distance d from a vertical barrier fixed to the surface at right-angles to their direction of motion. Given that B hits the barrier, and that the coefficient of restitution between B and the barrier is $\frac{11}{16}$,

d find the distance of A from the barrier at the instant that B hits the barrier,

e show that, after B rebounds from the barrier, it collides with A again at a distance $\frac{5}{32}d$ from the barrier.

Solution:

a



$$2m \times 2u + 3m \times u = 2mv_A + 3mv_B$$ Conservation of momentum.

$$7u = 2v_A + 3v_B \quad (1)$$

$$e(2u - u) = v_B - v_A$$ Newton's Law of Restitution.

$$eu = v_B - v_A \quad (2)$$

$(1) + 2 \times (2)$

$$7u + 2eu = 3v_B + 2v_B$$

$$v_B = \frac{1}{5}u(7 + 2e)$$

b Using (2)

$$v_A = v_B - eu$$

$$= \frac{1}{5}u(7 + 2e) - eu$$

$$= \frac{1}{5}u(7 - 3e)$$

c $\frac{1}{5}u(7 - 3e) = \frac{11u}{10}$

$$14u - 6eu = 11u$$

$$6eu = 3u \quad e = \frac{1}{2}$$

d For B:

Distance to barrier $= d$

Speed $= \frac{1}{5}u(7+1) = \frac{8u}{5}$

$\therefore$ Time to barrier $= d \div \frac{8u}{5} = \frac{5d}{8u}$

Distance moved by A in this time:

$= \frac{1}{5}u\left(7 - \frac{3}{2}\right) \times \frac{5d}{8u}$

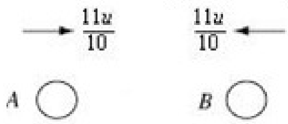
$= \frac{11u}{5 \times 2} \times \frac{5d}{8u} = \frac{11d}{16}$

$\therefore$ A is $d - \frac{11d}{16} = \frac{5d}{16}$ from the barrier.

Use $e = \frac{1}{2}$ now.

e After B hits the barrier:

Speed of $B = \frac{11}{16} \times \frac{8u}{5} = \frac{11u}{10}$



Equal speeds, opposite directions.

$\therefore$ A and B will collide at mid-point of the distance from A to the barrier at the instant B hits the barrier, i.e. they collide at distance $\frac{5d}{32}$ from the barrier.

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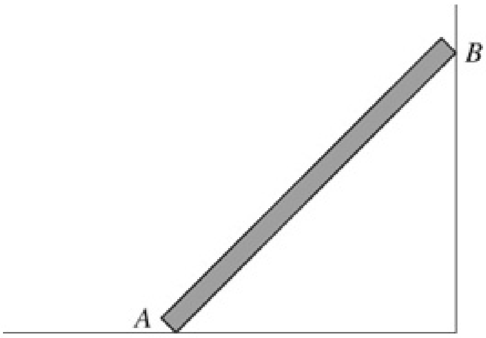
2 Review Exercise
Exercise A, Question 21

Question:



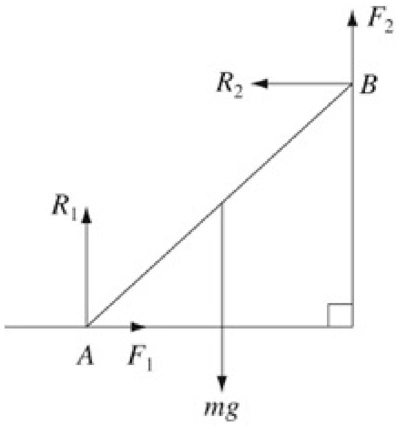
The diagram shows a uniform heavy plank of wood AB, of mass m, whose lower end A is resting on rough horizontal ground and whose upper end B is resting against a rough vertical wall. The coefficient of friction between the plank and the ground and between the plank and the wall is $\frac{2}{3}$.

The plank is about to slip at both ends.

a Suggest a suitable model for the plank so that the forces exerted on it by the ground and the wall can be found.

b Show that the horizontal component of the force exerted by the wall on the plank is $\frac{6mg}{13}$.

Solution:



a uniform rod

b

$$R(\rightarrow) \quad R_2 = F_1$$

At the ground $F_1 = \frac{2}{3}R_1$ — Friction is limiting.

$$\therefore R_2 = \frac{2}{3}R_1$$

$$R(\uparrow) \quad F_2 + R_1 = mg$$

At the wall $F_2 = \frac{2}{3}R_2$ — Friction is limiting.

$$\therefore \frac{2}{3}R_2 + \frac{3}{2}R_2 = mg$$

Substitute for F_2 and R_1 in $F_2 + R_1 = mg$.

$$\frac{13}{6}R_2 = mg$$

$$R_2 = \frac{6}{13}mg$$

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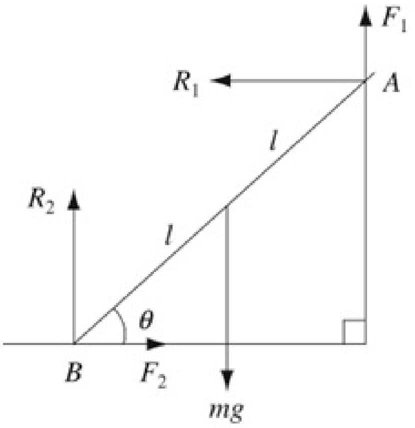
2 Review Exercise
Exercise A, Question 22

Question:

A uniform rod, of mass m, rests with one end A against a rough vertical wall and the other end B on a rough horizontal floor. The vertical plane through the rod is perpendicular to the wall. The coefficient of friction between the wall and the rod is μ_1. The coefficient of friction between the floor and the rod is μ_2. Given that θ is the inclination of the rod to the floor when the rod is on the point of slipping, show that

$$2\mu_2 \tan\theta = 1 - \mu_1\mu_2.$$

Solution:



Let $AB = 2l$

$F_1 = \mu_1 R_1 \quad F_2 = \mu_2 R_2$

The rod must be on the point of slipping at both ends.

$$\text{M}(B) \quad mgl\cos\theta = F_1 \times 2l\cos\theta + R_1 \times 2l\sin\theta$$

$$mg\cos\theta = 2\mu_1 R_1\cos\theta + 2R_1\sin\theta \quad (1)$$

$$\text{R}(\uparrow) \quad F_1 + R_2 = mg$$

Resolving will not use the angle θ.

$$\mu_1 R_1 + R_2 = mg \quad (2)$$

$$\text{R}(\rightarrow) \quad F_2 = R_1$$

$$\therefore R_1 = \mu_2 R_2 \quad (3)$$

From (2) and (3)

$$R_1 = \mu_2(mg - \mu_1 R_1)$$

R_1 is needed to substitute in equation (1).

$$R_1(1 + \mu_1\mu_2) = \mu_2 mg$$

$$R_1 = \frac{\mu_2 mg}{(1 + \mu_1\mu_2)}$$

Substitute in (1)

$$mg\cos\theta = 2\mu_1\cos\theta \times \frac{\mu_2 mg}{(1 + \mu_1\mu_2)}$$

$$+2\sin\theta \times \frac{\mu_2 mg}{(1 + \mu_1\mu_2)}$$

$$1 = \frac{2\mu_1\mu_2}{(1 + \mu_1\mu_2)} + \frac{2\tan\theta \times \mu_2}{(1 + \mu_1\mu_2)}$$

Cancel mg and divide by $\cos\theta$ using $\frac{\sin\theta}{\cos\theta} = \tan\theta$.
R_1 is needed to substitute in equation (1).

$$1 + \mu_1\mu_2 = 2\mu_1\mu_2 + 2\mu_2\tan\theta$$

$$2\mu_2\tan\theta = 1 - \mu_1\mu_2$$

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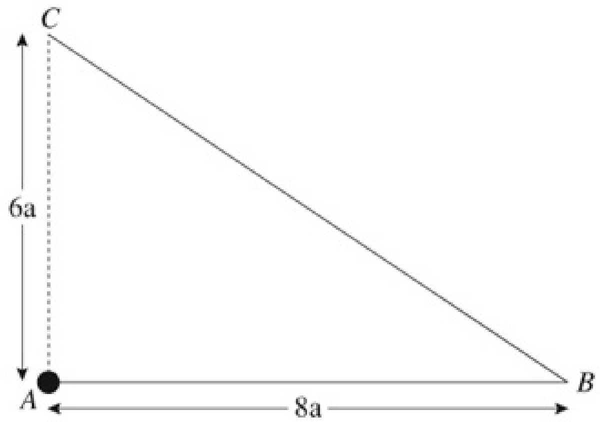
2 Review Exercise
Exercise A, Question 23

Question:

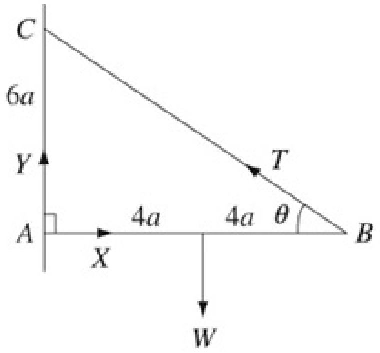


A uniform rod AB, of length $8a$ and weight W, is free to rotate in a vertical plane about a smooth pivot at A. One end of a light inextensible string is attached to B. The other end is attached to point C which is vertically above A, with $AC = 6a$. The rod is in equilibrium with AB horizontal, as shown in the diagram.

a By taking moments about A, or otherwise, show that the tension in the string is $\frac{5}{6}W$.

b Calculate the magnitude of the horizontal component of the force exerted by the pivot on the rod.

Solution:



a $\mathrm{M}(A)\quad T\sin\theta\times 8a = W\times 4a$

$BC = 10a \Rightarrow \sin\theta = \frac{3}{5}$

By Pythagoras or recognition of a (3, 4, 5) Δ.

$$T\times\frac{3}{5}\times 8 = 4w$$

$$T = \frac{5w}{6}$$

b $\mathrm{R}(\rightarrow)\ X = T\cos\theta$

$$X = \frac{5w}{6}\times\frac{4}{5}$$

$$X = \frac{2}{3}W$$

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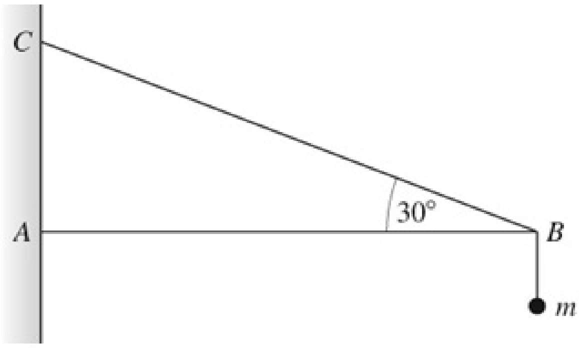
2 Review Exercise
Exercise A, Question 24

Question:

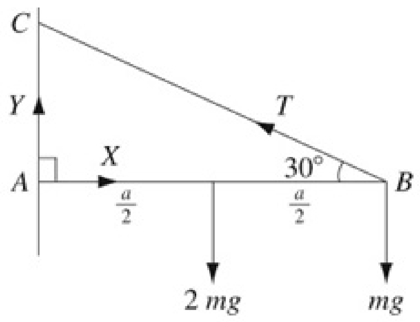


A uniform rod AB has mass $2m$ and length a. The end A is smoothly hinged at a fixed point. A particle of mass m is suspended from the rod at the end B. The loaded rod is held in equilibrium in a horizontal position by a light string, one end of which is attached to the rod at B, the other end being fixed to a point C vertically above A, as shown in the diagram. The string makes an angle of $30°$ with the horizontal.

a Show that the tension in the string is $4mg$.

b Find the magnitude of the force exerted by the hinge on the rod at A.

Solution:



a M(A) $2mg \times \frac{a}{2} + mga = Ta \sin 30°$

$2mg = T \times \frac{1}{2}$

$T = 4mg$

Taking moments about A gives an equation without X or Y.

b R($\uparrow$) $Y + T \sin 30° = 2mg + mg$

$Y = 3mg - 4mg \sin 30°$

$Y = 3mg - 4mg \times \frac{1}{2}$

$Y = mg$

R($\rightarrow$) $X = T \cos 30°$

$X = 4mg \times \frac{\sqrt{3}}{2}$

$X = 2mg\sqrt{3}$

Resultant $= \sqrt{\left[1 + (2\sqrt{3})^2\right]}mg$

$= mg\sqrt{13}$

Use the exact value of $\cos 30°$ for an exact final answer.

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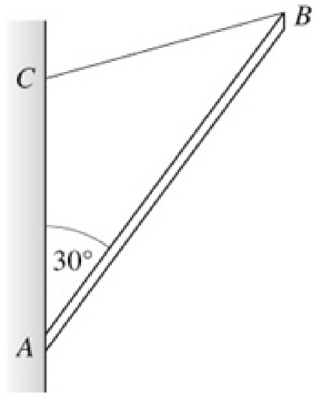
2 Review Exercise
Exercise A, Question 25

Question:



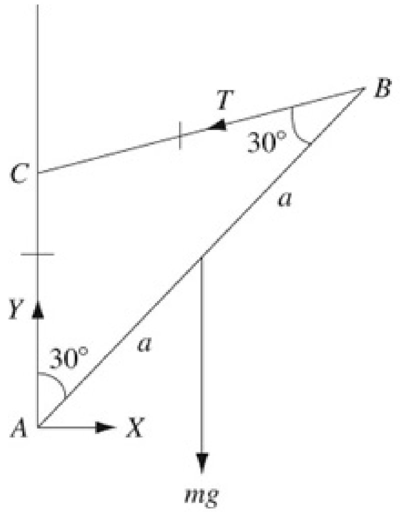
A uniform rod AB of mass m and length $2a$ is smoothly hinged to a vertical wall at A and is supported in equilibrium by a rope which is modelled as a light string. One end of the rope is attached to the end B of the rod and the other end is attached to a point C of the wall, where C is vertically above A, $AC = CB$, and $\angle CAB = 30°$, as shown in the diagram.

a Show that the tension in the rope is $0.5mg$.

b Find the magnitude of the vertical component of the force acting on the rod at A.

c If the rope were not modelled as a light string, state how this would affect the tension throughout the rope.

Solution:



ΔABC is isoceles.

a M(A) $\quad mga\sin 30° = T \times 2a\sin 30°$

$$2T = mg$$

$$T = 0.5mg$$

Taking moments about A gives an equation in which the only unknown is T.

b R($\uparrow$) $\quad Y = mg + T\cos 60°$

$$Y = mg + \frac{1}{2}mg \times \frac{1}{2}$$

$$Y = \frac{5mg}{4}$$

c The tension would not be constant throughout the length of the string.

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2 Review Exercise

Exercise A, Question 26

Question:

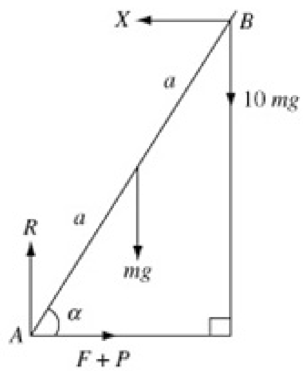
A uniform ladder AB, of mass m and length $2a$, has one end A on rough horizontal ground. The coefficient of friction between the ladder and the ground is 0.6. The other end B of the ladder rests against a smooth vertical wall.

A builder of mass $10m$ stands at the top of the ladder. To prevent the ladder from slipping, the builder's friend pushes the bottom of the ladder horizontally towards the wall with a force of magnitude P. This force acts in a direction perpendicular to the wall. The ladder rests in equilibrium in a vertical plane perpendicular to the wall and makes an angle α with the horizontal, where $\tan\alpha = \frac{3}{2}$.

a Show that the reaction of the wall on the ladder has magnitude $7mg$.

b Find, in terms of m and g, the range of values of P for which the ladder remains in equilibrium.

Solution:



a M(A) $X \times 2a \sin\alpha = 10mg \times 2a\cos\alpha + mg \times a\cos\alpha$

$$2X\tan\alpha = 20mg + mg$$

Divide by $\cos\alpha$ as you know the value of $\tan\alpha$.

$$2X \times \frac{3}{2} = 20mg + mg$$

$$3X = 21mg$$

$$X = 7mg$$

b R($\uparrow$) $R = 10mg + mg = 11mg$

R($\rightarrow$) $F + P = X$

$$P = X - F$$

P is minimum when F acts towards the wall and has its maximum magnitude.

$$F = \mu R = 0.6 \times 11mg = 6.6mg$$

$$\therefore P_{\min} = 7mg - 6.6mg$$

$$= 0.4mg$$

P is maximum when F acts away from the wall and has its maximum magnitude.

$$P_{\max} = 7mg - (-6.6mg)$$

$$= 13.6mg$$

$$\therefore 0.4mg \le P \le 13.6mg$$

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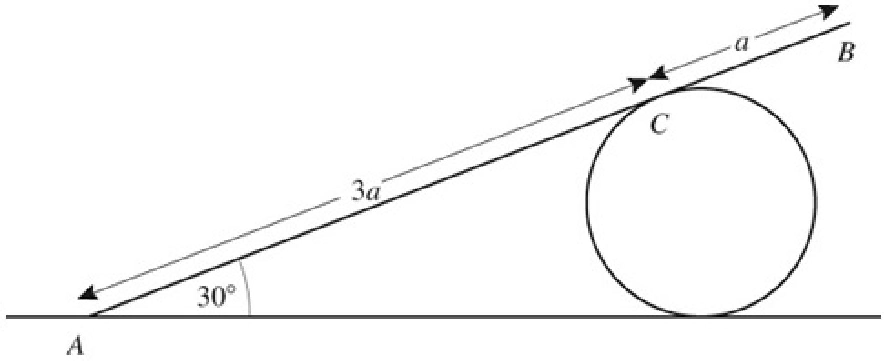
2 Review Exercise
Exercise A, Question 27

Question:



A piece of equipment used in an acrobatic show consists of a smooth cylinder which is fixed, with its axis horizontal, to a rough horizontal plane. A plank, which is modelled as a uniform rod AB of mass m and length $4a$, rests in equilibrium on the cylinder at the point C, where $AC = 3a$.

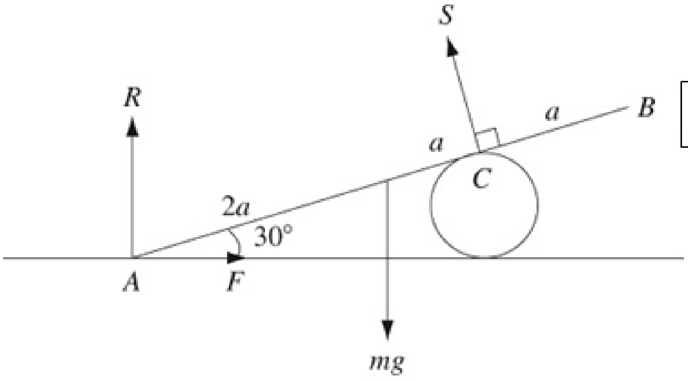
The end A of the plank rests on the plane and AB makes an angle of $30°$ with the horizontal, as shown in the diagram. The points A, B and C lie in a vertical plane which is perpendicular to the axis of the cylinder.

a Find the magnitude of the force exerted on the plank by the cylinder at the point C.

Given that the plank is in limiting equilibrium and that the coefficient of friction between the plank and the plane is μ,

b show that $\mu = \frac{1}{3}\sqrt{3}$.

Solution:



There is no frictional force at C.

a M(A) $S\times 3a = mg\times 2a\cos 30°$

$3S = 2mg\times\frac{\sqrt{3}}{2}$

$S = mg\frac{\sqrt{3}}{2}$

Take moments about A for an equation without R and F.

b $F = \mu R$

R(↑) $R = mg - S\cos 30°$

$R = mg - mg\frac{\sqrt{3}}{3}\times\frac{\sqrt{3}}{2}$

$R = mg - \frac{1}{2}mg = \frac{1}{2}mg$

R(→) $F = S\cos 60°$

$F = mg\frac{\sqrt{3}}{3}\times\frac{1}{2}$

Friction is limiting.

$\mu = \frac{F}{R} = \frac{\sqrt{3}}{6}\div\frac{1}{2} = \frac{\sqrt{3}}{3}$

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2 Review Exercise
Exercise A, Question 28

Question:

A non-uniform ladder AB, of length $4a$ and weight W, has its centre of mass at a distance a from B. The ladder rests with A against a rough vertical wall and with its lower end B on smooth horizontal ground.

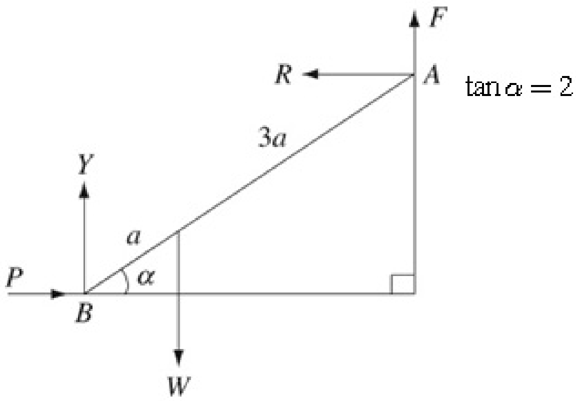
The coefficient of friction between the wall and the ladder is μ. The ladder is in a vertical plane perpendicular to the wall and makes an angle α with the horizontal where $\tan\alpha = 2$. A man can just prevent the ladder from slipping down by applying a horizontal force of magnitude P, perpendicular to the wall, at B. The ladder is modelled as a non-uniform rod.

a Draw a diagram showing all the forces acting on the ladder.

b Find an expression for P in terms of W and μ.

Solution:

a



b

M(A) $W \times 3a\cos\alpha + P \times 4a\sin\alpha = Y \times 4a\cos\alpha$

$$\begin{aligned}
3W + 4P\tan\alpha &= 4Y \\
3W + 8P &= 4Y \quad (1) \\
\text{R}(\rightarrow) \quad P &= R \quad (2) \\
\text{R}(\uparrow) \quad F + Y &= W \quad (3) \\
F &= \mu R \quad (4)
\end{aligned}$$

Divide by $\cos\alpha$ and use $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$.

Friction is limiting.

Using (3) and (4)

$Y = W - \mu R$

Using this in (1)

$$\begin{aligned}
3W + 8P &= 4(W - \mu R) \\
\text{and since } P &= R \quad \text{(from (2))} \\
3W + 8P &= 4(W - \mu P) \\
3W + 8P &= 4W - 4\mu P \\
P(4\mu + 8) &= W \\
P &= \frac{W}{4\mu + 8}
\end{aligned}$$

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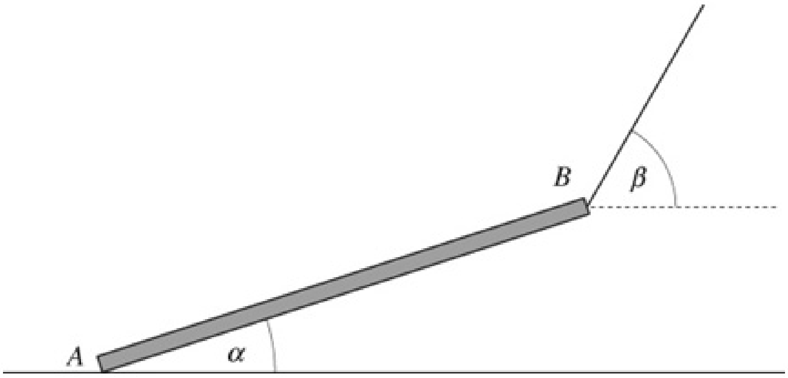
2 Review Exercise
Exercise A, Question 29

Question:



A straight log AB has weight W and length $2a$. A cable is attached to one end B of the log. The cable lifts the end B off the ground. The end A remains in contact with the ground, which is rough and horizontal. The log is in limiting equilibrium. The log makes an angle α to the horizontal, where $\tan\alpha = \frac{5}{12}$. The cable makes an angle β to the horizontal, as shown in the diagram. The coefficient of friction between the log and the ground is 0.6. The log is modelled as a uniform rod and the cable as light.

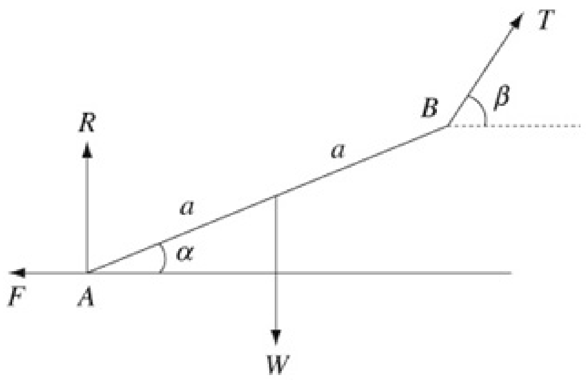
a Show that the normal reaction on the log at A is $\frac{2}{5}W$.

b Find the value of β.

The tension in the cable is kW.

c Find the value of k.

Solution:



a

$$\begin{aligned}
\text{M}(B)\ R\times 2a\cos\alpha &+ F\times 2a\sin\alpha \\
&= W\times a\cos\alpha \\
2R+2F\tan\alpha &= W \\
2R+2F\times\tfrac{5}{12} &= W \\
2R+\tfrac{5}{6}F &= W \\
F=\mu R &= 0.6R \\
\therefore 2R+\tfrac{5}{6}\times 0.6R &= W \\
2.5R &= W \\
R &= \frac{W}{2.5}=\frac{2W}{5}
\end{aligned}$$

T is not wanted.

Divide by $\cos\alpha$ as the value of $\tan\alpha$ is known.

Friction is limiting.

b

$\text{R}(\rightarrow)\qquad T\cos\beta = F$

As $F=0.6R$ and $R=\frac{2}{5}W$

$$\begin{aligned}
T\cos\beta &= 0.6\times\tfrac{2}{5}W \\
T\cos\beta &= \tfrac{6}{25}W \quad (1) \\
\text{R}(\uparrow)\quad T\sin\beta+R &= W \\
T\sin\beta &= W-\tfrac{2}{5}W \\
T\sin\beta &= \tfrac{3}{5}W \quad (2) \\
(2)\div(1)\quad \frac{T\sin\beta}{T\cos\beta} &= \frac{3}{5}W\times\frac{25}{6W} \\
\tan\beta &= \tfrac{5}{2} \\
\beta &= 68.2^\circ
\end{aligned}$$

c

$$\begin{aligned}
T &= \frac{3}{5}W\div\sin 68.2 \\
&= 0.646W \\
k &= 0.646
\end{aligned}$$

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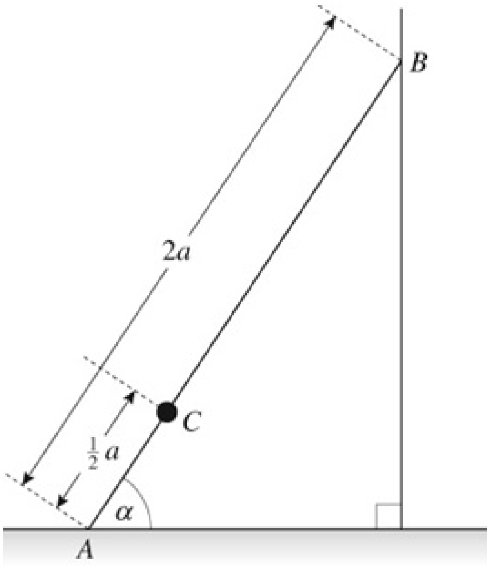
2 Review Exercise
Exercise A, Question 30

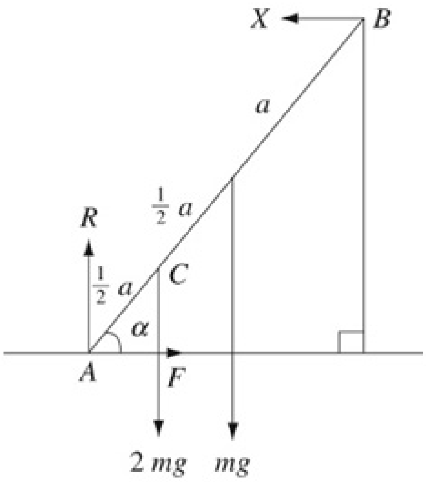
Question:



A uniform ladder AB, of mass m and length $2a$, has one end A on rough horizontal ground. The other end B rests against a smooth vertical wall. The ladder is in a vertical plane perpendicular to the wall. The ladder makes an angle α with the horizontal, where $\tan\alpha = \frac{4}{3}$. A child of mass $2m$ stands on the ladder at C where $AC = \frac{1}{2}a$, as shown in the diagram. The ladder and the child are in equilibrium.

By modelling the ladder as a rod and the child as a particle, calculate the least possible value of the coefficient of friction between the ladder and the ground.

Solution:



$\tan\alpha = \frac{4}{3}$

M(A)

$2mg \times \frac{1}{2}a\cos\alpha + mg \times a\cos\alpha = X \times 2a\sin\alpha$

$mg + mg = 2X\tan\alpha$

Divide by $\cos\alpha$ and use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$.

$2mg = 2X \times \frac{4}{3}$

$X = \frac{3}{4}mg$

R(↑) $R = 2mg + mg$

$R = 3mg$

R(→) $F = X$

$\therefore F = \frac{3}{4}mg$

$F \le \mu R$

$\frac{3}{4}mg \le \mu \times 3mg$

$\frac{1}{4} \le \mu$

The least value of μ is $\frac{1}{4}$.

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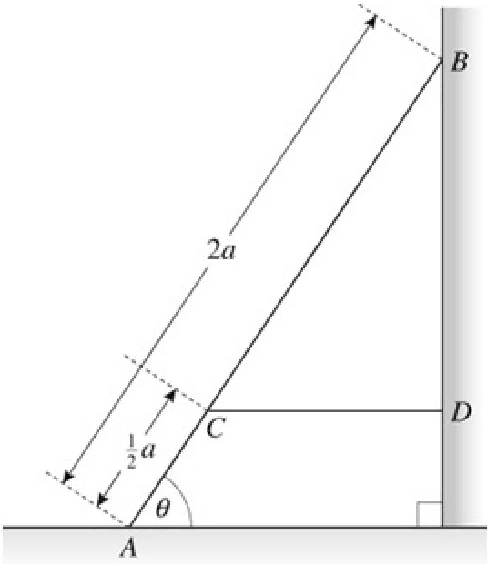
2 Review Exercise
Exercise A, Question 31

Question:



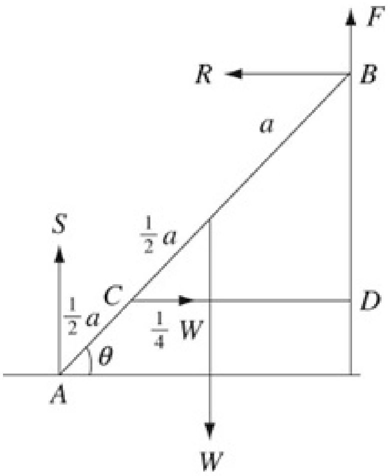
A uniform ladder, of weight W and length $2a$, rests in equilibrium with one end A on a smooth horizontal floor and the other end B on a rough vertical wall. The ladder is in a vertical plane perpendicular to the wall. The coefficient of friction between the wall and the ladder is μ. The ladder makes an angle θ with the floor, where $\tan\theta = 2$. A horizontal light inextensible string CD is attached to the ladder at the point C, where $AC = \frac{1}{2}a$. The string is attached to the wall at the point D, with BD vertical, as shown in the diagram. The tension in the string is $\frac{1}{4}W$. By modelling the ladder as a rod,

a find the magnitude of the force of the floor on the ladder,

b show that $\mu \geq \frac{1}{2}$.

c State how you have used the modelling assumption that the ladder is a rod.

Solution:



$\tan\theta = 2$

a $\mathrm{M}(B)\quad S \times 2a\cos\theta$

$$= \frac{1}{4}W \times \frac{3}{2}a\sin\theta + W \times a\cos\theta$$

Taking moments about B gives an equation with S the only unknown quantity.

$$2S = \frac{3}{8}W\tan\theta + W$$

Divide by $\cos\theta$ and use $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

$$2S = \frac{3W}{4} + W$$

$\tan\theta = 2$

$$S = \frac{7W}{8}$$

b $\mathrm{R}(\rightarrow)\quad R = \frac{1}{4}W$

$\mathrm{R}(\uparrow)\quad F + S = W$

$$F = W - \frac{7W}{8} = \frac{1}{8}W$$

$$F \le \mu R$$

$$\frac{1}{8}W \le \frac{1}{4}W \times \mu$$

$\therefore \mu \ge \frac{1}{2}$

c The ladder will be straight.

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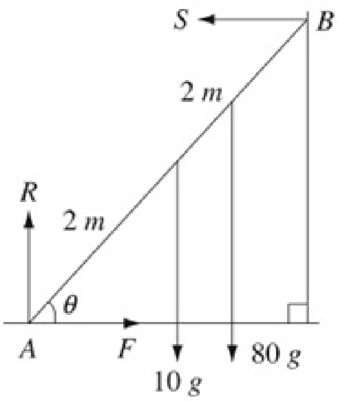
2 Review Exercise
Exercise A, Question 32

Question:

A uniform ladder AB, of mass 10 kg and length 4 m, rests in equilibrium with the end A on rough horizontal ground. The end B of the ladder rests against a smooth vertical wall, the ladder being in a vertical plane perpendicular to the wall. The coefficient of friction between the ladder and the ground is $\frac{1}{3}$. The ladder is inclined at an angle θ to the horizontal, where $\tan\theta = 2$. A man of mass 80 g stands on the ladder at a point which is a distance x metres from A.

Find the range of possible values of x.

Solution:



$\mu = \frac{1}{3}$

$\tan\theta = 2$

M(A) $10g \times 2\cos\theta + 80gx\cos\theta = S \times 4\sin\theta$

$20g + 80gx = 4S\tan\theta$

Divide by $\cos\theta$ and use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$20g + 80gx = 8S$

$\tan\theta = 2$

R(↑) $R = 80g + 10g$

$R = 90g$

R(→) $F = S$

$\therefore F = \frac{1}{8}(20g + 80gx)$

For equilibrium

$F \le \mu R$

$F \le \frac{1}{3}R$

$\therefore \frac{1}{8}(20g + 80gx) \le \frac{1}{3} \times 90g$

$\frac{1}{4} + x \le 3$

$x \le \frac{11}{4}$

x must be positive (or zero)

$\therefore 0 \le x \le \frac{11}{4}$

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2 Review Exercise
Exercise A, Question 33

Question:

A uniform ladder, of mass M and length 5 m, has one end on rough horizontal ground with the other end placed against a smooth vertical wall. The coefficient of friction between the ladder and the ground is 0.3. The highest point of the wall is higher than the highest point on the ladder. Given that the top of the ladder is 4 m vertically above the level of the ground,

a show that the ladder cannot remain in equilibrium in this position.

A brick is placed on the bottom rung of the ladder in order to enable it to stay in equilibrium in the position described above. Assuming that the brick is at the very bottom of the ladder and does not touch the ground,

b show that the horizontal frictional force exerted on the ladder by the ground is independent of the mass of the brick.

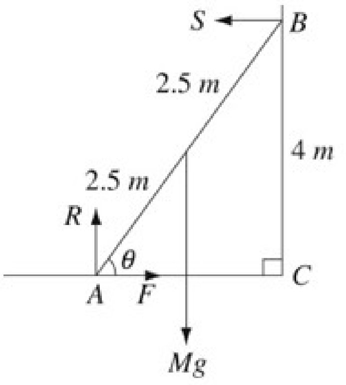
c Find, in terms of M, the smallest mass of the brick which will enable the ladder to remain in equilibrium.

The ladder, without the brick, is now extended so that the top of the ladder is higher than the top of the wall.

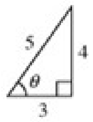
d Draw a diagram showing the forces acting on the ladder in this situation.

Solution:

a



$\mu = 0.3$



ΔABC is a (3,4,5) Δ.

Assume equilibrium:

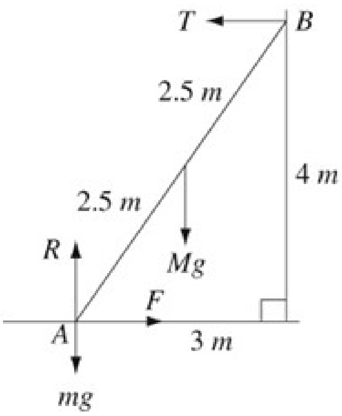
$$
\begin{aligned}
M(B) \quad R \times 3 &= Mg \times 1.5 + F \times 4 \\
\mathrm{R}(\uparrow) \quad R &= Mg \\
\therefore 4F &= 3Mg - 1.5Mg \\
F &= \tfrac{3}{8}Mg = 0.375Mg
\end{aligned}
$$

but $\mu = 0.3$

$\therefore$ maximum possible friction force
$= 0.3R = 0.3Mg$

$\therefore$ Equilibrium is not possible.

b



$$
\begin{aligned}
\mathrm{M}(A) \quad 4T &= 1.5Mg \\
\mathrm{R}(\rightarrow) \quad T &= F \\
\therefore F &= \frac{1.5}{4}Mg = \frac{3}{8}Mg
\end{aligned}
$$

$\therefore$ independent of mass of brick.

a

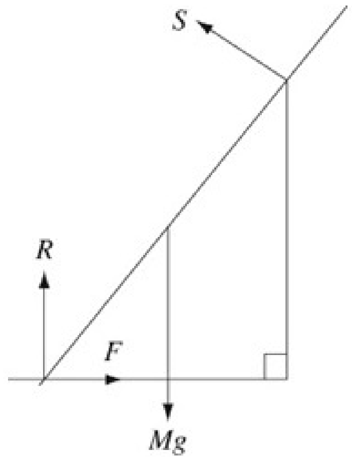
c R(↑) $R = Mg + mg$

For equilibrium

$$\begin{aligned} F &\le \mu R \\ \tfrac{3}{8}Mg &\le 0.3(Mg + mg) \\ 0.375M &\le 0.3M + 0.3m \\ 0.075M &\le 0.3m \\ m &\ge \frac{0.075M}{0.3} = 0.25M \end{aligned}$$

The smallest mass of the brick is $\frac{1}{4}M$.

d



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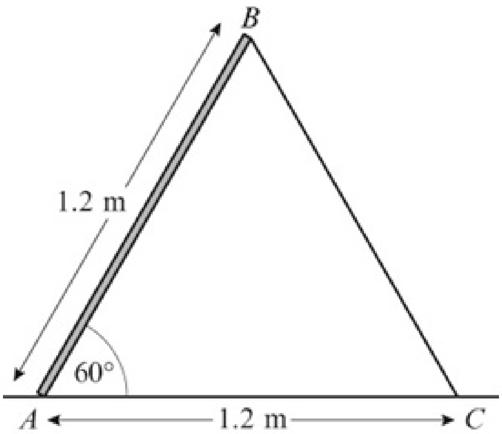
2 Review Exercise
Exercise A, Question 34

Question:



A trap door is propped open at $60°$ to the horizontal by a pole. The trap door is modelled as a uniform rod AB, of mass 10 kg and length 1.2 m, smoothly hinged at A. The pole is modelled as a light rod BC, smoothly hinged to AB at B. The points A and C are at the same horizontal level, $AC = 1.2$ m and the plane ABC is vertical, as shown in the diagram.

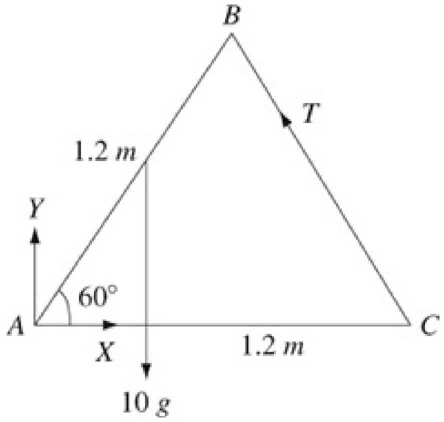
Find, to 3 significant figures,

a the thrust in BC,

b the magnitude of the force acting on the rod AB at A.

Solution:

a



$$\text{M}(A) \quad T \times 1.2\cos 30° = 10g \times 0.6\cos 60°$$

ΔABC is equilateral.

$$T = \frac{10 \times 9.8 \times 0.6\cos 60°}{1.2\cos 30°}$$

$$T = 28.29\ldots$$

$$T = 28.3 \text{ N}$$

Keep a record of T to at least 4 significant figures for use later.

b $\text{R}(\uparrow) \quad Y + T\cos 30° = 10g$

$$Y = 10 \times 9.8 - 28.29\cos 30°$$

$$Y = 73.50$$

$$\text{R}(\rightarrow) \quad X = T\cos 60°$$

$$= 14.145$$

Magnitude of resultant

$$= \sqrt{(73.50^2 + 14.145^2)}$$

$$= 74.84$$

$$= 74.8 \text{ N}$$

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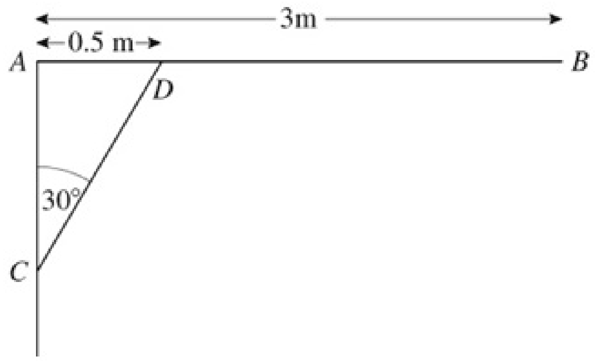
2 Review Exercise
Exercise A, Question 35

Question:



A uniform pole AB, of mass 30 kg and length 3 m, is smoothly hinged to a vertical wall at one end A. The pole is held in equilibrium in a horizontal position by a light rod CD. One end C of the rod is fixed to the wall vertically below A. The other end D is freely jointed to the pole so that $\angle ACD = 30°$ and $AD = 0.5$ m, as shown in the diagram. Find

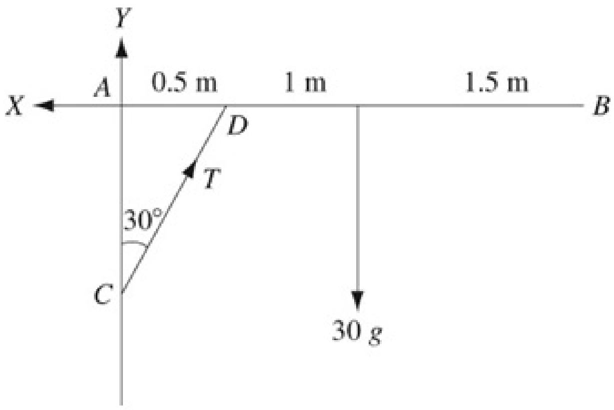
a the thrust in the rod CD,

b the magnitude of the force exerted by the wall on the pole at A.

The rod CD is removed and replaced by a longer light rod CM, where M is the midpoint of AB. The rod is freely jointed to the pole at M. The pole AB remains in equilibrium in a horizontal position.

c Show that the force exerted by the wall on the pole at A now acts horizontally.

Solution:



a M(A)

$$T \times 0.5\cos 30° = 30g \times 1.5$$

Take moments about A to get an equation with T as the only unknown.

$$T = \frac{30 \times 9.8 \times 1.5}{0.5\cos 30°}$$

$$T = 1018$$

$$T = 1020 \text{ N}$$

b R(↑) $Y + T\cos 30° = 30g$

$$Y = 30g - T\cos 30°$$

$$Y = -587.6$$

R(→) $X = T\cos 60°$

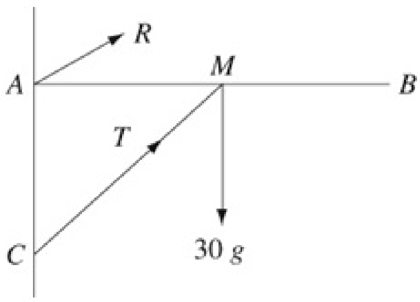
$$= 509.2$$

Magnitude of resultant

$$= \sqrt{(509.2^2 + (-587.6)^2)}$$

$$= 777.5 = 778 \text{ N}$$

c



Consider the moments of the forces about M.

Moment of T = 0

Moment of $30g$ = 0

So, for equilibrium, the moment of R must be zero.
Hence R must pass through M and so is horizontal.

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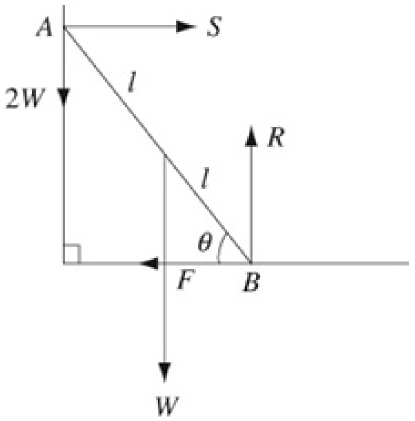
2 Review Exercise
Exercise A, Question 36

Question:

A uniform ladder rests with its lower end on a rough horizontal path and its upper end against a smooth vertical wall. The ladder rests in a vertical plane perpendicular to the wall. A woman stands on the top of this ladder, and the ladder is in limiting equilibrium. The weight of the woman is twice the weight of the ladder, and the coefficient of friction between the path and the ladder is $\frac{5}{12}$. By modelling the ladder as a uniform rod and the woman as a particle, find, to the nearest degree, the angle between the ladder and the horizontal.

Solution:



$\mu = \frac{5}{12}$

Ladder is uniform, so let its length be $2l$.

$$\begin{aligned}
\text{R}(\uparrow)\quad R &= 2W + W = 3W \\
F &= \mu R = \frac{5}{12} \times 3W \\
F &= \frac{5}{4} W
\end{aligned}$$

Friction is limiting.

M(A)

$$\begin{aligned}
W \times l\cos\theta + F \times 2l\sin\theta &= R \times 2l\cos\theta \\
W\cos\theta + \frac{5}{4}W \times 2\sin\theta &= 3W \times 2\cos\theta \\
W + \frac{5}{2}W\tan\theta &= 6W \\
\frac{5}{2}\tan\theta &= 5 \\
\tan\theta &= 2 \\
\theta &= 63.4\ldots \\
\theta &= 63^\circ \text{ (nearest degree)}
\end{aligned}$$

S is not known and not needed.

Divide by $\cos\theta$ and use $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

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2 Review Exercise

Exercise A, Question 37

Question:

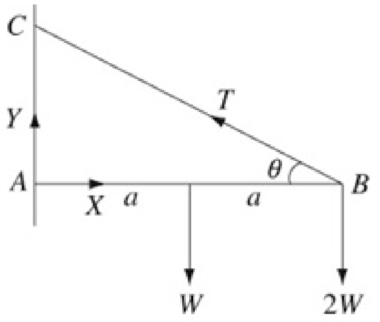
A uniform rod AB, of length $2a$ and weight W, is hinged to a vertical post at A and is supported in a horizontal position by a string attached to B and to a point C vertically above A, where $\angle ABC = \theta$.

A load of weight $2W$ is hung from B.

Find the tension in the string and the horizontal and vertical resolved parts of the force exerted by the hinge on the rod.

Show that, if the reaction of the hinge at A is at right angles to BC, then $AC = 2a\sqrt{5}$.

Solution:



$$\mathrm{M}(A) \quad T \times 2a\sin\theta = W \times a + 2W \times 2a$$

$$2T\sin\theta = 5W$$

$$T = \frac{5W}{2\sin\theta}$$

W and θ were both given in the question and are therefore acceptable in the answer.

$$\mathrm{R}(\uparrow) \quad Y + T\sin\theta = W + 2W$$

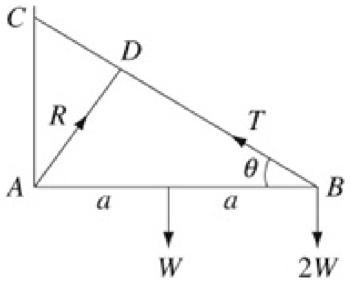
$$Y = 3W - \frac{5W}{2}$$

$$Y = \frac{W}{2}$$

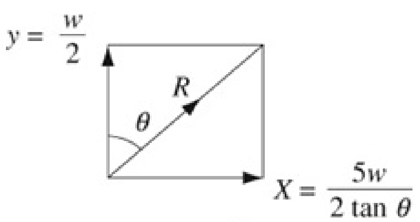
$$\mathrm{R}(\rightarrow) \quad X = T\cos\theta$$

$$X = \frac{5W}{2\sin\theta} \times \cos\theta$$

$$X = \frac{5W}{2\tan\theta}$$



When R is perpendicular to BC
$\angle CAD = \theta$.



$$\tan\theta = \frac{X}{Y} = \frac{\frac{5W}{2\tan\theta}}{\frac{W}{2}} = \frac{5}{\tan\theta}$$

Using the components of R to find $\angle CAD$.

$$\tan^2\theta = 5$$

$$\tan\theta = \sqrt{5}$$

$$AC = 2a\tan\theta$$

Using ΔACB.

$$\therefore AC = 2a\sqrt{5}$$

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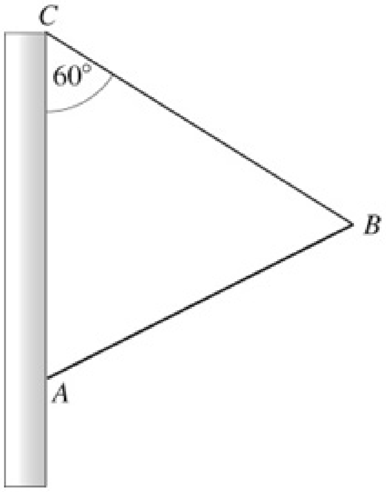
2 Review Exercise
Exercise A, Question 38

Question:

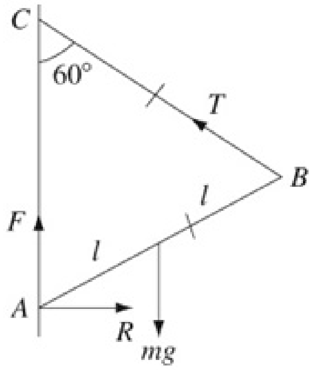


A uniform rod AB of mass m rests in equilibrium with A in contact with a rough vertical wall. The coefficient of friction between the rod and the wall is μ. A light string is attached to B and to a point C of the wall, where C is vertically above A. The plane ABC is perpendicular to the wall, $BC = BA$ and $\angle ACB = 60°$, as shown in the diagram.

a Show that the tension in the string is $\frac{1}{2}mg$.

b Show that $\mu \geq \sqrt{3}$.

Solution:



$\triangle ABC$ is equilateral.

a M(A) $T \times 2l \sin 60° = mg \times l \sin 60°$

$2T = mg$

$T = \frac{1}{2}mg$

b R(↑) $F + T\cos 60° = mg$

$F = mg - \frac{1}{2}mg\cos 60°$

$F = mg - \frac{1}{2}mg \times \frac{1}{2}$

$F = \frac{3}{4}mg$

R(→) $R = T\cos 30°$

$R = \frac{1}{2}mg \times \frac{\sqrt{3}}{2}$ — Use the exact value for an exact final answer.

$R = mg\frac{\sqrt{3}}{4}$

$F \le \mu R$ — Equilibrium condition.

$\frac{3}{4}mg \le \mu mg\frac{\sqrt{3}}{4}$

$\mu \ge \frac{3}{\sqrt{3}}$

$\mu \ge \sqrt{3}$

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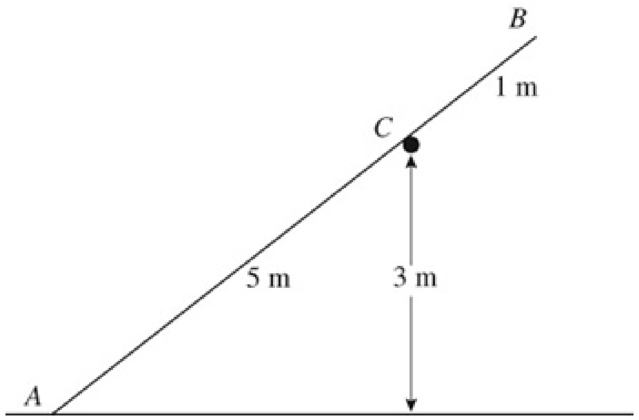
2 Review Exercise
Exercise A, Question 39

Question:



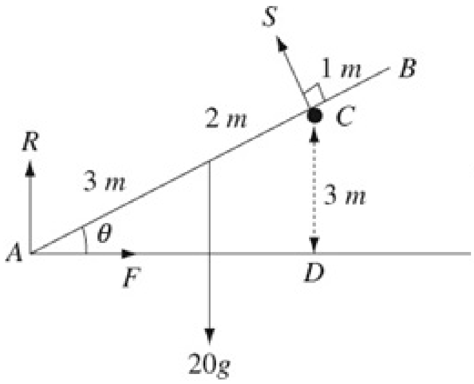
A smooth horizontal rail is fixed at a height of 3 m above a horizontal playground whose surface is rough. A straight uniform pole AB, of mass 20 kg and length 6 m, is placed to rest at a point C on the rail with the end A on the playground. The vertical plane containing the pole is at right angles to the rail. The distance AC is 5 m and the pole rests in limiting equilibrium as shown in the diagram. Calculate

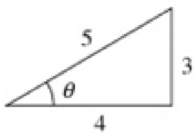
a the magnitude of the force exerted by the rail on the pole, giving your answer to the nearest N,

b the coefficient of friction between the pole and the playground, giving your answer to 2 decimal places,

c the magnitude of the force exerted by the playground on the pole giving your answer to the nearest N.

Solution:





ΔACD is a (3, 4, 5) Δ.

a M(A)
$$5S = 20g \times 3\cos\theta$$
$$5S = 20g \times 3 \times \frac{4}{5}$$
$$5S = 48g$$
$$S = \frac{48 \times 9.8}{5}$$
$$S = 94.08$$
$$S = 94 \text{ N}$$

b R($\rightarrow$)
$$F = S\sin\theta$$
$$F = 94.08 \times \frac{3}{5}$$
$$F = 56.448$$

Keep extra figures for intermediate answers (or use the memory function of your calculator).

R($\uparrow$)
$$R + S\cos\theta = 20g$$
$$R = 20 \times 9.8 - 94.08 \times \frac{4}{5}$$
$$R = 120.74$$

$$F = \mu R$$

Friction is limiting.

$$\mu = \frac{F}{R} = \frac{56.448}{120.74}$$
$$\mu = 0.467\ldots$$
$$\mu = 0.47 \text{ (2 d.p.)}$$

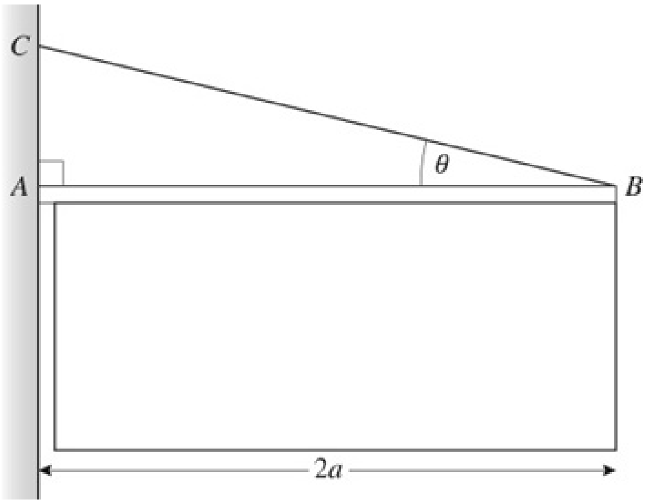
c Magnitude of force
$$= \sqrt{(120.74^2 + 56.448^2)}$$
$$= 133.2$$
$$= 133 \text{ N (nearest N)}$$

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 40

Question:

A pole of mass m and length $2a$ is used to display a light banner. The pole is modelled as a uniform rod AB, freely hinged to a vertical wall at the point A. It is held in a horizontal position by a light wire. One end of the wire is attached to the end B of the rod and the other end is attached to the wall at a point C which is vertically above A such that $\angle ABC$ is θ, where $\tan\theta = \frac{1}{2}$, as shown in the diagram.

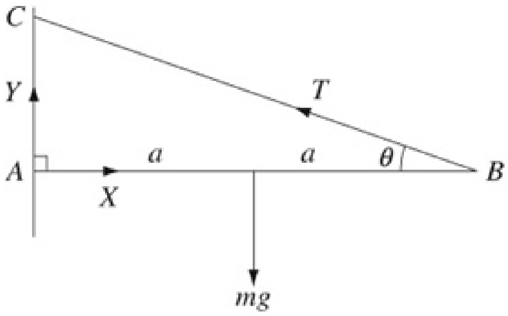


a Show that the tension in the wire is $\dfrac{mg}{2\sin\theta}$.

b Find, in terms of m and g, the magnitude of the force exerted by the wall on the rod at A.

c State, briefly, where in your calculation you have used the modelling assumption that the pole is a rod.

Solution:



$\tan\theta = \frac{1}{2}$

a M(A) $\quad T \times 2a\sin\theta = mg \times a$

$$T = \frac{mg}{2\sin\theta}$$

Take moments about A for an equation with T as the only unknown quantity.

b R(↑) $\quad Y + T\sin\theta = mg$

$$Y = mg - \frac{mg}{2\sin\theta} \times \sin\theta$$

$$Y = \frac{1}{2}mg$$

R(→) $\quad X = T\cos\theta$

$$X = \frac{mg}{2\sin\theta} \times \cos\theta$$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$X = \frac{mg}{2\tan\theta} = mg$$

$\tan\theta = \frac{1}{2}$

Magnitude of resultant

$$= \sqrt{\left[\left(\tfrac{1}{2}mg\right)^2 + (mg)^2\right]}$$

$$= \sqrt{\left(\tfrac{1}{4} + 1\right)}mg = mg\frac{\sqrt{5}}{4}$$

c In the moments equation.