Solutionbank

Edexcel AS and A Level Modular Mathematics

Partial fractions

Exercise A, Question 1

Question:

Express the following as a single fraction:

$$\frac{1}{3} + \frac{1}{4}$$

Solution:

$$\frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

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Partial fractions

Exercise A, Question 2

Question:

Express the following as a single fraction:

$$\frac{3}{4} - \frac{2}{5}$$

Solution:

$$\frac{3}{4} - \frac{2}{5} = \frac{15}{20} - \frac{8}{20} = \frac{7}{20}$$

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Partial fractions

Exercise A, Question 3

Question:

Express the following as a single fraction:

$$\frac{3}{x} - \frac{2}{x+1}$$

Solution:

$$\frac{3}{x} - \frac{2}{x+1}$$

$$= \frac{3(x+1)}{x(x+1)} - \frac{2x}{x(x+1)}$$

$$= \frac{3(x+1) - 2x}{x(x+1)}$$

$$= \frac{3x+3-2x}{x(x+1)}$$

$$= \frac{x+3}{x(x+1)}$$

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Partial fractions

Exercise A, Question 4

Question:

Express the following as a single fraction:

$$\frac{2}{(x-1)}+\frac{3}{(x+2)}$$

Solution:

$$\frac{2}{(x-1)}+\frac{3}{(x+2)}$$

$$=\frac{2(x+2)}{(x-1)(x+2)}+\frac{3(x-1)}{(x-1)(x+2)}$$

$$=\frac{2(x+2)+3(x-1)}{(x-1)(x+2)}$$

$$=\frac{2x+4+3x-3}{(x-1)(x+2)}$$

$$=\frac{5x+1}{(x-1)(x+2)}$$

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Partial fractions

Exercise A, Question 5

Question:

Express the following as a single fraction:

$$\frac{4}{(2x+1)}+\frac{2}{(x-1)}$$

Solution:

$$\frac{4}{(2x+1)}+\frac{2}{(x-1)}$$

$$=\frac{4(x-1)}{(2x+1)(x-1)}+\frac{2(2x+1)}{(2x+1)(x-1)}$$

$$=\frac{4(x-1)+2(2x+1)}{(2x+1)(x-1)}$$

$$=\frac{4x-4+4x+2}{(2x+1)(x-1)}$$

$$=\frac{8x-2}{(2x+1)(x-1)}$$

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Partial fractions

Exercise A, Question 6

Question:

Express the following as a single fraction:

$$\frac{7}{(x-3)} - \frac{2}{(x+4)}$$

Solution:

$$\frac{7}{(x-3)} - \frac{2}{(x+4)}$$

$$= \frac{7(x+4)}{(x-3)(x+4)} - \frac{2(x-3)}{(x-3)(x+4)}$$

$$= \frac{7(x+4) - 2(x-3)}{(x-3)(x+4)}$$

$$= \frac{7x + 28 - 2x + 6}{(x-3)(x+4)}$$

$$= \frac{5x + 34}{(x-3)(x+4)}$$

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Partial fractions

Exercise A, Question 7

Question:

Express the following as a single fraction:

$$\frac{3}{2x} - \frac{6}{(x-1)}$$

Solution:

$$\frac{3}{2x} - \frac{6}{(x-1)}$$

$$= \frac{3(x-1)}{2x(x-1)} - \frac{6 \times 2x}{2x(x-1)}$$

$$= \frac{3(x-1) - 12x}{2x(x-1)}$$

$$= \frac{3x - 3 - 12x}{2x(x-1)}$$

$$= \frac{-9x - 3}{2x(x-1)}$$

or $-\dfrac{9x+3}{2x(x-1)}$

or $-\dfrac{3(3x+1)}{2x(x-1)}$

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Partial fractions
Exercise A, Question 8

Question:

Express the following as a single fraction:

$$\frac{3}{x} + \frac{2}{(x+1)} + \frac{1}{(x+2)}$$

Solution:

$$\frac{3}{x} + \frac{2}{(x+1)} + \frac{1}{(x+2)}$$

$$= \frac{3(x+1)(x+2)}{x(x+1)(x+2)} + \frac{2x(x+2)}{x(x+1)(x+2)} + \frac{1x(x+1)}{x(x+1)(x+2)}$$

$$= \frac{3(x+1)(x+2) + 2x(x+2) + 1x(x+1)}{x(x+1)(x+2)}$$ Add numerators

$$= \frac{3(x^2+3x+2) + 2x^2+4x+x^2+x}{x(x+1)(x+2)}$$ Expand brackets

$$= \frac{3x^2+9x+6+2x^2+4x+x^2+x}{x(x+1)(x+2)}$$ Simplify terms

$$= \frac{6x^2+14x+6}{x(x+1)(x+2)}$$ Add like terms

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Partial fractions
Exercise A, Question 9

Question:

Express the following as a single fraction:

$$\frac{4}{3x} - \frac{2}{(x-2)} + \frac{1}{(2x+1)}$$

Solution:

$$\frac{4}{3x} - \frac{2}{(x-2)} + \frac{1}{(2x+1)}$$

$$= \frac{4(x-2)(2x+1)}{3x(x-2)(2x+1)} - \frac{2 \times 3x(2x+1)}{3x(x-2)(2x+1)} + \frac{3x(x-2)}{3x(x-2)(2x+1)}$$

$$= \frac{4(x-2)(2x+1) - 2 \times 3x(2x+1) + 3x(x-2)}{3x(x-2)(2x+1)}$$ Add numerators

$$= \frac{4(2x^2 - 3x - 2) - 6x(2x+1) + 3x^2 - 6x}{3x(x-2)(2x+1)}$$ Expand brackets

$$= \frac{8x^2 - 12x - 8 - 12x^2 - 6x + 3x^2 - 6x}{3x(x-2)(2x+1)}$$ Simplify terms

$$= \frac{-1x^2 - 24x - 8}{3x(x-2)(2x+1)}$$ Add like terms

or $$-\frac{x^2 + 24x + 8}{3x(x-2)(2x+1)}$$

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Edexcel AS and A Level Modular Mathematics

Partial fractions

Exercise A, Question 10

Question:

Express the following as a single fraction:

$$\frac{3}{(x-1)}+\frac{2}{(x+1)}+\frac{4}{(x-3)}$$

Solution:

$$\frac{3}{(x-1)}+\frac{2}{(x+1)}+\frac{4}{(x-3)}$$

$$=\frac{3(x+1)(x-3)}{(x-1)(x+1)(x-3)}+\frac{2(x-1)(x-3)}{(x-1)(x+1)(x-3)}+\frac{4(x-1)(x+1)}{(x-1)(x+1)(x-3)}$$

$$=\frac{3(x+1)(x-3)+2(x-1)(x-3)+4(x-1)(x+1)}{(x-1)(x+1)(x-3)}$$ Add numerators

$$=\frac{3(x^2-2x-3)+2(x^2-4x+3)+4(x^2-1)}{(x-1)(x+1)(x-3)}$$ Expand brackets

$$=\frac{3x^2-6x-9+2x^2-8x+6+4x^2-4}{(x-1)(x+1)(x-3)}$$ Simplify terms

$$=\frac{9x^2-14x-7}{(x-1)(x+1)(x-3)}$$ Add like terms

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Partial fractions

Exercise B, Question 1

Question:

Express the following as partial fractions:

(a) $\frac{6x-2}{(x-2)(x+3)}$

(b) $\frac{2x+11}{(x+1)(x+4)}$

(c) $\frac{-7x-12}{2x(x-4)}$

(d) $\frac{2x-13}{(2x+1)(x-3)}$

(e) $\frac{6x+6}{x^2-9}$

(f) $\frac{7-3x}{x^2-3x-4}$

(g) $\frac{8-x}{x^2+4x}$

(h) $\frac{2x-14}{x^2+2x-15}$

Solution:

(a) Let $\frac{6x-2}{(x-2)(x+3)} \equiv \frac{A}{(x-2)} + \frac{B}{(x+3)}$ Add the fractions

$\Rightarrow \quad \frac{6x-2}{(x-2)(x+3)} \equiv \frac{A(x+3)+B(x-2)}{(x-2)(x+3)}$

So $6x-2 \equiv A(x+3)+B(x-2)$ Set numerators equal

Substitute $x=2 \quad \Rightarrow \quad 6 \times 2 - 2 = A(2+3) + B(2-2)$

$\Rightarrow \quad 10 = 5A$

$\Rightarrow \quad A = 2$

Substitute $x = -3 \quad \Rightarrow \quad 6 \times (-3) - 2 = A(-3 + 3) + B$
$(-3 - 2)$

$\Rightarrow \quad -20 = B \times -5$

$\Rightarrow \quad B = 4$

Hence $\frac{6x - 2}{(x - 2)(x + 3)} \equiv \frac{2}{(x - 2)} + \frac{4}{(x + 3)}$

(b) Let $\frac{2x + 11}{(x + 1)(x + 4)} \equiv \frac{A}{(x + 1)} + \frac{B}{(x + 4)}$ Add the fractions

$\Rightarrow \quad \frac{2x + 11}{(x + 1)(x + 4)} \equiv \frac{A(x + 4) + B(x + 1)}{(x + 1)(x + 4)}$

So $2x + 11 \equiv A(x + 4) + B(x + 1)$ Set numerators equal

Substitute $x = -4 \quad \Rightarrow \quad 2 \times (-4) + 11 = A(-4 + 4) + B$
$(-4 + 1)$

$\Rightarrow \quad 3 = -3B$

$\Rightarrow \quad B = -1$

Substitute $x = -1 \quad \Rightarrow \quad 2 \times -1 + 11 = A(-1 + 4) + B(-1 + 1)$

$\Rightarrow \quad 9 = 3A$

$\Rightarrow \quad A = 3$

Hence $\frac{2x + 11}{(x + 1)(x + 4)} \equiv \frac{3}{(x + 1)} + \frac{(-1)}{(x + 4)} \equiv \frac{3}{(x + 1)} - \frac{1}{(x + 4)}$

(c) Let $\frac{-7x - 12}{2x(x - 4)} \equiv \frac{A}{2x} + \frac{B}{(x - 4)}$ Add the fractions

$\Rightarrow \quad \frac{-7x - 12}{2x(x - 4)} \equiv \frac{A(x - 4) + B \times 2x}{2x(x - 4)}$

So $-7x - 12 \equiv A(x - 4) + 2Bx$ Set numerators equal

Substitute $x = 4 \quad \Rightarrow \quad -7 \times 4 - 12 = A(4 - 4) + 2B \times 4$

$\Rightarrow \quad -40 = 8B$

$\Rightarrow \quad B = -5$

Substitute $x = 0 \quad \Rightarrow \quad -7 \times 0 - 12 = A(0 - 4) + 2B \times 0$

$\Rightarrow \quad -12 = -4A$

$\Rightarrow \quad A = 3$

Hence $\frac{-7x - 12}{2x(x - 4)} \equiv \frac{3}{2x} + \frac{-5}{(x - 4)} \equiv \frac{3}{2x} - \frac{5}{(x - 4)}$

(d) Let $\frac{2x-13}{(2x+1)(x-3)} \equiv \frac{A}{(2x+1)} + \frac{B}{(x-3)}$ Add the fractions

$\Rightarrow \frac{2x-13}{(2x+1)(x-3)} \equiv \frac{A(x-3)+B(2x+1)}{(2x+1)(x-3)}$

So $2x-13 \equiv A(x-3)+B(2x+1)$ Set numerators equal

Substitute $x=3 \Rightarrow 2\times 3-13 = A(3-3)+B(2\times 3+1)$

$\Rightarrow -7 = B\times 7$

$\Rightarrow B = -1$

Substitute $x = -\frac{1}{2} \Rightarrow 2\times\left(-\frac{1}{2}\right) - 13 = A\left(-\frac{1}{2}-3\right) + B$ $\left(2\times\left(-\frac{1}{2}\right)+1\right)$

$\Rightarrow -14 = A\times -3\frac{1}{2}$

$\Rightarrow A = 4$

Hence $\frac{2x-13}{(2x+1)(x-3)} \equiv \frac{4}{(2x+1)} + \frac{-1}{(x-3)} \equiv \frac{4}{(2x+1)} - \frac{1}{(x-3)}$

(e) $\frac{6x+6}{x^2+9} \equiv \frac{6x+6}{(x+3)(x-3)}$ Factorise denominator

Let $\frac{6x+6}{(x+3)(x-3)} \equiv \frac{A}{(x+3)} + \frac{B}{(x-3)}$ Add fractions

$\Rightarrow \frac{6x+6}{(x+3)(x-3)} \equiv \frac{A(x-3)+B(x+3)}{(x+3)(x-3)}$

So $6x+6 \equiv A(x-3)+B(x+3)$ Set numerators equal

Substitute $x=3 \Rightarrow 6\times 3+6 = A(3-3)+B(3+3)$

$\Rightarrow 24 = B\times 6$

$\Rightarrow B = 4$

Substitute $x = -3 \Rightarrow 6\times(-3)+6 = A(-3-3)+B$ $(-3+3)$

$\Rightarrow -12 = A\times -6$

$\Rightarrow A = 2$

Hence $\frac{6x+6}{x^2-9} \equiv \frac{2}{(x+3)} + \frac{4}{(x-3)}$

(f) $\frac{7-3x}{x^2-3x-4} \equiv \frac{7-3x}{(x-4)(x+1)}$ Factorise denominator

Let $\frac{7-3x}{(x-4)(x+1)} \equiv \frac{A}{(x-4)} + \frac{B}{(x+1)}$ Add fractions

$\Rightarrow \quad \frac{7-3x}{(x-4)(x+1)} \equiv \frac{A(x+1)+B(x-4)}{(x-4)(x+1)}$

So $7-3x \equiv A(x+1)+B(x-4)$ Set numerators equal

Substitute $x=-1 \Rightarrow 7-3\times(-1)=A(-1+1)+B(-1-4)$

$\Rightarrow \quad 10=B\times -5$

$\Rightarrow \quad B=-2$

Substitute $x=4 \Rightarrow 7-3\times4=A(4+1)+B(4-4)$

$\Rightarrow \quad -5=A\times5$

$\Rightarrow \quad A=-1$

Hence $\frac{7-3x}{x^2-3x-4} \equiv \frac{-1}{(x-4)} + \frac{-2}{(x+1)} \equiv -\frac{1}{(x-4)} - \frac{2}{(x+1)}$

(g) $\frac{8-x}{x^2+4x} \equiv \frac{8-x}{x(x+4)}$ Factorise denominator

Let $\frac{8-x}{x(x+4)} \equiv \frac{A}{x} + \frac{B}{(x+4)}$ Add fractions

$\Rightarrow \quad \frac{8-x}{x(x+4)} \equiv \frac{A(x+4)+Bx}{x(x+4)}$

So $8-x \equiv A(x+4)+Bx$ Set numerators equal

Substitute $x=0 \Rightarrow 8-0=A(0+4)+B\times0$

$\Rightarrow \quad 8=4A$

$\Rightarrow \quad A=2$ Substitute $x=-4 \Rightarrow 8-\left(-4\right)=A\left(-4+4\right)+B\times\left(-4\right)$

$\Rightarrow \quad 12=-4B$

$\Rightarrow \quad B=-3$

Hence $\frac{8-x}{x^2+4x} \equiv \frac{2}{x} + \frac{-3}{(x+4)} \equiv \frac{2}{x} - \frac{3}{(x+4)}$

(h) $\frac{2x-14}{x^2+2x-15} \equiv \frac{2x-14}{(x+5)(x-3)}$ Factorise denominator

Let $\frac{2x-14}{(x+5)(x-3)} \equiv \frac{A}{(x+5)} + \frac{B}{(x-3)}$ Add fractions

$\Rightarrow \quad \frac{2x-14}{(x+5)(x-3)} \equiv \frac{A(x-3)+B(x+5)}{(x+5)(x-3)}$

So $2x - 14 \equiv A(x-3) + B(x+5)$ Set numerators equal

Substitute $x = 3 \quad \Rightarrow \quad 2 \times 3 - 14 = A(3-3) + B(3+5)$

$\Rightarrow \quad -8 = B \times 8$

$\Rightarrow \quad B = -1$

Substitute $x = -5 \quad \Rightarrow \quad 2 \times (-5) - 14 = A(-5-3) + B$
$(-5+5)$

$\Rightarrow \quad -24 = A \times (-8)$

$\Rightarrow \quad A = 3$

Hence $\frac{2x-14}{x^2+2x-15} \equiv \frac{3}{(x+5)} + \frac{-1}{(x-3)} \equiv \frac{3}{(x+5)} - \frac{1}{(x-3)}$

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Edexcel AS and A Level Modular Mathematics

Partial fractions

Exercise B, Question 2

Question:

Show that $\frac{-2x-5}{(4+x)(2-x)}$ can be written in the form $\frac{A}{(4+x)}+\frac{B}{(2-x)}$ where A and B are constants to be found.

Solution:

Let $\frac{-2x-5}{(4+x)(2-x)} \equiv \frac{A}{(4+x)}+\frac{B}{(2-x)} \equiv \frac{A(2-x)+B(4+x)}{(4+x)(2-x)}$

So $-2x-5 \equiv A(2-x)+B(4+x)$

Substitute $x=2 \quad \Rightarrow \quad -2\times 2-5=A(2-2)+B(4+2)$

$\Rightarrow \quad -9=B\times 6$

$\Rightarrow \quad B=\frac{-3}{2}$

Substitute $x=-4 \quad \Rightarrow \quad -2\times\left(-4\right)-5=A\left(2-\left(-4\right)\right)+B\left(4+\left(-4\right)\right)$

$\Rightarrow \quad 3=A\times 6$

$\Rightarrow \quad \frac{1}{2}=A$

Hence $\frac{-2x-5}{(4+x)(2-x)} \equiv \frac{A}{(4+x)}+\frac{B}{(2-x)}$ when $A=\frac{1}{2}$ and $B=\frac{-3}{2}$.

or $\frac{-2x-5}{(4+x)(2-x)}=\frac{1}{2(4+x)}-\frac{3}{2(2-x)}$

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Edexcel AS and A Level Modular Mathematics

Partial fractions

Exercise C, Question 1

Question:

Express the following as partial fractions:

(a) $\frac{2x^2 - 12x - 26}{(x+1)(x-2)(x+5)}$

(b) $\frac{-10x^2 - 8x + 2}{x(2x+1)(3x-2)}$

(c) $\frac{-5x^2 - 19x - 32}{(x+1)(x+2)(x-5)}$

Solution:

(a) Let $\frac{2x^2 - 12x - 26}{(x+1)(x-2)(x+5)} \equiv \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x+5)}$ Add fractions

$$\Rightarrow \quad \frac{2x^2 - 12x - 26}{(x+1)(x-2)(x+5)} \equiv$$

$$\frac{A(x-2)(x+5) + B(x+1)(x+5) + C(x+1)(x-2)}{(x+1)(x-2)(x+5)}$$

So $2x^2 - 12x - 26 \equiv A(x-2)(x+5) + B(x+1)(x+5) + C(x+1)(x-2)$

Substitute $x = 2 \quad \Rightarrow \quad 8 - 24 - 26 = A \times 0 + B \times 3 \times 7 + C \times 0$

$\Rightarrow \quad -42 = 21B$

$\Rightarrow \quad B = -2$

Substitute $x = -1 \quad \Rightarrow \quad 2 + 12 - 26 = A \times (-3) \times 4 + B \times 0 + C \times 0$

$\Rightarrow \quad -12 = -12A$

$\Rightarrow \quad A = 1$

Substitute $x = -5 \quad \Rightarrow \quad 50 + 60 - 26 = A \times 0 + B \times 0 + C \times 28$

$\Rightarrow \quad 84 = 28C$

$\Rightarrow \quad C = 3$

Hence $\frac{2x^2 - 12x - 26}{(x + 1)(x - 2)(x + 5)} \equiv \frac{1}{(x + 1)} - \frac{2}{(x - 2)} + \frac{3}{(x + 5)}$

(b) Let $\frac{-10x^2 - 8x + 2}{x(2x + 1)(3x - 2)} \equiv \frac{A}{x} + \frac{B}{(2x + 1)} + \frac{C}{(3x - 2)}$ Add fractions

$\Rightarrow \quad \frac{-10x^2 - 8x + 2}{x(2x + 1)(3x - 2)} \equiv \frac{A(2x + 1)(3x - 2) + Bx(3x - 2) + Cx(2x + 1)}{x(2x + 1)(3x - 2)}$

So $-10x^2 - 8x + 2 \equiv A(2x + 1)(3x - 2) + Bx(3x - 2) + Cx$
$(2x + 1)$

Substitute $x = 0 \quad \Rightarrow \quad -0 - 0 + 2 = A \times 1 \times (-2) + B \times 0 + C \times 0$

$\Rightarrow \quad 2 = -2A$

$\Rightarrow \quad A = -1$

Substitute $x = -\frac{1}{2} \quad \Rightarrow \quad \frac{-10}{4} + 4 + 2 = A \times 0 + B \times \frac{-1}{2} \times$
$\frac{-7}{2} + C \times 0$

$\Rightarrow \quad \frac{7}{2} = B \times \frac{7}{4}$

$\Rightarrow \quad B = 2$

Equate coefficients in x^2: $\quad -10 = 6A + 3B + 2C$

$\Rightarrow \quad -10 = -6 + 6 + 2C$

$\Rightarrow \quad -10 = 2C$

$\Rightarrow \quad -5 = C$

Hence $\frac{-10x^2 - 8x + 2}{x(2x + 1)(3x - 2)} \equiv \frac{-1}{x} + \frac{2}{(2x + 1)} - \frac{5}{(3x - 2)}$

(c) Let $\frac{-5x^2 - 19x - 32}{(x + 1)(x + 2)(x - 5)} \equiv \frac{A}{(x + 1)} + \frac{B}{(x + 2)} + \frac{C}{(x - 5)}$

$\Rightarrow \quad \frac{-5x^2 - 19x - 32}{(x + 1)(x + 2)(x - 5)} \equiv$
$\frac{A(x + 2)(x - 5) + B(x + 1)(x - 5) + C(x + 1)(x + 2)}{(x + 1)(x + 2)(x - 5)}$

So $-5x^2 - 19x - 32 \equiv A(x + 2)(x - 5) + B(x + 1)(x - 5) + C$
$(x + 1)(x + 2)$

Substitute $x = -1 \quad \Rightarrow \quad -5 + 19 - 32 = A \times 1 \times (-6)$
$+ B \times 0 + C \times 0$

$\Rightarrow \quad -18 = -6A$

$\Rightarrow \quad A = 3$

Substitute $x = 5 \quad \Rightarrow \quad -125 - 95 - 32 = A \times 0 + B \times 0 + C \times 6 \times 7$

$\Rightarrow \quad -252 = 42C$

$\Rightarrow \quad C = -6$

Substitute $x = -2 \quad \Rightarrow \quad -20 + 38 - 32 = A \times 0 + B \times \left(-1 \right) \times \left(-7 \right) + C \times 0$

$\Rightarrow \quad -14 = 7B$

$\Rightarrow \quad B = -2$

Hence $\frac{-5x^2 - 19x - 32}{(x+1)(x+2)(x-5)} \equiv \frac{3}{(x+1)} - \frac{2}{(x+2)} - \frac{6}{(x-5)}$

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Edexcel AS and A Level Modular Mathematics

Partial fractions
Exercise C, Question 2

Question:

By firstly factorising the denominator, express the following as partial fractions:

(a) $\frac{6x^2+7x-3}{x^3-x}$

(b) $\frac{5x^2+15x+8}{x^3+3x^2+2x}$

(c) $\frac{5x^2-15x-8}{x^3-4x^2+x+6}$

Solution:

(a) $x^3-x \equiv x(x^2-1) \equiv x(x+1)(x-1)$

So $\frac{6x^2+7x-3}{x^3-x} \equiv \frac{6x^2+7x-3}{x(x+1)(x-1)} \equiv \frac{A}{x}+\frac{B}{(x+1)}+\frac{C}{(x-1)}$

$\equiv \frac{A(x+1)(x-1)+Bx(x-1)+Cx(x+1)}{x(x+1)(x-1)}$

Setting numerators equal gives $6x^2+7x-3 \equiv A\left(x+1\right)\left(x-1\right)+Bx\left(x-1\right)+Cx\left(x+1\right)$

Substitute $x=0$ $\Rightarrow$ $0+0-3=A\times 1\times(-1)+B\times 0+C\times 0$

$\Rightarrow$ $-3=-1A$

$\Rightarrow$ $A=3$

Substitute $x=1$ $\Rightarrow$ $6+7-3=A\times 0+B\times 0+C\times 1\times 2$

$\Rightarrow$ $10=2C$

$\Rightarrow$ $C=5$

Substitute $x=-1$ $\Rightarrow$ $6-7-3=A\times 0+B\times(-1)\times(-2)+C\times 0$

$\Rightarrow$ $-4=2B$

$\Rightarrow$ $B=-2$

Hence $\frac{6x^2+7x-3}{x^3-x} \equiv \frac{3}{x}-\frac{2}{(x+1)}+\frac{5}{(x-1)}$

(b) $x^3+3x^2+2x \equiv x(x^2+3x+2) \equiv x(x+1)(x+2)$

So $\frac{5x^2+15x+8}{x^3+3x^2+2x} \equiv \frac{5x^2+15x+8}{x(x+1)(x+2)} \equiv \frac{A}{x}+\frac{B}{(x+1)}+\frac{C}{(x+2)}$

$$\equiv \frac{A(x+1)(x+2)+Bx(x+2)+Cx(x+1)}{x(x+1)(x+2)}$$

Setting numerators equal gives $5x^2+15x+8 \equiv A\left(x+1\right)\left(x+2\right)+Bx\left(x+2\right)+Cx\left(x+1\right)$

Substitute $x=0 \quad \Rightarrow \quad 0+0+8=A\times 1\times 2+B\times 0+C\times 0$

$\Rightarrow \quad 8=2A$

$\Rightarrow \quad A=4$

Substitute $x=-1 \quad \Rightarrow \quad 5-15+8=A\times 0+B\times(-1)\times 1+C\times 0$

$\Rightarrow \quad -2=-1B$

$\Rightarrow \quad B=2$

Substitute $x=-2 \quad \Rightarrow \quad 20-30+8=A\times 0+B\times 0+C\times\left(-2\right)\times\left(-1\right)$

$\Rightarrow \quad -2=2C$

$\Rightarrow \quad C=-1$

Hence $\frac{5x^2+15x+8}{x^3+3x^2+2x} \equiv \frac{4}{x}+\frac{2}{(x+1)}-\frac{1}{(x+2)}$

(c) Consider $f(x)=x^3-4x^2+x+6$

$f(-1)=-1-4-1+6=0$

Hence $(x+1)$ is a factor

By inspection

$f(x)=x^3-4x^2+x+6=\left(x+1\right)\left(x^2-5x+6\right)=\left(x+1\right)\left(x-2\right)\left(x-3\right)$

Note. This last part could have been found by division.

$$\begin{array}{r} x^2-5x+6 \\ x+1\overline{)x^3-4x^2+x+6} \\ \underline{x^3+x^2} \\ -5x^2+x \\ \underline{-5x^2-5x} \\ 6x+6 \\ \underline{6x+6} \\ \underline{0} \end{array}$$

Hence $\frac{5x^2-15x-8}{x^3-4x^2+x+6} \equiv \frac{5x^2-15x-8}{(x+1)(x-2)(x-3)} \equiv \frac{A}{(x+1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$$\equiv \frac{A(x-2)(x-3)+B(x+1)(x-3)+C(x+1)(x-2)}{(x+1)(x-2)(x-3)}$$

Setting numerators equal gives

$$5x^2 - 15x - 8 \equiv A(x-2)(x-3) + B(x+1)(x-3) + C(x+1)(x-2)$$

Substitute $x = 2 \Rightarrow 20 - 30 - 8 = A \times 0 + B \times 3 \times (-1) + C \times 0$

$\Rightarrow -18 = -3B$

$\Rightarrow B = 6$

Substitute $x = -1 \Rightarrow 5 + 15 - 8 = A \times (-3) \times (-4) + B \times 0 + C \times 0$

$\Rightarrow 12 = 12A$

$\Rightarrow A = 1$

Substitute $x = 3 \Rightarrow 45 - 45 - 8 = A \times 0 + B \times 0 + C \times 4 \times 1$

$\Rightarrow -8 = 4C$

$\Rightarrow C = -2$

Hence $\dfrac{5x^2 - 15x - 8}{x^3 - 4x^2 + x + 6} \equiv \dfrac{1}{(x+1)} + \dfrac{6}{(x-2)} - \dfrac{2}{(x-3)}$

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Edexcel AS and A Level Modular Mathematics

Partial fractions
Exercise D, Question 1

Question:

Put the following into partial fraction form:

$$\frac{3x^2 + x + 2}{x^2(x+1)}$$

Solution:

Let $\frac{3x^2 + x + 2}{x^2(x+1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)}$

$$\equiv \frac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)}$$

Set the numerators equal:

$3x^2 + x + 2 \equiv Ax(x+1) + B(x+1) + Cx^2$

Substitute $x = 0 \quad \Rightarrow \quad 0 + 0 + 2 = A \times 0 + B \times 1 + C \times 0$

$\Rightarrow \quad 2 = 1B$

$\Rightarrow \quad B = 2$

Substitute $x = -1 \quad \Rightarrow \quad 3 - 1 + 2 = A \times 0 + B \times 0 + C \times 1$

$\Rightarrow \quad 4 = 1C$

$\Rightarrow \quad C = 4$

Equate coefficients in x^2: $\quad 3 = A + C \quad$ Substitute $C = 4$

$\Rightarrow \quad 3 = A + 4$

$\Rightarrow \quad A = -1$

Hence $\frac{3x^2 + x + 2}{x^2(x+1)} \equiv \frac{-1}{x} + \frac{2}{x^2} + \frac{4}{(x+1)}$

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Edexcel AS and A Level Modular Mathematics

Partial fractions
Exercise D, Question 2

Question:

Put the following into partial fraction form:

$$\frac{-x^2-10x-5}{(x+1)^2(x-1)}$$

Solution:

Let $\frac{-x^2-10x-5}{(x+1)^2(x-1)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)}$

$$\equiv \frac{A(x+1)(x-1)+B(x-1)+C(x+1)^2}{(x+1)^2(x-1)}$$

Set the numerators equal:

$-x^2-10x-5 \equiv A(x+1)(x-1)+B(x-1)+C(x+1)^2$

Substitute $x=1 \Rightarrow -1-10-5=A\times 0+B\times 0+C\times 4$

$\Rightarrow -16=4C$

$\Rightarrow C=-4$

Substitute $x=-1 \Rightarrow -1+10-5=A\times 0+B\times(-2)+C\times 0$

$\Rightarrow 4=-2B$

$\Rightarrow B=-2$

Equate coefficients in x^2: $-1=A+C$ Substitute $C=-4$

$\Rightarrow -1=A-4$

$\Rightarrow A=3$

Hence $\frac{-x^2-10x-5}{(x+1)^2(x-1)} \equiv \frac{3}{(x+1)} - \frac{2}{(x+1)^2} - \frac{4}{(x-1)}$

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Edexcel AS and A Level Modular Mathematics

Partial fractions

Exercise D, Question 3

Question:

Put the following into partial fraction form:

$$\frac{2x^2+2x-18}{x(x-3)^2}$$

Solution:

$$\text{Let } \frac{2x^2+2x-18}{x(x-3)^2} \equiv \frac{A}{x}+\frac{B}{(x-3)}+\frac{C}{(x-3)^2}$$

$$\equiv \frac{A(x-3)^2+Bx(x-3)+Cx}{x(x-3)^2}$$

Set the numerators equal:

$2x^2+2x-18 \equiv A(x-3)^2+Bx(x-3)+Cx$

Substitute $x=0 \quad \Rightarrow \quad 0+0-18=A\times 9+B\times 0+C\times 0$

$\Rightarrow \quad -18=9A$

$\Rightarrow \quad A=-2$

Substitute $x=3 \quad \Rightarrow \quad 18+6-18=A\times 0+B\times 0+C\times 3$

$\Rightarrow \quad 6=3C$

$\Rightarrow \quad C=2$

Equate coefficients in x^2: $\quad 2=A+B \quad$ Substitute $A=-2$

$\Rightarrow \quad 2=-2+B$

$\Rightarrow \quad B=4$

$$\text{Hence } \frac{2x^2+2x-18}{x(x-3)^2} \equiv -\frac{2}{x}+\frac{4}{(x-3)}+\frac{2}{(x-3)^2}$$

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Partial fractions

Exercise D, Question 4

Question:

Put the following into partial fraction form:

$$\frac{7x^2-42x+64}{x(x-4)^2}$$

Solution:

$$\text{Let } \frac{7x^2-42x+64}{x(x-4)^2} \equiv \frac{A}{x}+\frac{B}{(x-4)}+\frac{C}{(x-4)^2}$$

$$\equiv \frac{A(x-4)^2+Bx(x-4)+Cx}{x(x-4)^2}$$

Set the numerators equal:

$7x^2-42x+64 \equiv A(x-4)^2+Bx(x-4)+Cx$

Substitute $x=0$ $\Rightarrow$ $0-0+64=A\times 16+B\times 0+C\times 0$

$\Rightarrow$ $64=16A$

$\Rightarrow$ $A=4$

Substitute $x=4$ $\Rightarrow$ $112-168+64=A\times 0+B\times 0+C\times 4$

$\Rightarrow$ $8=4C$

$\Rightarrow$ $C=2$

Equate coefficients in x^2: $7=A+B$ Substitute $A=4$

$\Rightarrow$ $7=4+B$

$\Rightarrow$ $B=3$

$$\text{Hence } \frac{7x^2-42x+64}{x(x-4)^2} \equiv \frac{4}{x}+\frac{3}{(x-4)}+\frac{2}{(x-4)^2}$$

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Partial fractions

Exercise D, Question 5

Question:

Put the following into partial fraction form:

$$\frac{5x^2 - 2x - 1}{x^3 - x^2}$$

Solution:

$x^3 - x^2 \equiv x^2 (x - 1)$

So $\frac{5x^2 - 2x - 1}{x^3 - x^2} \equiv \frac{5x^2 - 2x - 1}{x^2 (x - 1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x - 1)}$

$\equiv \frac{Ax (x - 1) + B (x - 1) + Cx^2}{x^2 (x - 1)}$

Set the numerators equal:

$5x^2 - 2x - 1 \equiv Ax (x - 1) + B (x - 1) + Cx^2$

Substitute $x = 1$ $\Rightarrow$ $5 - 2 - 1 = A \times 0 + B \times 0 + C \times 1$

$\Rightarrow$ $2 = 1C$

$\Rightarrow$ $C = 2$

Substitute $x = 0$ $\Rightarrow$ $0 - 0 - 1 = A \times 0 + B \times (- 1) + C \times 0$

$\Rightarrow$ $- 1 = - 1B$

$\Rightarrow$ $B = 1$

Equate coefficients in x^2: $5 = A + C$ Substitute $C = 2$

$\Rightarrow$ $5 = A + 2$

$\Rightarrow$ $A = 3$

Hence $\frac{5x^2 - 2x - 1}{x^3 - x^2} \equiv \frac{3}{x} + \frac{1}{x^2} + \frac{2}{(x - 1)}$

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Partial fractions
Exercise D, Question 6

Question:

Put the following into partial fraction form:

$$\frac{2x^2 + 2x - 18}{x^3 - 6x^2 + 9x}$$

Solution:

$x^3 - 6x^2 + 9x \equiv x(x^2 - 6x + 9) \equiv x(x - 3)^2$

$$\text{So } \frac{2x^2 + 2x - 18}{x^3 - 6x^2 + 9x} \equiv \frac{2x^2 + 2x - 18}{x(x - 3)^2} \equiv \frac{A}{x} + \frac{B}{(x - 3)} + \frac{C}{(x - 3)^2}$$

$$\equiv \frac{A(x - 3)^2 + Bx(x - 3) + Cx}{x(x - 3)^2}$$

Set the numerators equal:

$2x^2 + 2x - 18 \equiv A(x - 3)^2 + Bx(x - 3) + Cx$

Substitute $x = 0 \Rightarrow 0 + 0 - 18 = A \times 9 + B \times 0 + C \times 0$

$\Rightarrow -18 = 9A$

$\Rightarrow A = -2$

Substitute $x = 3 \Rightarrow 18 + 6 - 18 = A \times 0 + B \times 0 + C \times 3$

$\Rightarrow 6 = 3C$

$\Rightarrow C = 2$

Equate coefficients in x^2: $2 = A + B$ Substitute $A = -2$

$\Rightarrow 2 = -2 + B$

$\Rightarrow B = 4$

$$\text{Hence } \frac{2x^2 + 2x - 18}{x^3 - 6x^2 + 9x} \equiv -\frac{2}{x} + \frac{4}{(x - 3)} + \frac{2}{(x - 3)^2}$$

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Partial fractions

Exercise D, Question 7

Question:

Put the following into partial fraction form:

$$\frac{2x}{(x+2)^2}$$

Solution:

Let $\frac{2x}{(x+2)^2} \equiv \frac{A}{(x+2)} + \frac{B}{(x+2)^2} \equiv \frac{A(x+2)+B}{(x+2)^2}$

Set the numerators equal: $2x = A(x+2)+B$

Substitute $x = -2 \Rightarrow -4 = A \times 0 + B \Rightarrow B = -4$

Equate coefficients in x: $2 = A$

Hence $\frac{2x}{(x+2)^2} \equiv \frac{2}{(x+2)} - \frac{4}{(x+2)^2}$

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Partial fractions

Exercise D, Question 8

Question:

Put the following into partial fraction form:

$$\frac{x^2 + 5x + 7}{(x + 2)^3}$$

Solution:

Let $\frac{x^2 + 5x + 7}{(x + 2)^3} \equiv \frac{A}{(x + 2)} + \frac{B}{(x + 2)^2} + \frac{C}{(x + 2)^3}$

$\equiv \frac{A(x + 2)^2 + B(x + 2) + C}{(x + 2)^3}$

Set the numerators equal:

$x^2 + 5x + 7 \equiv A(x + 2)^2 + B(x + 2) + C$

Substitute $x = -2 \quad \Rightarrow \quad 4 - 10 + 7 = A \times 0 + B \times 0 + C \quad \Rightarrow \quad C = 1$

Equate coefficients in x^2: $\quad 1 = A$

$\Rightarrow \quad A = 1$

Equate coefficients in x: $\quad 5 = 4A + B \quad$ Substitute $A = 1$

$\Rightarrow \quad 5 = 4 + B$

$\Rightarrow \quad B = 1$

Hence $\frac{x^2 + 5x + 7}{(x + 2)^3} \equiv \frac{1}{(x + 2)} + \frac{1}{(x + 2)^2} + \frac{1}{(x + 2)^3}$

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Partial fractions
Exercise E, Question 1

Question:

Express the following improper fractions as a partial fraction:

(a) $\frac{x^2+3x-2}{(x+1)(x-3)}$

(b) $\frac{x^2-10}{(x-2)(x+1)}$

(c) $\frac{x^3-x^2-x-3}{x(x-1)}$

(d) $\frac{2x^2-1}{(x+1)^2}$

Solution:

(a) $\frac{x^2+3x-2}{(x+1)(x-3)} \equiv \frac{x^2+3x-2}{x^2-2x-3}$

Divide the numerator by the denominator:

$$\begin{array}{r} 1 \\ x^2-2x-3\overline{\smash{)}\,x^2+3x-2} \\ \underline{x^2-2x-3} \\ \underline{5x+1} \end{array} \leftarrow \text{Remainder}$$

Therefore $\frac{x^2+3x-2}{(x+1)(x-3)} \equiv 1+\frac{5x+1}{(x+1)(x-3)}$

Let $\frac{5x+1}{(x+1)(x-3)} \equiv \frac{A}{(x+1)}+\frac{B}{(x-3)}$ Add fractions

$$\equiv \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

Set the numerators equal: $5x+1 \equiv A(x-3)+B(x+1)$

Substitute $x=3$ $\Rightarrow$ $5\times3+1=A\times0+B\times4$

$\Rightarrow$ $16=4B$

$\Rightarrow$ $B=4$

Substitute $x=-1$ $\Rightarrow$ $5\times(-1)+1=A\times(-4)+B\times0$

$\Rightarrow$ $-4=-4A$

$\Rightarrow$ $A=1$

Hence

$$\frac{x^2+3x-2}{(x+1)(x-3)} \equiv 1+\frac{5x+1}{(x+1)(x-3)} \equiv 1+\frac{1}{(x+1)}+\frac{4}{(x-3)}$$

(b) $\dfrac{x^2-10}{(x-2)(x+1)} \equiv \dfrac{x^2-10}{x^2-x-2} \equiv \dfrac{x^2+0x-10}{x^2-x-2}$

Divide the numerator by the denominator:

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2-x-2 & x^2+0x-10 \\ & \underline{x^2-\ x-\ 2} \\ & \underline{\quad\ \ x-\ 8} \quad \leftarrow \text{Remainder} \end{array}$$

Therefore $\dfrac{x^2-10}{(x-2)(x+1)} \equiv 1+\dfrac{x-8}{(x-2)(x+1)}$

Let $\dfrac{x-8}{(x-2)(x+1)} \equiv \dfrac{A}{(x-2)}+\dfrac{B}{(x+1)}$ Add fractions

$$\equiv \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

Set the numerators equal: $x-8 \equiv A(x+1)+B(x-2)$

Substitute $x=2$ $\Rightarrow$ $2-8=A\times 3+B\times 0$

$\Rightarrow$ $-6=3A$

$\Rightarrow$ $A=-2$

Substitute $x=-1$ $\Rightarrow$ $-1-8=A\times 0+B\times(-3)$

$\Rightarrow$ $-9=-3B$

$\Rightarrow$ $B=3$

Hence

$$\frac{x^2-10}{(x-2)(x+1)} \equiv 1+\frac{x-8}{(x-2)(x+1)} \equiv 1+\frac{-2}{(x-2)}+\frac{3}{(x+1)}$$

$$\equiv 1-\frac{2}{(x-2)}+\frac{3}{(x+1)}$$

(c) $\dfrac{x^3-x^2-x-3}{x(x-1)} \equiv \dfrac{x^3-x^2-x-3}{x^2-x}$

Divide the numerator by the denominator:

$$\begin{array}{r|l} & -3x+2 \\ \hline x^2+2x-3 & -3x^3+4x^2-19x+8 \\ & \underline{-3x^3-6x^2-9x} \\ & \qquad\quad 2x^2+10x+8 \\ & \qquad\quad \underline{2x^2+4x-6} \\ & \qquad\qquad\quad \underline{6x+14} \quad \leftarrow \text{Remainder} \end{array}$$

Therefore $\dfrac{x^3-x^2-x-3}{x(x-1)} \equiv x+\dfrac{-x-3}{x(x-1)}$

Let $\frac{-x-3}{x(x-1)} \equiv \frac{A}{x} + \frac{B}{(x-1)}$ Add fractions

$\equiv \frac{A(x-1)+Bx}{x(x-1)}$

Set the numerators equal: $-x-3 \equiv A(x-1)+Bx$

Substitute $x=1$ ⇒ $-1-3=A\times 0+B\times 1$

⇒ $-4=1B$

⇒ $B=-4$

Substitute $x=0$ ⇒ $-3=A\times(-1)+B\times 0$

⇒ $A=3$

Hence $\frac{x^3-x^2-x-3}{x(x-1)} \equiv x+\frac{-x-3}{x(x-1)} \equiv x+\frac{3}{x}-\frac{4}{(x-1)}$

(d) $\frac{2x^2-1}{(x+1)^2} \equiv \frac{2x^2-1}{(x+1)(x+1)} \equiv \frac{2x^2-1}{x^2+2x+1} \equiv \frac{2x^2+0x-1}{x^2+2x+1}$

Divide the numerator by the denominator:

$$\begin{array}{r|l} & 2 \\ \hline x^2+2x+1 & 2x^2+0x-1 \\ & \underline{2x^2+4x+2} \\ & \quad\ \underline{-4x-3} \leftarrow \text{Remainder} \end{array}$$

Therefore $\frac{2x^2-1}{(x+1)^2} \equiv 2+\frac{-4x-3}{(x+1)^2}$

Let $\frac{-4x-3}{(x+1)^2} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2}$ Add fractions

$\equiv \frac{A(x+1)+B}{(x+1)^2}$

Set the numerators equal: $-4x-3 \equiv A(x+1)+B$

Substitute $x=-1$ ⇒ $-4\times(-1)-3=A\times 0+B$

⇒ $1=B$

⇒ $B=1$

Equate coefficients in x: $-4=A$

⇒ $A=-4$

Hence $\frac{2x^2-1}{(x+1)^2} \equiv 2+\frac{-4x-3}{(x+1)^2} \equiv 2-\frac{4}{(x+1)}+\frac{1}{(x+1)^2}$

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Edexcel AS and A Level Modular Mathematics

Partial fractions

Exercise E, Question 2

Question:

By factorising the denominator, express the following as partial fraction:

(a) $\frac{4x^2 + 17x - 11}{x^2 + 3x - 4}$

(b) $\frac{x^4 - 4x^3 + 9x^2 - 17x + 12}{x^3 - 4x^2 + 4x}$

Solution:

(a) Divide the numerator by the denominator:

$$\begin{array}{r l} & \quad 4 \\ x^2+3x-4 \big) & \overline{4x^2+17x-11} \\ & \underline{4x^2+12x-16} \\ & \qquad \underline{5x+\ 5} \quad \leftarrow \text{Remainder} \end{array}$$

Therefore $\frac{4x^2 + 17x - 11}{x^2 + 3x - 4} \equiv 4 + \frac{5x + 5}{x^2 + 3x - 4}$ Factorise denominator

$$\equiv 4 + \frac{5x + 5}{(x + 4) (x - 1)}$$

Let $\frac{5x + 5}{(x + 4) (x - 1)} \equiv \frac{A}{(x + 4)} + \frac{B}{(x - 1)}$ Add fractions

$$\equiv \frac{A (x - 1) + B (x + 4)}{(x + 4) (x - 1)}$$

Set the numerators equal: $5x + 5 \equiv A (x - 1) + B (x + 4)$

Substitute $x = 1$ $\Rightarrow$ $5 \times 1 + 5 = A \times 0 + B \times 5$

$\Rightarrow$ $10 = 5B$

$\Rightarrow$ $B = 2$

Substitute $x = - 4$ $\Rightarrow$ $5 \times (- 4) + 5 = A \times (- 5) + B \times 0$

$\Rightarrow$ $- 15 = - 5A$

$\Rightarrow$ $A = 3$

Hence

$$\frac{4x^2 + 17x - 11}{x^2 + 3x - 4} \equiv 4 + \frac{5x + 5}{(x + 4) (x - 1)} \equiv 4 + \frac{3}{(x + 4)} + \frac{2}{(x - 1)}$$

(b) Divide the numerator by the denominator:

$$\begin{array}{r} x \\ 4x^2+4x+0\overline{)x^4-4x^3+9x^2-17x+12} \\ \underline{x^4-4x^3+4x^2+0x} \\ \underline{5x^2-17x+12} \end{array} \leftarrow \text{Remainder}$$

Therefore $\dfrac{x^4-4x^3+9x^2-17x+12}{x^3-4x^2+4x}$

$\equiv x+\dfrac{5x^2-17x+12}{x^3-4x^2+4x}$ Take out a factor of x in the denominator

$\equiv x+\dfrac{5x^2-17x+12}{x(x^2-4x+4)}$ Factorise the denominator fully

$\equiv x+\dfrac{5x^2-17x+12}{x(x-2)^2}$

Let $\dfrac{5x^2-17x+12}{x(x-2)^2}\equiv\dfrac{A}{x}+\dfrac{B}{(x-2)}+\dfrac{C}{(x-2)^2}$ Add fractions

$\equiv\dfrac{A(x-2)^2+Bx(x-2)+Cx}{x(x-2)^2}$

Set the numerators equal: $5x^2-17x+12\equiv A(x-2)^2+Bx(x-2)+Cx$

Substitute $x=0$ $\Rightarrow$ $0-0+12=A\times 4+B\times 0+C\times 0$

$\Rightarrow$ $12=4A$

$\Rightarrow$ $A=3$

Substitute $x=2$ $\Rightarrow$ $20-34+12=A\times 0+B\times 0+C\times 2$

$\Rightarrow$ $-2=2C$

$\Rightarrow$ $C=-1$

Equate coefficients in x^2: $5=A+B$

$\Rightarrow$ $5=3+B$

$\Rightarrow$ $B=2$

Hence

$$\frac{x^4-4x^3+9x^2-17x+12}{x^3-4x^2+4x}\equiv x+\frac{5x^2-17x+12}{x(x-2)^2}$$

$$\equiv x+\frac{3}{x}+\frac{2}{(x-2)}-\frac{1}{(x-2)^2}$$

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Partial fractions
Exercise E, Question 3

Question:

Show that $\frac{-3x^3-4x^2+19x+8}{x^2+2x-3}$ can be expressed in the form

$A+Bx+\frac{C}{(x-1)}+\frac{D}{(x+3)}$, where A, B, C and D are constants to be found.

Solution:

Divide the numerator by the denominator:

$$\begin{array}{r} -3x+2 \\ x^2+2x-3\overline{\big)-3x^3-4x^2+19x+8} \\ \underline{-3x^3-6x^2+9x} \\ 2x^2+10x+8 \\ \underline{2x^2+\ 4x-6} \\ \underline{6x+14} \end{array} \leftarrow \text{Remainder}$$

Therefore $\frac{-3x^3-4x^2+19x+8}{x^2+2x-3} \equiv -3x+2+\frac{6x+14}{x^2+2x-3}$ Factorise denominator

$$\equiv -3x+2+\frac{6x+14}{(x-1)(x+3)}$$

Let $\frac{6x+14}{(x-1)(x+3)} \equiv \frac{C}{(x-1)}+\frac{D}{(x+3)} \equiv \frac{C(x+3)+D(x-1)}{(x-1)(x+3)}$

Set the numerators equal: $6x+14 \equiv C(x+3)+D(x-1)$

Substitute $x=1$ $\Rightarrow$ $6+14=C\times 4+D\times 0$

$\Rightarrow$ $20=4C$

$\Rightarrow$ $5=C$

Substitute $x=-3$ $\Rightarrow$ $6\times(-3)+14=C\times 0+D\times(-4)$

$\Rightarrow$ $-4=-4D$

$\Rightarrow$ $D=1$

Hence

$$\frac{-3x^3-4x^2+19x+8}{x^2+2x-3} \equiv -3x+2+\frac{6x+14}{(x-1)(x+3)}$$

$$\equiv -3x+2+\frac{5}{(x-1)}+\frac{1}{(x+3)}$$

So $A=2$, $B=-3$, $C=5$ and $D=1$

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Partial fractions
Exercise F, Question 1

Question:

Express the following as a partial fraction:

(a) $\frac{x-3}{x(x-1)}$

(b) $\frac{7x^2+2x-2}{x^2(x+1)}$

(c) $\frac{-15x+21}{(x-2)(x+1)(x-5)}$

(d) $\frac{x^2+1}{x(x-2)}$

Solution:

(a) Let $\frac{x-3}{x(x-1)} \equiv \frac{A}{x} + \frac{B}{(x-1)}$ Add the fractions

$\equiv \frac{A(x-1)+Bx}{x(x-1)}$

Set the numerators equal: $x-3 \equiv A(x-1)+Bx$

Substitute $x=1$ $\Rightarrow$ $1-3=A\times0+B\times1$

$\Rightarrow$ $B=-2$

Substitute $x=0$ $\Rightarrow$ $0-3=A\times(-1)+B\times0$

$\Rightarrow$ $-3=-1A$

$\Rightarrow$ $A=3$

Hence $\frac{x-3}{x(x-1)} \equiv \frac{3}{x} - \frac{2}{(x-1)}$

(b) Let $\frac{7x^2+2x-2}{x^2(x+1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)}$ Add the fractions

$\equiv \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}$

Set the numerators equal:

$7x^2+2x-2 \equiv Ax(x+1)+B(x+1)+Cx^2$

Substitute $x=0$ $\Rightarrow$ $0+0-2=A\times0+B\times1+C\times0$

$\Rightarrow$ $-2=1B$

$\Rightarrow$ $B=-2$

Substitute $x=-1$ $\Rightarrow$ $7-2-2=A\times0+B\times0+C\times1$

$\Rightarrow \quad 3 = 1C$

$\Rightarrow \quad C = 3$

Equate coefficients in x^2: $\quad 7 = A + C \quad$ Substitute $C = 3$

$\Rightarrow \quad 7 = A + 3$

$\Rightarrow \quad A = 4$

Hence $\dfrac{7x^2 + 2x - 2}{x^2 (x + 1)} \equiv \dfrac{4}{x} - \dfrac{2}{x^2} + \dfrac{3}{(x + 1)}$

(c) Let $\dfrac{- 15x + 21}{(x - 2) (x + 1) (x - 5)} \equiv \dfrac{A}{(x - 2)} + \dfrac{B}{(x + 1)} + \dfrac{C}{(x - 5)}$ Add the fractions

$\equiv \dfrac{A (x + 1) (x - 5) + B (x - 2) (x - 5) + C (x - 2) (x + 1)}{(x - 2) (x + 1) (x - 5)}$

Set the numerators equal:

$- 15x + 21 \equiv A (x + 1) (x - 5) + B (x - 2) (x - 5) + C (x - 2) (x + 1)$

Substitute $x = - 1 \quad \Rightarrow \quad 15 + 21 = A \times 0 + B \times (- 3) \times (- 6) + C \times 0$

$\Rightarrow \quad 36 = 18B$

$\Rightarrow \quad B = 2$

Substitute $x = 5 \quad \Rightarrow \quad - 75 + 21 = A \times 0 + B \times 0 + C \times 3 \times 6$

$\Rightarrow \quad - 54 = 18C$

$\Rightarrow \quad C = - 3$

Substitute $x = 2 \quad \Rightarrow \quad - 30 + 21 = A \times 3 \times (- 3) + B \times 0 + C \times 0$

$\Rightarrow \quad - 9 = - 9A$

$\Rightarrow \quad A = 1$

Hence $\dfrac{- 15x + 21}{(x - 2) (x + 1) (x - 5)} \equiv \dfrac{1}{(x - 2)} + \dfrac{2}{(x + 1)} - \dfrac{3}{(x - 5)}$

(d) $\dfrac{x^2 + 1}{x (x - 2)} \equiv \dfrac{x^2 + 1}{x^2 - 2x} \equiv \dfrac{x^2 + 0x + 1}{x^2 - 2x + 0}$

Divide the numerator by the denominator:

$$\begin{array}{r} 1 \\ x^2-2x+0\overline{\big)x^2+0x+1} \\ \underline{x^2-2x+0} \\ \underline{2x+1} \end{array} \leftarrow \text{Remainder}$$

Therefore $\dfrac{x^2 + 1}{x (x - 2)} \equiv 1 + \dfrac{2x + 1}{x (x - 2)}$

Let $\dfrac{2x + 1}{x (x - 2)} \equiv \dfrac{A}{x} + \dfrac{B}{(x - 2)} \equiv \dfrac{A (x - 2) + Bx}{x (x - 2)}$

Set the numerators equal: $\quad 2x + 1 \equiv A (x - 2) + Bx$

Substitute $x = 0 \quad \Rightarrow \quad 1 = A \times (- 2) + B \times 0$

$\Rightarrow \quad 1 = - 2A$

$\Rightarrow \quad A = -\frac{1}{2}$

Substitute $x = 2 \quad \Rightarrow \quad 2 \times 2 + 1 = A \times 0 + B \times 2$

$\Rightarrow \quad 5 = 2B$

$\Rightarrow \quad B = \frac{5}{2}$

Hence

$$\frac{x^2 + 1}{x(x-2)} \equiv 1 + \frac{2x+1}{x(x-2)} \equiv 1 + \frac{-\frac{1}{2}}{x} + \frac{\frac{5}{2}}{(x-2)}$$

$$\equiv 1 - \frac{1}{2x} + \frac{5}{2(x-2)}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Partial fractions

Exercise F, Question 2

Question:

Write the following algebraic fractions as a partial fraction:

(a) $\frac{3x+1}{x^2+2x+1}$

(b) $\frac{2x^2+2x-8}{x^2+2x-3}$

(c) $\frac{3x^2+12x+8}{(x+2)^3}$

(d) $\frac{x^4}{x^2-2x+1}$

Solution:

(a) $\frac{3x+1}{x^2+2x+1} \equiv \frac{3x+1}{(x+1)^2}$ Repeated factor in denominator

Let $\frac{3x+1}{(x+1)^2} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} \equiv \frac{A(x+1)+B}{(x+1)^2}$

Set the numerators equal: $3x+1 \equiv A(x+1)+B$

Substitute $x=-1$ $\Rightarrow$ $-3+1=A\times 0+B$

$\Rightarrow$ $B=-2$

Equate coefficients of x: $3=A$

$\Rightarrow$ $A=3$

Hence $\frac{3x+1}{x^2+2x+1} \equiv \frac{3x+1}{(x+1)^2} \equiv \frac{3}{(x+1)} - \frac{2}{(x+1)^2}$

(b) $\frac{2x^2+2x-8}{x^2+2x-3}$ is an **improper fraction**

Dividing gives

$$\begin{array}{r l}
 & 2 \\
x^2+2x-3 \, \big) & 2x^2+2x-8 \\
 & \underline{2x^2+4x-6} \\
 & \underline{-2x-2} \quad \leftarrow \text{Remainder}
\end{array}$$

Therefore $\frac{2x^2+2x-8}{x^2+2x-3} \equiv 2 + \frac{-2x-2}{x^2+2x-3}$ Factorise the denominator

$$\equiv 2 + \frac{-2x-2}{(x+3)(x-1)}$$

Let $\frac{-2x-2}{(x+3)(x-1)} \equiv \frac{A}{(x+3)} + \frac{B}{(x-1)} \equiv \frac{A(x-1)+B(x+3)}{(x+3)(x-1)}$

Set the numerators equal: $-2x-2 \equiv A(x-1)+B(x+3)$

Substitute $x=1$ ⇒ $-2-2 = A \times 0 + B \times 4$

⇒ $-4 = 4B$

⇒ $B = -1$

Substitute $x = -3$ ⇒ $6-2 = A \times (-4) + B \times 0$

⇒ $4 = -4A$

⇒ $A = -1$

Hence

$$\frac{2x^2+2x-8}{x^2+2x-3} \equiv 2 + \frac{-2x-2}{(x+3)(x-1)}$$

$$\equiv 2 - \frac{1}{(x+3)} - \frac{1}{(x-1)}$$

(c) Let $\frac{3x^2+12x+8}{(x+2)^3} \equiv \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$

$$\equiv \frac{A(x+2)^2+B(x+2)+C}{(x+2)^3}$$

Set the numerators equal:

$3x^2+12x+8 \equiv A(x+2)^2+B(x+2)+C$

Substitute $x=-2$ ⇒ $12-24+8 = A \times 0 + B \times 0 + C$

⇒ $C = -4$

Equate coefficients in x^2: $3 = A$

⇒ $A = 3$

Equate coefficients in x: $12 = 4A + B$ Substitute $A = 3$

⇒ $12 = 12 + B$

⇒ $B = 0$

Hence $\frac{3x^2+12x+8}{(x+2)^3} \equiv \frac{3}{(x+2)} - \frac{4}{(x+2)^3}$

(d) $\frac{x^4}{x^2-2x+1} \equiv \frac{x^4+0x^3+0x^2+0x+0}{x^2-2x+1}$

Divide the numerator by the denominator:

$$
\begin{array}{rl}
 & \,x^2+2x+3 \\
\cline{2-2}
x^2-2x+1 & \big)\,x^4+0x^3+0x^2+0x+0 \\
 & \,\underline{x^4-2x^3+x^2} \\
 & \qquad 2x^3-x^2+0x \\
 & \qquad \underline{2x^3-4x^2+2x} \\
 & \qquad\qquad 3x^2-2x+0 \\
 & \qquad\qquad \underline{3x^2-6x+3} \\
 & \qquad\qquad \underline{\qquad 4x-3} \quad \leftarrow \text{Remainder}
\end{array}
$$

Therefore

$$\frac{x^4}{x^2-2x+1} \equiv x^2+2x+3+\frac{4x-3}{x^2-2x+1}$$ Factorise the denominator

$$\equiv x^2+2x+3+\frac{4x-3}{(x-1)^2}$$

Let $\frac{4x-3}{(x-1)^2} \equiv \frac{A}{(x-1)} + \frac{B}{(x-1)^2} \equiv \frac{A(x-1)+B}{(x-1)^2}$

Set the numerators equal: $4x-3 \equiv A(x-1)+B$

Substitute $x = 1 \Rightarrow 4-3 = B \Rightarrow B = 1$

Equate coefficients in x: $4 = A$

Hence

$$\frac{x^4}{x^2-2x+1} \equiv x^2+2x+3+\frac{4x-3}{(x-1)^2} \equiv x^2+2x+3+\frac{4}{(x-1)}+\frac{1}{(x-1)^2}$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Partial fractions
Exercise F, Question 3

Question:

Given that $f(x) = 2x^3 + 9x^2 + 10x + 3$:

(a) Show that -3 is a root of $f(x)$.

(b) Express $\frac{10}{f(x)}$ as partial fractions.

Solution:

(a) $f(-3) = 2 \times (-27) + 9 \times 9 + 10 \times (-3) + 3 = -54 + 81 - 30 + 3 = 0$

Therefore -3 is a root $\Rightarrow$ $(x+3)$ is a factor

(b) $f(x) = 2x^3 + 9x^2 + 10x + 3$ $\quad (x+3)$ is a factor

$= (x+3)(2x^2 + 3x + 1)$ $\quad$ By inspection

$= (x+3)(2x+1)(x+1)$

$$\frac{10}{f(x)} \equiv \frac{10}{(x+3)(2x+1)(x+1)} \equiv \frac{A}{(x+3)} + \frac{B}{(2x+1)} + \frac{C}{(x+1)}$$

$$\equiv \frac{A(2x+1)(x+1) + B(x+3)(x+1) + C(x+3)(2x+1)}{(x+3)(2x+1)(x+1)}$$

Set the numerators equal:

$10 \equiv A(2x+1)(x+1) + B(x+3)(x+1) + C(x+3)(2x+1)$

Substitute $x = -1$ $\Rightarrow$ $10 = A \times 0 + B \times 0 + C \times 2 \times (-1)$

$\Rightarrow$ $10 = -2C$

$\Rightarrow$ $C = -5$

Substitute $x = -3$ $\Rightarrow$ $10 = A \times (-5) \times (-2) + B \times 0 + C \times 0$

$\Rightarrow$ $10 = 10A$

$\Rightarrow$ $A = 1$

Substitute $x = -\frac{1}{2}$ $\Rightarrow$ $10 = A \times 0 + B \times \left(2\frac{1}{2}\right) \times \left(\frac{1}{2}\right) + C \times 0$

$$\Rightarrow \quad 10 = 1.25B$$

$$\Rightarrow \quad B = 8$$

Hence $\frac{10}{\text{f}(x)} \equiv \frac{1}{(x+3)} + \frac{8}{(2x+1)} - \frac{5}{(x+1)}$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane

Exercise A, Question 1

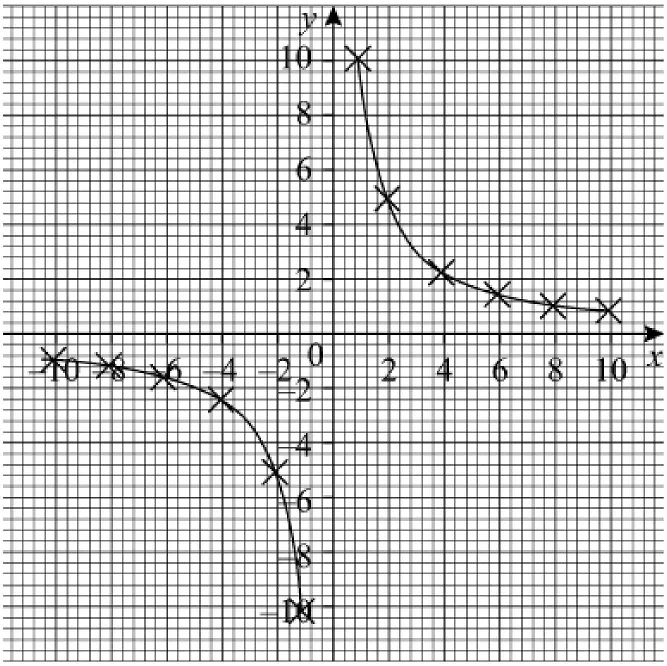
Question:

A curve is given by the parametric equations $x = 2t$, $y = \frac{5}{t}$ where $t \neq 0$. Complete the table and draw a graph of the curve for $-5 \leq t \leq 5$.

t	-5	-4	-3	-2	-1	-0.5	0.5	1	2	3	4	5
$x = 2t$	-10	-8				-1						
$y = \frac{5}{t}$	-1	-1.25					10					

Solution:

t	-5	-4	-3	-2	-1	-0.5	0.5	1	2	3	4	5
$x=2t$	-10	-8	-6	-4	-2	-1	1	2	4	6	8	10
$y=\frac{5}{t}$	-1	-1.25	-1.67	-2.5	-5	-10	10	5	2.5	1.67	1.25	1



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Coordinate geometry in the (x, y) plane

Exercise A, Question 2

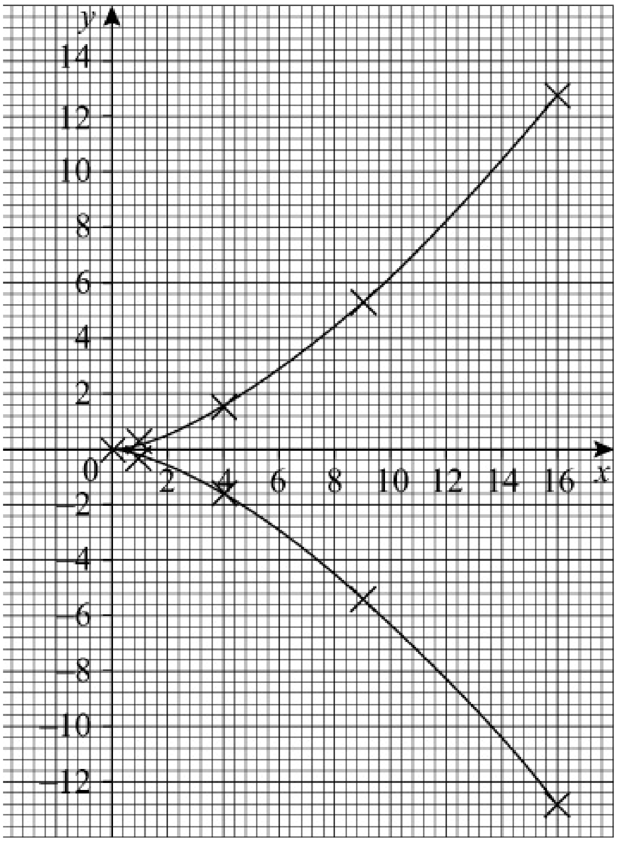
Question:

A curve is given by the parametric equations $x = t^2$, $y = \frac{t^3}{5}$. Complete the table and draw a graph of the curve for $-4 \leq t \leq 4$.

t	−4	−3	−2	−1	0	1	2	3	4
$x = t^2$	16								
$y = \frac{t^3}{5}$	−12.8								

Solution:

t	−4	−3	−2	−1	0	1	2	3	4
$x = t^2$	16	9	4	1	0	1	4	9	16
$y = \frac{t^3}{5}$	−12.8	−5.4	−1.6	−0.2	0	0.2	1.6	5.4	12.8



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Coordinate geometry in the (x, y) plane
Exercise A, Question 3

Question:

Sketch the curves given by these parametric equations:

(a) $x = t - 2$, $y = t^2 + 1$ for $-4 \leq t \leq 4$

(b) $x = t^2 - 2$, $y = 3 - t$ for $-3 \leq t \leq 3$

(c) $x = t^2$, $y = t(5 - t)$ for $0 \leq t \leq 5$

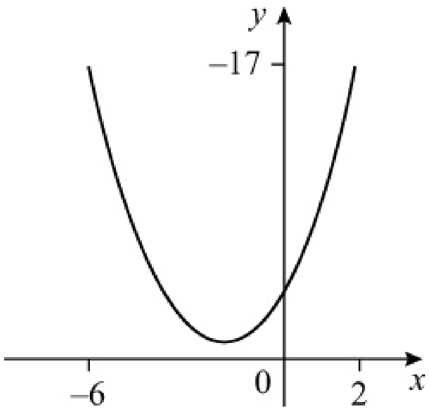
(d) $x = 3\sqrt{t}$, $y = t^3 - 2t$ for $0 \leq t \leq 2$

(e) $x = t^2$, $y = (2 - t)(t + 3)$ for $-5 \leq t \leq 5$

Solution:

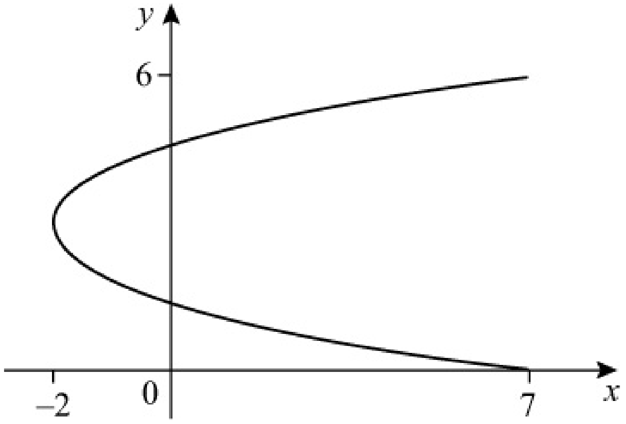
(a)

t	−4	−3	−2	−1	0	1	2	3	4
$x = t - 2$	−6	−5	−4	−3	−2	−1	0	1	2
$y = t^2 + 1$	17	10	5	2	1	2	5	10	17



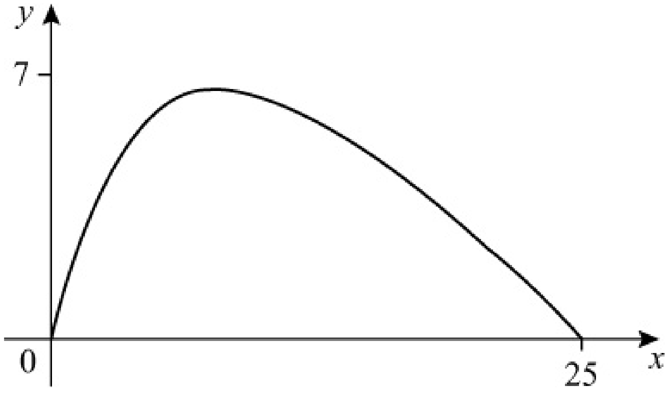
(b)

t	−3	−2	−1	0	1	2	3
$x = t^2 - 2$	7	2	−1	−2	−1	2	7
$y = 3 - t$	6	5	4	3	2	1	0



(c)

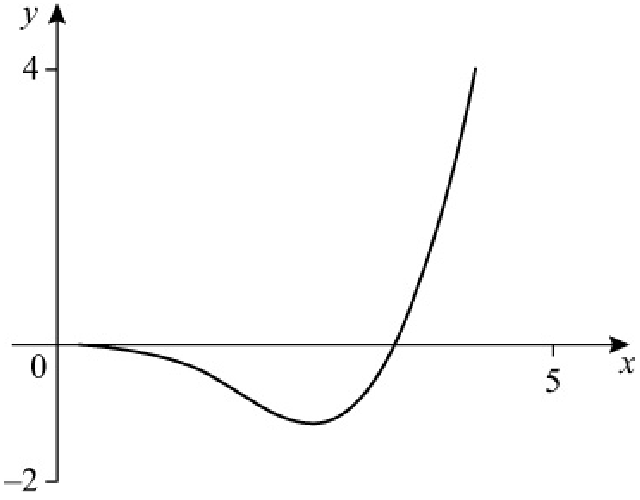
t	0	1	2	3	4	5
$x = t^2$	0	1	4	9	16	25
$y = t(5 - t)$	0	4	6	6	4	0



(d)

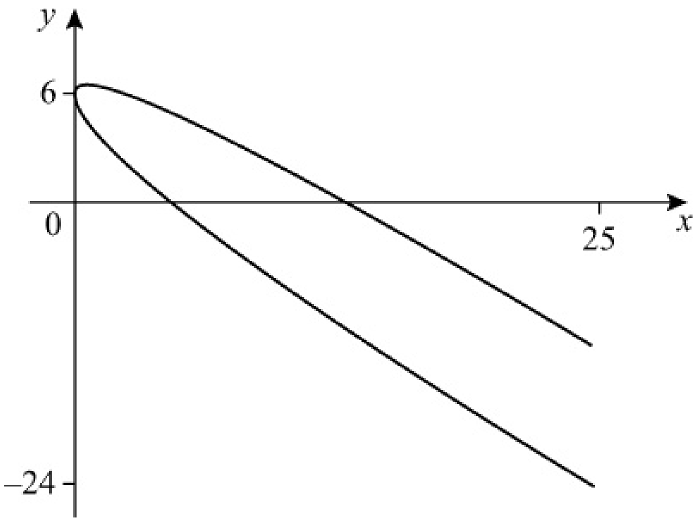
t	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$x = 3\sqrt{t}$	0	1.5	2.12	2.60	3	3.35	3.67	3.97	4.24
$y = t^3 - 2t$	0	−0.48	−0.88	−1.08	−1	−0.55	0.38	1.86	4

Answers have been rounded to 2 d.p.



(e)

t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$x = t^2$	25	16	9	4	1	0	1	4	9	16	25
$y = (2-t)(t+3)$	-14	-6	0	4	6	6	4	0	-6	-14	-24



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Coordinate geometry in the (x, y) plane

Exercise A, Question 4

Question:

Find the cartesian equation of the curves given by these parametric equations:

(a) $x = t - 2, y = t^2$

(b) $x = 5 - t, y = t^2 - 1$

(c) $x = \frac{1}{t}, y = 3 - t, t \neq 0$

(d) $x = 2t + 1, y = \frac{1}{t}, t \neq 0$

(e) $x = 2t^2 - 3, y = 9 - t^2$

(f) $x = \sqrt{t}, y = t (9 - t)$

(g) $x = 3t - 1, y = (t - 1) (t + 2)$

(h) $x = \frac{1}{t - 2}, y = t^2, t \neq 2$

(i) $x = \frac{1}{t + 1}, y = \frac{1}{t - 2}, t \neq - 1, t \neq 2$

(j) $x = \frac{t}{2t - 1}, y = \frac{t}{t + 1}, t \neq - 1, t \neq \frac{1}{2}$

Solution:

(a) $x = t - 2, y = t^2$

$x = t - 2$

$t = x + 2$

Substitute $t = x + 2$ into $y = t^2$

$y = (x + 2) ^2$

So the cartesian equation of the curve is $y = (x + 2) ^2$.

(b) $x = 5 - t, y = t^2 - 1$

$x = 5 - t$
$t = 5 - x$

Substitute $t = 5 - x$ into $y = t^2 - 1$

$y = (5 - x)^2 - 1$
$y = 25 - 10x + x^2 - 1$
$y = x^2 - 10x + 24$

So the cartesian equation of the curve is $y = x^2 - 10x + 24$.

(c) $x = \frac{1}{t}, y = 3 - t$

$x = \frac{1}{t}$

$t = \frac{1}{x}$

Substitute $t = \frac{1}{x}$ into $y = 3 - t$

$y = 3 - \frac{1}{x}$

So the cartesian equation of the curve is $y = 3 - \frac{1}{x}$.

(d) $x = 2t + 1, y = \frac{1}{t}$

$x = 2t + 1$
$2t = x - 1$

$t = \frac{x - 1}{2}$

Substitute $t = \frac{x - 1}{2}$ into $y = \frac{1}{t}$

$$y = \frac{1}{\left(\frac{x - 1}{2}\right)}$$

$$y = \frac{2}{x - 1} \qquad \left[\text{Note: This uses } \frac{1}{\left(\frac{a}{b}\right)} = \frac{b}{a} \right]$$

So the cartesian equation of the curve is $y = \frac{2}{x - 1}$.

(e) $x = 2t^2 - 3 , y = 9 - t^2$

$x = 2t^2 - 3$

$2t^2 = x + 3$

$t^2 = \frac{x + 3}{2}$

Substitute $t^2 = \frac{x + 3}{2}$ into $y = 9 - t^2$

$y = 9 - \frac{x + 3}{2}$

$y = \frac{18 - (x + 3)}{2}$

$y = \frac{15 - x}{2}$

So the cartesian equation is $y = \frac{15 - x}{2}$.

(f) $x = \sqrt{t} , y = t (9 - t)$

$x = \sqrt{t}$

$t = x^2$

Substitute $t = x^2$ into $y = t (9 - t)$

$y = x^2 (9 - x^2)$

So the cartesian equation is $y = x^2 (9 - x^2)$.

(g) $x = 3t - 1 , y = (t - 1) (t + 2)$

$x = 3t - 1$

$3t = x + 1$

$t = \frac{x + 1}{3}$

Substitute $t = \frac{x + 1}{3}$ into $y = (t - 1) (t + 2)$

$$y = \left(\frac{x + 1}{3} - 1 \right) \left(\frac{x + 1}{3} + 2 \right)$$

$$y = \left(\frac{x + 1}{3} - \frac{3}{3} \right) \left(\frac{x + 1}{3} + \frac{6}{3} \right)$$

$$y = \left(\frac{x + 1 - 3}{3} \right) \left(\frac{x + 1 + 6}{3} \right)$$

$$y = \left(\frac{x - 2}{3} \right) \left(\frac{x + 7}{3} \right)$$

$$y = \frac{1}{9}\left(x - 2 \right)\left(x + 7 \right)$$

So the cartesian equation of the curve is $y = \frac{1}{9}\left(x - 2 \right)\left(x + 7 \right)$.

(h) $x = \frac{1}{t-2}, y = t^2$

$$x = \frac{1}{t-2}$$

$$x (t - 2) = 1$$

$$t - 2 = \frac{1}{x}$$

$$t = \frac{1}{x} + 2$$

$$t = \frac{1}{x} + \frac{2x}{x}$$

$$t = \frac{1 + 2x}{x}$$

Substitute $t = \frac{1+2x}{x}$ into $y = t^2$

$$y = \left(\frac{1+2x}{x} \right)^2$$

So the cartesian equation of the curve is $y = \left(\frac{1+2x}{x} \right)^2$.

(i) $x = \frac{1}{t+1}, y = \frac{1}{t-2}$

$$x = \frac{1}{t+1}$$

$$(t + 1) x = 1$$

$$t + 1 = \frac{1}{x}$$

$$t = \frac{1}{x} - 1$$

Substitute $t = \frac{1}{x} - 1$ into $y = \frac{1}{t-2}$

$$y = \frac{1}{\left(\frac{1}{x} - 1 \right) - 2}$$

$$y = \frac{1}{\frac{1}{x} - 3}$$

$$y = \frac{1}{\frac{1}{x} - \frac{3x}{x}}$$

$$y = \frac{1}{\left(\frac{1 - 3x}{x} \right)}$$

$$y = \frac{x}{1 - 3x} \qquad \left[\text{ Note: This uses } \frac{1}{\left(\frac{a}{b} \right)} = \frac{b}{a} \right]$$

So the cartesian equation of the curve is $y = \frac{x}{1 - 3x}$.

(j) $x = \frac{t}{2t - 1}, y = \frac{t}{t + 1}$

$x = \frac{t}{2t - 1}$

$x \times \left(2t - 1 \right) = \frac{t}{2t - 1} \times \left(2t - 1 \right)$ Multiply each side by $(2t - 1)$

$x (2t - 1) = t$ Simplify

$2tx - x = t$ Expand the brackets

$2tx = t + x$ Add x to each side

$2tx - t = x$ Subtract $2t$ from each side

$t (2x - 1) = x$ Factorise t

$\frac{t (2x - 1)}{(2x - 1)} = \frac{x}{2x - 1}$ Divide each side by $(2x - 1)$

$t = \frac{x}{2x - 1}$ Simplify

Substitute $t = \frac{x}{2x - 1}$ into $y = \frac{t}{t + 1}$

$$y = \frac{\left(\frac{x}{2x-1} \right)}{\left(\frac{x}{2x-1} + 1 \right)}$$

$$y = \frac{\left(\frac{x}{2x-1} \right)}{\left(\frac{x}{2x-1} + \frac{2x-1}{2x-1} \right)}$$

$$y = \frac{\left(\frac{x}{2x-1} \right)}{\left(\frac{x+2x-1}{2x-1} \right)}$$

$$y = \frac{\left(\frac{x}{2x-1} \right)}{\left(\frac{3x-1}{2x-1} \right)}$$

$$y = \frac{x}{3x-1} \qquad \left[\text{Note: This uses } \frac{\left(\frac{a}{b} \right)}{\left(\frac{c}{b} \right)} = \frac{a}{c} \right]$$

So the cartesian equation of the curve is $y = \frac{x}{3x-1}$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise A, Question 5

Question:

Show that the parametric equations:
(i) $x = 1 + 2t, y = 2 + 3t$

(ii) $x = \frac{1}{2t-3}, y = \frac{t}{2t-3}, t \neq \frac{3}{2}$

represent the same straight line.

Solution:

(i) $x = 1 + 2t, y = 2 + 3t$

$x = 1 + 2t$

$2t = x - 1$

$t = \frac{x-1}{2}$

Substitute $t = \frac{x-1}{2}$ into $y = 2 + 3t$

$y = 2 + 3\left(\frac{x-1}{2}\right)$

$y = 2 + 3\left(\frac{x}{2} - \frac{1}{2}\right)$

$y = 2 + \frac{3x}{2} - \frac{3}{2}$

$y = \frac{3x}{2} + \frac{1}{2}$

(ii) $x = \frac{1}{2t-3}, y = \frac{t}{2t-3}$

$\frac{y}{x} = \frac{\left(\frac{t}{2t-3}\right)}{\left(\frac{1}{2t-3}\right)}$ Note: $\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{b}\right)} = \frac{a}{c}$

$\frac{y}{x} = t$

Substitute $t = \frac{y}{x}$ into $x = \frac{1}{2t-3}$

$$x = \frac{1}{2\left(\frac{y}{x}\right) - 3}$$

$$x\left[2\left(\frac{y}{x}\right) - 3\right] = 1$$

$$2y - 3x = 1$$

$$2y = 3x + 1$$

$$y = \frac{3}{2}x + \frac{1}{2}$$

The cartesian equations of (i) and (ii) are the same, so they represent the same straight line.

Solutionbank
Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise B, Question 1

Question:

Find the coordinates of the point(s) where the following curves meet the x-axis:

(a) $x = 5 + t, y = 6 - t$

(b) $x = 2t + 1, y = 2t - 6$

(c) $x = t^2, y = (1 - t) (t + 3)$

(d) $x = \frac{1}{t}, y = \sqrt{(t - 1) (2t - 1)}, t \neq 0$

(e) $x = \frac{2t}{1 + t}, y = t - 9, t \neq - 1$

Solution:

(a) $x = 5 + t, y = 6 - t$
When $y = 0$
$6 - t = 0$
so $t = 6$
Substitute $t = 6$ into $x = 5 + t$
$x = 5 + 6$
$x = 11$
So the curve meets the x-axis at (11, 0).

(b) $x = 2t + 1, y = 2t - 6$
When $y = 0$
$2t - 6 = 0$
$2t = 6$
so $t = 3$
Substitute $t = 3$ into $x = 2t + 1$
$x = 2 (3) + 1$
$x = 6 + 1$
$x = 7$
So the curve meets the x-axis at (7, 0).

(c) $x = t^2, y = (1 - t) (t + 3)$
When $y = 0$

$(1 - t) (t + 3) = 0$

so $t = 1$ and $t = -3$

(1) Substitute $t = 1$ into $x = t^2$

$x = 1^2$

$x = 1$

(2) Substitute $t = -3$ into $x = t^2$

$x = (-3)^2$

$x = 9$

So the curve meets the x-axis at (1, 0) and (9, 0).

(d) $x = \frac{1}{t}, y = \sqrt{(t - 1) (2t - 1)}$

When $y = 0$

$\sqrt{(t - 1) (2t - 1)} = 0$

$(t - 1) (2t - 1) = 0$

so $t = 1$ and $t = \frac{1}{2}$

(1) Substitute $t = 1$ into $x = \frac{1}{t}$

$x = \frac{1}{(1)}$

$x = 1$

(2) Substitute $t = \frac{1}{2}$ into $x = \frac{1}{t}$

$x = \frac{1}{\left(\frac{1}{2} \right)}$

$x = 2$

So the curve meets the x-axis at (1, 0) and (2, 0).

(e) $x = \frac{2t}{1 + t}, y = t - 9$

When $y = 0$

$t - 9 = 0$

so $t = 9$

Substitute $t = 9$ into $x = \frac{2t}{1 + t}$

$x = \frac{2 (9)}{1 + (9)}$

$$x = \frac{18}{10}$$

$$x = \frac{9}{5}$$

So the curve meets the x-axis at $\left(\frac{9}{5} , \ 0 \right)$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise B, Question 2

Question:

Find the coordinates of the point(s) where the following curves meet the y-axis:

(a) $x = 2t, y = t^2 - 5$

(b) $x = \sqrt{(3t - 4)} , y = \frac{1}{t^2}, t \neq 0$

(c) $x = t^2 + 2t - 3, y = t (t - 1)$

(d) $x = 27 - t^3, y = \frac{1}{t - 1}, t \neq 1$

(e) $x = \frac{t - 1}{t + 1}, y = \frac{2t}{t^2 + 1}, t \neq - 1$

Solution:

(a) When $x = 0$

$2t = 0$

so $t = 0$

Substitute $t = 0$ into $y = t^2 - 5$

$y = (0) ^2 - 5$

$y = - 5$

So the curve meets the y-axis at $(0 , - 5)$.

(b) When $x = 0$

$\sqrt{3t - 4} = 0$

$3t - 4 = 0$

$3t = 4$

so $t = \frac{4}{3}$

Substitute $t = \frac{4}{3}$ into $y = \frac{1}{t^2}$

$$y = \frac{1}{\left(\frac{4}{3}\right)^2}$$

$$y = \frac{1}{\left(\frac{16}{9}\right)}$$

$$y = \frac{9}{16} \quad \left[\text{Note: This uses } \frac{1}{\left(\frac{a}{b}\right)} = \frac{b}{a} \right]$$

So the curve meets the y-axis at $\left(0 , \frac{9}{16} \right)$.

(c) When $x = 0$

$t^2 + 2t - 3 = 0$

$(t + 3) (t - 1) = 0$

so $t = -3$ and $t = 1$

(1) Substitute $t = -3$ into $y = t (t - 1)$

$y = (-3) [(-3) - 1]$

$y = (-3) \times (-4)$

$y = 12$

(2) Substitute $t = 1$ into $y = t (t - 1)$

$y = 1 (1 - 1)$

$y = 1 \times 0$

$y = 0$

So the curve meets the y-axis at (0, 0) and (0, 12).

(d) When $x = 0$

$27 - t^3 = 0$

$t^3 = 27$

$t = \sqrt[3]{27}$

so $t = 3$

Substitute $t = 3$ into $y = \frac{1}{t - 1}$

$$y = \frac{1}{(3) - 1}$$

$$y = \frac{1}{2}$$

So the curve meets the y-axis at $\left(0 , \frac{1}{2} \right)$.

(e) When $x = 0$

$$\frac{t-1}{t+1} = 0$$

$$t - 1 = 0 \qquad \left[\text{Note: } \frac{a}{b} = 0 \Rightarrow a = 0 \right]$$

So $t = 1$

Substitute $t = 1$ into $y = \frac{2t}{t^2 + 1}$

$$y = \frac{2 (1)}{(1)^2 + 1}$$

$$y = \frac{2}{2}$$

$y = 1$

So the curve meets the y-axis at (0, 1).

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise B, Question 3

Question:

A curve has parametric equations $x = 4at^2$, $y = a\left(2t - 1\right)$, where a is a constant. The curve passes through the point (4, 0). Find the value of a.

Solution:

When $y = 0$
$a\left(2t - 1\right) = 0$
$2t - 1 = 0$
$2t = 1$
$t = \frac{1}{2}$

When $t = \frac{1}{2}$, $x = 4$

So substitute $t = \frac{1}{2}$ and $x = 4$ into $x = 4at^2$

$4a\left(\frac{1}{2}\right)^2 = 4$

$4a \times \frac{1}{4} = 4$

$a = 4$
So the value of a is 4.

Solutionbank
Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise B, Question 4

Question:

A curve has parametric equations $x = b\left(2t - 3\right)$, $y = b\left(1 - t^2\right)$, where b is a constant. The curve passes through the point $\left(0, -5\right)$. Find the value of b.

Solution:

When $x = 0$
$b\left(2t - 3\right) = 0$
$2t - 3 = 0$
$2t = 3$
$t = \frac{3}{2}$

When $t = \frac{3}{2}$, $y = -5$

So substitute $t = \frac{3}{2}$ and $y = -5$ into $y = b\left(1 - t^2\right)$

$$b\left[1 - \left(\frac{3}{2}\right)^2\right] = -5$$

$$b\left(1 - \frac{9}{4}\right) = -5$$

$$b\left(\frac{-5}{4}\right) = -5$$

$$b = \frac{-5}{\left(\frac{-5}{4}\right)}$$

$b = 4$
So the value of b is 4.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane

Exercise B, Question 5

Question:

A curve has parametric equations $x = p(2t - 1)$, $y = p(t^3 + 8)$, where p is a constant. The curve meets the x-axis at (2, 0) and the y-axis at A.

(a) Find the value of p.

(b) Find the coordinates of A.

Solution:

(a) When $y = 0$

$p(t^3 + 8) = 0$

$t^3 + 8 = 0$

$t^3 = -8$

$t = \sqrt[3]{-8}$

$t = -2$

When $t = -2$, $x = 2$

So substitute $t = -2$ and $x = 2$ into $x = p(2t - 1)$

$p[2(-2) - 1] = 2$

$p(-4 - 1) = 2$

$p(-5) = 2$

$p = -\frac{2}{5}$

(b) When $x = 0$

$p(2t - 1) = 0$

$2t - 1 = 0$

$2t = 1$

$t = \frac{1}{2}$

When the curve meets the y-axis $t = \frac{1}{2}$

So substitute $t = \frac{1}{2}$ into $y = p(t^3 + 8)$

$y = p\left[\left(\frac{1}{2}\right)^3 + 8\right]$

but $p = -\frac{2}{5}$

So $y = -\frac{2}{5}\left[\left(\frac{1}{2}\right)^3 + 8\right] = -\frac{2}{5}\left(\frac{1}{8} + 8\right) = -\frac{2}{5} \times \frac{65}{8} = -\frac{13}{4}$

So the coordinates of A are $\left(0, -\frac{13}{4}\right)$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise B, Question 6

Question:

A curve is given parametrically by the equations $x = 3qt^2$, $y = 4 (t^3 + 1)$, where q is a constant. The curve meets the x-axis at X and the y-axis at Y. Given that $OX = 2OY$, where O is the origin, find the value of q.

Solution:

(1) When $y = 0$
$4 (t^3 + 1) = 0$
$t^3 + 1 = 0$
$t^3 = - 1$
$t = \sqrt[3]{- 1}$
$t = - 1$
Substitute $t = - 1$ into $x = 3qt^2$
$x = 3q (- 1)^2$
$x = 3q$
So the coordinates of X are $(3q, 0)$.
(2) When $x = 0$
$3qt^2 = 0$
$t^2 = 0$
$t = 0$
Substitute $t = 0$ into $y = 4 (t^3 + 1)$
$y = 4 [(0)^3 + 1]$
$y = 4$
So the coordinates of Y are $(0, 4)$.
(3) Now $OX = 3q$ and $OY = 4$
As $OX = 2OY$
$(3q) = 2 (4)$
$3q = 8$
$q = \frac{8}{3}$

So the value of q is $\frac{8}{3}$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise B, Question 7

Question:

Find the coordinates of the point of intersection of the line with parametric equations $x = 3t + 2, y = 1 - t$ and the line $y + x = 2$.

Solution:

(1) Substitute $x = 3t + 2$ and $y = 1 - t$ into $y + x = 2$

$(1 - t) + (3t + 2) = 2$

$1 - t + 3t + 2 = 2$

$2t + 3 = 2$

$2t = - 1$

$t = - \frac{1}{2}$

(2) Substitute $t = - \frac{1}{2}$ into $x = 3t + 2$

$x = 3 \left(- \frac{1}{2} \right) + 2 = - \frac{3}{2} + 2 = \frac{1}{2}$

(3) Substitute $t = - \frac{1}{2}$ into $y = 1 - t$

$y = 1 - \left(- \frac{1}{2} \right) = 1 + \frac{1}{2} = \frac{3}{2}$

So the coordinates of the point of intersection are $\left(\frac{1}{2} , \frac{3}{2} \right)$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane

Exercise B, Question 8

Question:

Find the coordinates of the points of intersection of the curve with parametric equations $x = 2t^2 - 1$, $y = 3(t + 1)$ and the line $3x - 4y = 3$.

Solution:

(1) Substitute $x = 2t^2 - 1$ and $y = 3(t + 1)$ into $3x - 4y = 3$

$3(2t^2 - 1) - 4[3(t + 1)] = 3$

$3(2t^2 - 1) - 12(t + 1) = 3$

$6t^2 - 3 - 12t - 12 = 3$

$6t^2 - 12t - 15 = 3$

$6t^2 - 12t - 18 = 0 \quad (\div 6)$

$t^2 - 2t - 3 = 0$

$(t - 3)(t + 1) = 0$

so $t = 3$ and $t = -1$

(2) Substitute $t = 3$ into $x = 2t^2 - 1$ and $y = 3(t + 1)$

$x = 2(3)^2 - 1 = 17$

$y = 3(3 + 1) = 12$

(3) Substitute $t = -1$ into $x = 2t^2 - 1$ and $y = 3(t + 1)$

$x = 2(-1)^2 - 1 = 1$

$y = 3(-1 + 1) = 0$

So the coordinates of the points of intersection are (17, 12) and (1, 0).

Solutionbank
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Coordinate geometry in the (x, y) plane
Exercise B, Question 9

Question:

Find the values of t at the points of intersection of the line $4x - 2y - 15 = 0$ with the parabola $x = t^2$, $y = 2t$ and give the coordinates of these points.

Solution:

(1) Substitute $x = t^2$ and $y = 2t$ into $4x - 2y - 15 = 0$

$4(t^2) - 2(2t) - 15 = 0$

$4t^2 - 4t - 15 = 0$

$(2t + 3)(2t - 5) = 0$

So $2t + 3 = 0 \quad \Rightarrow \quad 2t = -3 \quad \Rightarrow \quad t = \frac{-3}{2}$ and

$2t - 5 = 0 \quad \Rightarrow \quad 2t = 5 \quad \Rightarrow \quad t = \frac{5}{2}$

(2) Substitute $t = -\frac{3}{2}$ into $x = t^2$ and $y = 2t$

$x = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}$

$y = 2\left(-\frac{3}{2}\right) = -3$

(3) Substitute $t = \frac{5}{2}$ into $x = t^2$ and $y = 2t$

$x = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

$y = 2\left(\frac{5}{2}\right) = 5$

So the coordinates of the points of intersection are $\left(\frac{9}{4}, -3\right)$ and $\left(\frac{25}{4}, 5\right)$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane

Exercise B, Question 10

Question:

Find the points of intersection of the parabola $x = t^2$, $y = 2t$ with the circle $x^2 + y^2 - 9x + 4 = 0$.

Solution:

(1) Substitute $x = t^2$ and $y = 2t$ into $x^2 + y^2 - 9x + 4 = 0$

$\left(t^2 \right)^2 + \left(2t \right)^2 - 9 \left(t^2 \right) + 4 = 0$

$t^4 + 4t^2 - 9t^2 + 4 = 0$

$t^4 - 5t^2 + 4 = 0$

$\left(t^2 - 4 \right) \left(t^2 - 1 \right) = 0$

So $t^2 - 4 = 0 \quad \Rightarrow \quad t^2 = 4 \quad \Rightarrow \quad t = \sqrt{4} \quad \Rightarrow \quad t = \pm 2$ and

$t^2 - 1 = 0 \quad \Rightarrow \quad t^2 = 1 \quad \Rightarrow \quad t = \sqrt{1} \quad \Rightarrow \quad t = \pm 1$

(2) Substitute $t = 2$ into $x = t^2$ and $y = 2t$

$x = \left(2 \right)^2 = 4$

$y = 2 \left(2 \right) = 4$

(3) Substitute $t = -2$ into $x = t^2$ and $y = 2t$

$x = \left(-2 \right)^2 = 4$

$y = 2 \left(-2 \right) = -4$

(4) Substitute $t = 1$ into $x = t^2$ and $y = 2t$

$x = \left(1 \right)^2 = 1$

$y = 2 \left(1 \right) = 2$

(5) Substitute $t = -1$ into $x = t^2$ and $y = 2t$

$x = \left(-1 \right)^2 = 1$

$y = 2 \left(-1 \right) = -2$

So the coordinates of the points of intersection are (4, 4), $\left(4 , -4 \right)$, (1, 2) and $\left(1 , -2 \right)$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise C, Question 1

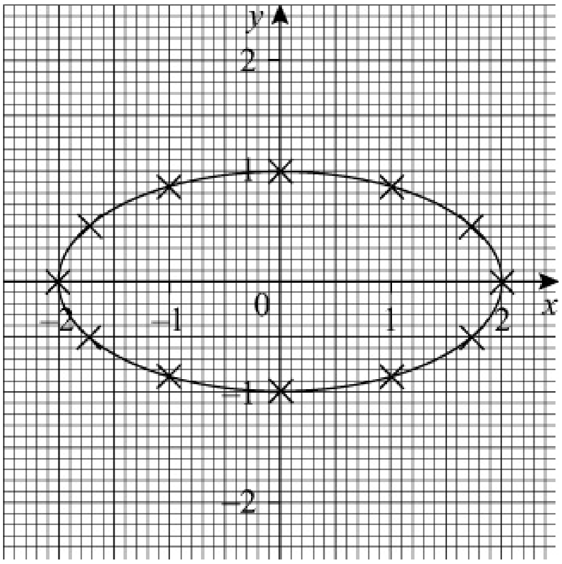
Question:

A curve is given by the parametric equations $x = 2 \sin t$, $y = \cos t$.
Complete the table and draw a graph of the curve for $0 \leq t \leq 2\pi$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$x = 2\sin t$			1.73		1.73			−1		−2			0
$y = \cos t$		0.87					−1		−0.5		0.5		

Solution:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$x = 2\sin t$	0	1	1.73	2	1.73	1	0	−1	−1.73	−2	−1.73	−1	0
$y = \cos t$	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1



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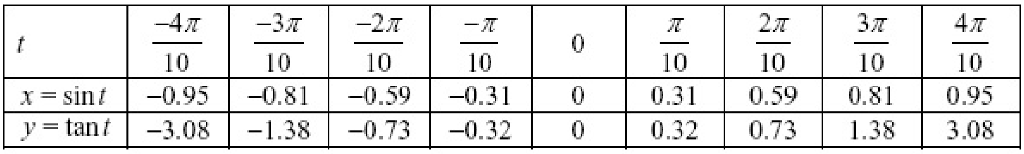
Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise C, Question 2

Question:

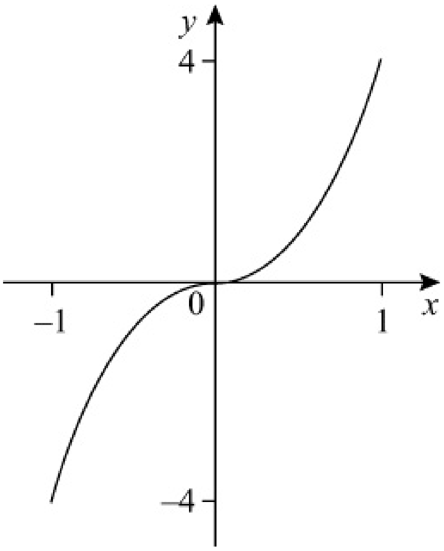
A curve is given by the parametric equations $x = \sin t$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Draw a graph of the curve.

Solution:



t	$\frac{-4\pi}{10}$	$\frac{-3\pi}{10}$	$\frac{-2\pi}{10}$	$\frac{-\pi}{10}$	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$
$x = \sin t$	−0.95	−0.81	−0.59	−0.31	0	0.31	0.59	0.81	0.95
$y = \tan t$	−3.08	−1.38	−0.73	−0.32	0	0.32	0.73	1.38	3.08

Answers are given to 2 d.p.



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Coordinate geometry in the (x, y) plane

Exercise C, Question 3

Question:

Find the cartesian equation of the curves given by the following parametric equations:

(a) $x = \sin t, y = \cos t$

(b) $x = \sin t - 3, y = \cos t$

(c) $x = \cos t - 2, y = \sin t + 3$

(d) $x = 2 \cos t, y = 3 \sin t$

(e) $x = 2 \sin t - 1, y = 5 \cos t + 4$

(f) $x = \cos t, y = \sin 2t$

(g) $x = \cos t, y = 2 \cos 2t$

(h) $x = \sin t, y = \tan t$

(i) $x = \cos t + 2, y = 4 \sec t$

(j) $x = 3 \cot t, y = \text{cosec}\, t$

Solution:

(a) $x = \sin t, y = \cos t$

$x^2 = \sin^2 t, y^2 = \cos^2 t$

As $\sin^2 t + \cos^2 t = 1$

$x^2 + y^2 = 1$

(b) $x = \sin t - 3, y = \cos t$

$\sin t = x + 3$

$\sin^2 t = (x + 3)^2$

$\cos t = y$

$\cos^2 t = y^2$

As $\sin^2 t + \cos^2 t = 1$

$(x + 3)^2 + y^2 = 1$

(c) $x = \cos t - 2,\, y = \sin t + 3$

$\cos t = x + 2$

$\sin t = y - 3$

As $\sin^2 t + \cos^2 t = 1$

$(y - 3)^2 + (x + 2)^2 = 1$ or $(x + 2)^2 + (y - 3)^2 = 1$

(d) $x = 2\ \cos t,\, y = 3\ \sin t$

$\sin t = \frac{y}{3}$

$\cos t = \frac{x}{2}$

As $\sin^2 t + \cos^2 t = 1$

$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{2}\right)^2 = 1$ or $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

(e) $x = 2\ \sin t - 1,\, y = 5\ \cos t + 4$

$2\ \sin t - 1 = x$

$2\ \sin t = x + 1$

$\sin t = \frac{x + 1}{2}$

and

$5\ \cos t + 4 = y$

$5\ \cos t = y - 4$

$\cos t = \frac{y - 4}{5}$

As $\sin^2 t + \cos^2 t = 1$

$\left(\frac{x + 1}{2}\right)^2 + \left(\frac{y - 4}{5}\right)^2 = 1$

(f) $x = \cos t,\, y = \sin 2t$

As $\sin 2t = 2\ \sin t\ \cos t$

$y = 2\ \sin t\ \cos t = (2\ \sin t)\quad x$

Now $\sin^2 t + \cos^2 t = 1$

So $\sin^2 t + x^2 = 1$

$\Rightarrow \quad \sin^2 t = 1 - x^2$

$\Rightarrow \quad \sin t = \sqrt{1 - x^2}$

So $y = (2\sqrt{1 - x^2})\quad x$ or $y = 2x\sqrt{1 - x^2}$

(g) $x = \cos t,\, y = 2\ \cos 2t$

As $\cos 2t = 2\ \cos^2 t - 1$

$y = 2 (2 \cos^2 t - 1)$

But $x = \cos t$

So $y = 2 (2x^2 - 1)$

$y = 4x^2 - 2$

(h) $x = \sin t, y = \tan t$

As $\tan t = \frac{\sin t}{\cos t}$

$y = \frac{\sin t}{\cos t}$

But $x = \sin t$

So $y = \frac{x}{\cos t}$

Now $\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - x^2}$ (from $\sin^2 t + \cos^2 t = 1$)

So $y = \frac{x}{\sqrt{1 - x^2}}$

(i) $x = \cos t + 2, y = 4 \sec t$

As $\sec t = \frac{1}{\cos t}$

$y = 4 \times \frac{1}{\cos t} = \frac{4}{\cos t}$

Now $x = \cos t + 2 \Rightarrow \cos t = x - 2$

So $y = \frac{4}{x - 2}$

(j) $x = 3 \cot t, y = \mathrm{cosec}\, t$

As $\sin^2 t + \cos^2 t = 1$

$$\frac{\sin^2 t}{\sin^2 t} + \frac{\cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \qquad \left(\div \sin^2 t \right)$$

$$1 + \left(\frac{\cos t}{\sin t} \right) ^2 = \left(\frac{1}{\sin t} \right) ^2$$

$1 + \cot^2 t = \mathrm{cosec}^2 t$

Now $x = 3 \cot t \Rightarrow \cot t = \frac{x}{3}$

and $y = \mathrm{cosec}\, t$

So $1 + \left(\frac{x}{3} \right) ^2 = (y) ^2$ (using $1 + \cos^2 t = \mathrm{cosec}^2 t$)

or $y^2 = 1 + \left(\frac{x}{3} \right)^2$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane

Exercise C, Question 4

Question:

A circle has parametric equations $x = \sin t - 5$, $y = \cos t + 2$.

(a) Find the cartesian equation of the circle.

(b) Write down the radius and the coordinates of the centre of the circle.

Solution:

(a) $x = \sin t - 5$, $y = \cos t + 2$

$\sin t = x + 5$ and $\cos t = y - 2$

As $\sin^2 t + \cos^2 t = 1$

$$\left(x + 5 \right)^2 + \left(y - 2 \right)^2 = 1$$

(b) This is a circle with centre $\left(-5 , 2 \right)$ and radius 1.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane

Exercise C, Question 5

Question:

A circle has parametric equations $x = 4 \sin t + 3$, $y = 4 \cos t - 1$. Find the radius and the coordinates of the centre of the circle.

Solution:

$x = 4 \sin t + 3$

$4 \sin t = x - 3$

$\sin t = \frac{x - 3}{4}$

and

$y = 4 \cos t - 1$

$4 \cos t = y + 1$

$\cos t = \frac{y + 1}{4}$

As $\sin^2 t + \cos^2 t = 1$

$$\left(\frac{x - 3}{4} \right)^2 + \left(\frac{y + 1}{4} \right)^2 = 1$$

$$\frac{(x - 3)^2}{4^2} + \frac{(y + 1)^2}{4^2} = 1$$

$$\frac{(x - 3)^2}{16} + \frac{(y + 1)^2}{16} = 1$$

$(x - 3)^2 + (y + 1)^2 = 16$ Multiply throughout by 16

So the centre of the circle is $(3, -1)$ and the radius is 4.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane

Exercise D, Question 1

Question:

The following curves are given parametrically. In each case, find an expression for $y\frac{dx}{dt}$ in terms of t.

(a) $x = t + 3, y = 4t - 3$

(b) $x = t^3 + 3t, y = t^2$

(c) $x = (2t - 3)^2, y = 1 - t^2$

(d) $x = 6 - \frac{1}{t}, y = 4t^3, t > 0$

(e) $x = \sqrt{t}, y = 6t^3, t \geq 0$

(f) $x = \frac{4}{t^2}, y = 5t^2, t < 0$

(g) $x = 5t^{\frac{1}{2}}, y = 4t^{-\frac{3}{2}}, t > 0$

(h) $x = t^{\frac{1}{3}} - 1, y = \sqrt{t}, t \geq 0$

(i) $x = 16 - t^4, y = 3 - \frac{2}{t}, t < 0$

(j) $x = 6t^{\frac{2}{3}}, y = t^2$

Solution:

(a) $x = t + 3, y = 4t - 3$

$$\frac{dx}{dt} = 1$$

$$\text{So } y\frac{dx}{dt} = \left(4t - 3\right) \times 1 = 4t - 3$$

(b) $x = t^3 + 3t, y = t^2$

$\frac{dx}{dt} = 3t^2 + 3$

So $y \frac{dx}{dt} = t^2 \left(3t^2 + 3 \right) = 3t^2 \left(t^2 + 1 \right)$ Factorise 3

(c) $x = \left(2t - 3 \right)^2, y = 1 - t^2$

$x = 4t^2 - 12t + 9$

$\frac{dx}{dt} = 8t - 12$

So $y \frac{dx}{dt} = \left(1 - t^2 \right) \left(8t - 12 \right) = 4 \left(1 - t^2 \right) \left(2t - 3 \right)$

Factorise 4

(d) $x = 6 - \frac{1}{t}, y = 4t^3$

$x = 6 - t^{-1}$

$\frac{dx}{dt} = t^{-2}$

So $y \frac{dx}{dt} = 4t^3 \times t^{-2} = 4t$

(e) $x = \sqrt{t}, y = 6t^3$

$x = t^{\frac{1}{2}}$

$\frac{dx}{dt} = \frac{1}{2} t^{-\frac{1}{2}}$

So $y \frac{dx}{dt} = 6t^3 \times \frac{1}{2} t^{-\frac{1}{2}} = 3t^{3 - \frac{1}{2}} = 3t^{\frac{5}{2}}$

(f) $x = \frac{4}{t^2}, y = 5t^2$

$x = 4t^{-2}$

$\frac{dx}{dt} = -8t^{-3}$

So $y \frac{dx}{dt} = 5t^2 \times -8t^{-3} = -40t^{2-3} = -40t^{-1} = -\frac{40}{t}$

(g) $x = 5t^{\frac{1}{2}}, y = 4t^{-\frac{3}{2}}$

$$\frac{dx}{dt} = 5 \times \frac{1}{2}t^{-\frac{1}{2}} = \frac{5}{2}t^{-\frac{1}{2}}$$

$$\text{So } y\frac{dx}{dt} = 4t^{-\frac{3}{2}} \times \frac{5}{2}t^{-\frac{1}{2}} = 10t^{-\frac{3}{2}-\frac{1}{2}} = 10t^{-2}$$

(h) $x = t^{\frac{1}{3}} - 1, y = \sqrt{t}$

$$\frac{dx}{dt} = \frac{1}{3}t^{\frac{1}{3}-1} = \frac{1}{3}t^{-\frac{2}{3}}$$

$$\text{So } y\frac{dx}{dt} = \sqrt{t} \times \frac{1}{3}t^{-\frac{2}{3}} = t^{\frac{1}{2}} \times \frac{1}{3}t^{-\frac{2}{3}} = \frac{1}{3}t^{\frac{1}{2}-\frac{2}{3}} = \frac{1}{3}t^{-\frac{1}{6}}$$

(i) $x = 16 - t^4, y = 3 - \frac{2}{t}$

$$\frac{dx}{dt} = -4t^3$$

$$\text{So } y\frac{dx}{dt} = \left(3 - \frac{2}{t}\right)\left(-4t^3\right)$$

$$= 3 \times \left(-4t^3\right) + \frac{2}{t} \times 4t^3$$

$$= -12t^3 + 8t^2 \quad [\text{or } 8t^2 - 12t^3 \text{ or } 4t^2 \ (2 - 3t)]$$

(j) $x = 6t^{\frac{2}{3}}, y = t^2$

$$\frac{dx}{dt} = 6 \times \frac{2}{3}t^{\frac{2}{3}-1} = 4t^{-\frac{1}{3}}$$

$$\text{So } y\frac{dx}{dt} = t^2 \times 4t^{-\frac{1}{3}} = 4t^{2-\frac{1}{3}} = 4t^{\frac{5}{3}}$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise D, Question 2

Question:

A curve has parametric equations $x = 2t - 5$, $y = 3t + 8$. Work out $\int_0^4 y \frac{dx}{dt} dt$.

Solution:

$$x = 2t - 5, y = 3t + 8$$

$$\frac{dx}{dt} = 2$$

$$\text{So } y \frac{dx}{dt} = \left(3t + 8 \right) \times 2 = 6t + 16$$

$$\int_0^4 y \frac{dx}{dt} dt = \int_0^4 6t + 16 \quad dt$$

$$= \left[3t^2 + 16t \right]_0^4$$

$$= \left[3 \left(4 \right)^2 + 16 \left(4 \right) \right] - \left[3 \left(0 \right)^2 + 16 \left(0 \right) \right]$$

$$= \left(3 \times 16 + 16 \times 4 \right) - 0$$

$$= 48 + 64$$

$$= 112$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise D, Question 3

Question:

A curve has parametric equations $x = t^2 - 3t + 1$, $y = 4t^2$. Work out $\int_{-1}^{5} y \frac{dx}{dt} dt$.

Solution:

$$x = t^2 - 3t + 1 , y = 4t^2$$

$$\frac{dx}{dt} = 2t - 3$$

$$\text{So } y \frac{dx}{dt} = 4t^2 \left(2t - 3 \right) = 8t^3 - 12t^2$$

$$\int_{-1}^{5} y \frac{dx}{dt} dt = \int_{-1}^{5} 8t^3 - 12t^2 \quad dt$$

$$= \left[2t^4 - 4t^3 \right]_{-1}^{5}$$

$$= \left[2 (5)^4 - 4 (5)^3 \right] - \left[2 (- 1)^4 - 4 (- 1)^3 \right]$$

$$= 750 - 6$$

$$= 744$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise D, Question 4

Question:

A curve has parametric equations $x = 3t^2, y = \frac{1}{t} + t^3, t > 0$. Work out $\int_{0.5}^{3} y \frac{dx}{dt} dt$.

Solution:

$$x = 3t^2, y = \frac{1}{t} + t^3$$

$$\frac{dx}{dt} = 6t$$

$$\text{So } y \frac{dx}{dt} = \left(\frac{1}{t} + t^3 \right) \times 6t = \frac{1}{t} \times 6t + t^3 \times 6t = 6 + 6t^4$$

$$\int_{0.5}^{3} y \frac{dx}{dt} dt = \int_{0.5}^{3} 6 + 6t^4 \quad dt$$

$$= \left[6t + \frac{6}{5} t^5 \right]_{0.5}^{3}$$

$$= \left[6 \left(3 \right) + \frac{6}{5} (3)^5 \right] - \left[6 \left(0.5 \right) + \frac{6}{5} (0.5)^5 \right]$$

$$= 309.6 - 3.0375$$

$$= 306.5625 \quad (\text{or } 306 \frac{9}{16})$$

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Coordinate geometry in the (x, y) plane
Exercise D, Question 5

Question:

A curve has parametric equations $x = t^3 - 4t$, $y = t^2 - 1$. Work out $\int_{-2}^{2} y \frac{dx}{dt} dt$.

Solution:

$$x = t^3 - 4t, y = t^2 - 1$$

$$\frac{dx}{dt} = 3t^2 - 4$$

$$\text{So } y \frac{dx}{dt} = \left(t^2 - 1 \right) \times \left(3t^2 - 4 \right) = 3t^4 - 4t^2 - 3t^2 + 4 = 3t^4 - 7t^2 + 4$$

$$\int_{-2}^{2} 3t^4 - 7t^2 + 4 \quad dt = \left[\frac{3}{5}t^5 - \frac{7}{3}t^3 + 4t \right]_{-2}^{2}$$

$$= \left[\frac{3}{5}(2)^5 - \frac{7}{3}(2)^3 + 4(2) \right] - \left[\frac{3}{5}(-2)^5 - \frac{7}{3}(-2)^3 + (-2) \right]$$

$$= 8\frac{8}{15} - \left(-8\frac{8}{15} \right)$$

$$= 17\frac{1}{15}$$

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Coordinate geometry in the (x, y) plane
Exercise D, Question 6

Question:

A curve has parametric equations $x = 9t^{\frac{4}{3}}, y = t^{-\frac{1}{3}}, t > 0$.

(a) Show that $y\frac{dx}{dt} = a$, where a is a constant to be found.

(b) Work out $\int_3^5 y\frac{dx}{dt}dt$.

Solution:

(a) $x = 9t^{\frac{4}{3}}, y = t^{-\frac{1}{3}}$

$$\frac{dx}{dt} = 9 \times \frac{4}{3}t^{\frac{4}{3}-1} = 9 \times \frac{4}{3}t^{\frac{1}{3}} = 12t^{\frac{1}{3}}$$

$$\text{So } y\frac{dx}{dt} = t^{-\frac{1}{3}} \times 12t^{\frac{1}{3}} = 12t^{-\frac{1}{3}+\frac{1}{3}} = 12t^0 = 12$$

So $a = 12$

(b) $\int_3^5 y\frac{dx}{dt}dt = \int_3^5 12 \quad dt = \left[12t \right]_3^5 = 12\left(5 \right) - 12\left(3 \right) = 24$

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Coordinate geometry in the (x, y) plane
Exercise D, Question 7

Question:

A curve has parametric equations $x = \sqrt{t}$, $y = 4\sqrt{t^3}$, $t > 0$.

(a) Show that $y \frac{dx}{dt} = pt$, where p is a constant to be found.

(b) Work out $\int_1^6 y \frac{dx}{dt} dt$.

Solution:

(a) $x = \sqrt{t}$, $y = 4\sqrt{t^3}$

$$x = t^{\frac{1}{2}}$$

$$\frac{dx}{dt} = \frac{1}{2} t^{\frac{1}{2} - 1} = \frac{1}{2} t^{-\frac{1}{2}}$$

$$y \frac{dx}{dt} = 4\sqrt{t^3} \times \frac{1}{2} t^{-\frac{1}{2}}$$

$$= 4t^{\frac{3}{2}} \times \frac{1}{2} t^{-\frac{1}{2}}$$

$$= 2t^{\frac{3}{2} - \frac{1}{2}}$$

$$= 2t^1$$

$$= 2t$$

So $p = 2$

(b) $\int_1^6 y \frac{dx}{dt} dt = \int_1^6 2t dt = \left[t^2 \right]_1^6 = (6)^2 - (1)^2 = 35$

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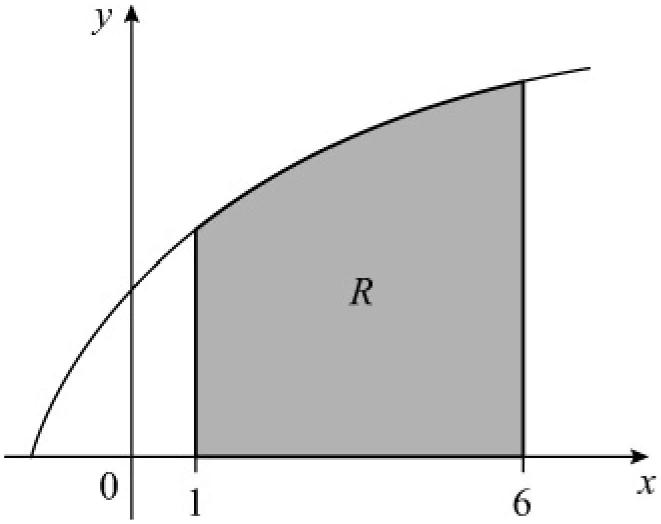
Coordinate geometry in the (x, y) plane
Exercise D, Question 8

Question:

The diagram shows a sketch of the curve with parametric equations $x = t^2 - 3$, $y = 3t$, $t > 0$. The shaded region R is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 6$.

(a) Find the value of t when
(i) $x = 1$
(ii) $x = 6$

(b) Find the area of R.



Solution:

(a) Substitute $x = 1$ into $x = t^2 - 3$

$t^2 - 3 = 1$

$t^2 = 4$

$t = 2 \quad$ (as $t > 0$)

Substitute $x = 6$ into $x = t^2 - 3$

$t^2 - 3 = 6$

$t^2 = 9$

$t = 3 \quad$ (as $t > 0$)

(b) $\int_1^6 y \mathrm{d}x = \int_2^3 y \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t$

$$\frac{dx}{dt} = 2t$$

So $y \frac{dx}{dt} = 3t \times 2t = 6t^2$

$$\int_2^3 y \frac{dx}{dt} dt = \int_2^3 6t^2 \quad dt$$

$$= \left[2t^3 \right]_2^3$$

$$= 2 (3)^3 - 2 (2)^3$$

$$= 54 - 16$$

$$= 38$$

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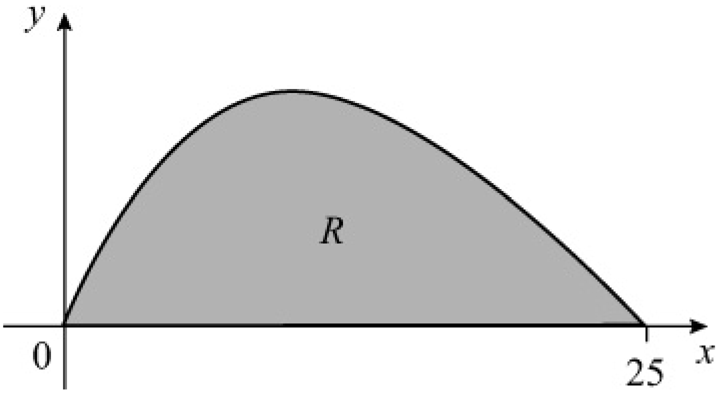
Coordinate geometry in the (x, y) plane
Exercise D, Question 9

Question:

The diagram shows a sketch of the curve with parametric equations $x = 4t^2$, $y = t(5 - 2t)$, $t \geq 0$. The shaded region R is bounded by the curve and the x-axis. Find the area of R.



Solution:

When $x = 0$

$4t^2 = 0$

$t^2 = 0$

$t = 0$

When $x = 25$

$4t^2 = 25$

$t^2 = \frac{25}{4}$

$t = \sqrt{\frac{25}{4}}$

$t = \frac{5}{2}$ (as $t \geq 0$)

So $\int_0^{25} y \mathrm{d}x = \int_0^{\frac{5}{2}} y \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 8t$

So $y \frac{\mathrm{d}x}{\mathrm{d}t} = t\left(5 - 2t\right) \times 8t = 8t^2\left(5 - 2t\right) = 40t^2 - 16t^3$

$$\int_0^{\frac{5}{2}} y \frac{dx}{dt} dt = \int_0^{\frac{5}{2}} 40t^2 - 16t^3 \quad dt$$

$$= \left[\frac{40}{3} t^3 - 4t^4 \right]_0^{\frac{5}{2}}$$

$$= \left[\frac{40}{3} \left(\frac{5}{2} \right)^3 - 4 \left(\frac{5}{2} \right)^4 \right] - \left[\frac{40}{3} (0)^3 - 4 (0)^4 \right]$$

$$= 52 \frac{1}{12} - 0$$

$$= 52 \frac{1}{12}$$

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Coordinate geometry in the (x, y) plane
Exercise D, Question 10

Question:

The region R is bounded by the curve with parametric equations $x = t^3$, $y = \frac{1}{3t^2}$, the x-axis and the lines $x = -1$ and $x = -8$.

(a) Find the value of t when
  (i) $x = -1$
  (ii) $x = -8$

(b) Find the area of R.

Solution:

(a) (i) Substitute $x = -1$ into $x = t^3$

$t^3 = -1$

$t = \sqrt[3]{-1}$

$t = -1$

(ii) Substitute $x = -8$ into $x = t^3$

$t^3 = -8$

$t = \sqrt[3]{-8}$

$t = -2$

(b) $R = \int_{-8}^{-1} y\,\mathrm{d}x = \int_{-2}^{-1} y \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$

So $y\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{3t^2} \times 3t^2 = 1$

$$\int_{-2}^{-1} y \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \int_{-2}^{-1} 1\,\mathrm{d}t = \left[t \right]_{-2}^{-1} = \left(-1 \right) - \left(-2 \right)$$

$= -1 + 2 = 1$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane

Exercise E, Question 1

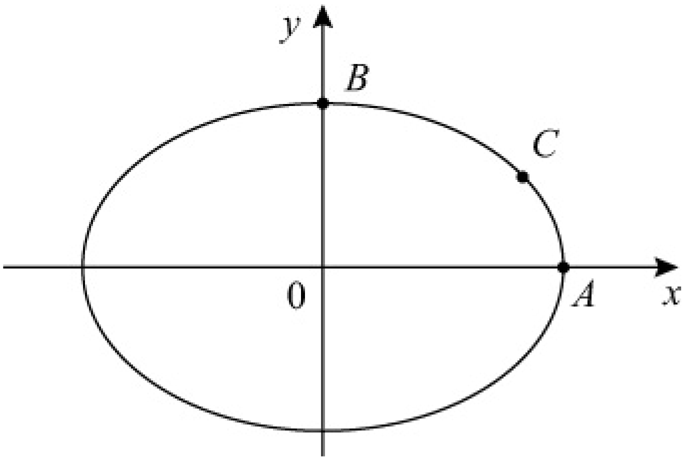
Question:

The diagram shows a sketch of the curve with parametric equations $x = 4 \cos t$, $y = 3 \sin t$, $0 \leq t < 2\pi$.

(a) Find the coordinates of the points A and B.

(b) The point C has parameter $t = \frac{\pi}{6}$. Find the exact coordinates of C.

(c) Find the cartesian equation of the curve.



Solution:

(a) (1) At A, $y = 0 \Rightarrow 3 \sin t = 0 \Rightarrow \sin t = 0$

So $t = 0$ and $t = \pi$

Substitute $t = 0$ and $t = \pi$ into $x = 4 \cos t$

$t = 0 \Rightarrow x = 4 \cos (0) = 4 \times 1 = 4$

$t = \pi \Rightarrow x = 4 \cos \pi = 4 \times (- 1) = - 4$

So the coordinates of A are (4, 0).

(2) At B, $x = 0 \Rightarrow 4 \cos t = 0 \Rightarrow \cos t = 0$

So $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$

Substitute $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ into $y = 3 \sin t$

$t = \frac{\pi}{2} \quad \Rightarrow \quad y = 3 \sin\left(\frac{\pi}{2}\right) = 3 \times 1 = 3$

$t = \frac{3\pi}{2} \quad \Rightarrow \quad y = 3 \sin\left(\frac{3\pi}{2}\right) = 3 \times -1 = -3$

So the coordinates of B are $(0, 3)$

(b) Substitute $t = \frac{\pi}{6}$ into $x = 4 \cos t$ and $y = 3 \sin t$

$x = 4 \cos\left(\frac{\pi}{6}\right) = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$y = 3 \sin\left(\frac{\pi}{6}\right) = 3 \times \frac{1}{2} = \frac{3}{2}$

So the coordinates of C are $\left(2\sqrt{3}, \frac{3}{2}\right)$

(c) $x = 4 \cos t, y = 3 \sin t$

$\cos t = \frac{x}{4}$ and $\sin t = \frac{y}{3}$

As $\sin^2 t + \cos^2 t = 1$

$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{4}\right)^2 = 1 \quad$ or $\quad \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

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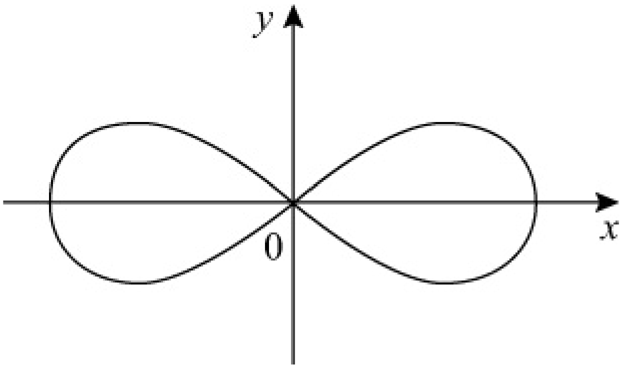
Coordinate geometry in the (x, y) plane
Exercise E, Question 2

Question:

The diagram shows a sketch of the curve with parametric equations $x = \cos t$, $y = \frac{1}{2} \sin 2t$.
$0 \leq t < 2\pi$. The curve is symmetrical about both axes.

(a) Copy the diagram and label the points having parameters $t = 0$, $t = \frac{\pi}{2}$, $t = \pi$ and $t = \frac{3\pi}{2}$.

(b) Show that the cartesian equation of the curve is $y^2 = x^2 (1 - x^2)$.



Solution:

(a) (1) Substitute $t = 0$ into $x = \cos t$ and $y = \frac{1}{2} \sin 2t$

$x = \cos 0 = 1$

$y = \frac{1}{2} \sin \left(2 \times 0 \right) = \frac{1}{2} \sin 0 = \frac{1}{2} \times 0 = 0$

So when $t = 0$, $(x , y) = (1 , 0)$

(2) Substitute $t = \frac{\pi}{2}$ into $x = \cos t$ and $y = \frac{1}{2} \sin 2t$

$x = \cos \frac{\pi}{2} = 0$

$y = \frac{1}{2} \sin \left(2 \times \frac{\pi}{2} \right) = \frac{1}{2} \sin \pi = \frac{1}{2} \times 0 = 0$

So when $t = \frac{\pi}{2}$, $(x , y) = (0 , 0)$

(3) Substitute $t = \pi$ into $x = \cos t$ and $y = \frac{1}{2} \sin 2t$

$x = \cos \pi = - 1$

$y = \frac{1}{2} \sin \left(2\pi \right) = \frac{1}{2} \times 0 = 0$

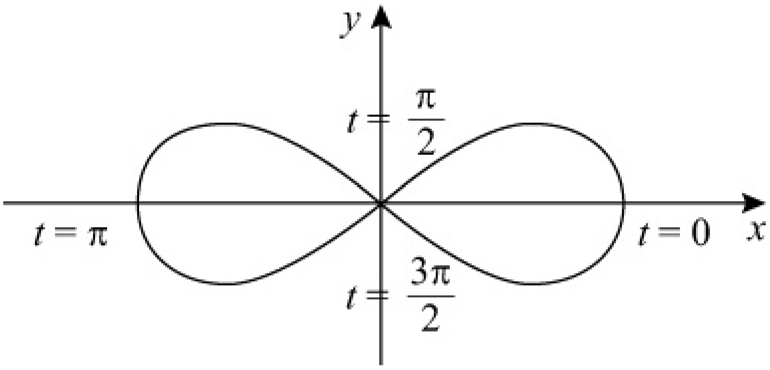
So when $t = \pi$, $(x , y) = (- 1 , 0)$

(4) Substitute $t = \frac{3\pi}{2}$ into $x = \cos t$ and $y = \frac{1}{2} \sin 2t$

$x = \cos \frac{3\pi}{2} = 0$

$y = \frac{1}{2} \sin \left(2 \times \frac{3\pi}{2} \right) = \frac{1}{2} \sin \left(3\pi \right) = \frac{1}{2} \times 0 = 0$

So when $t = \frac{3\pi}{2}$, $(x , y) = (0 , 0)$



(b) $y = \frac{1}{2} \sin 2t = \frac{1}{2} \times 2 \sin t \cos t = \sin t \cos t$

As $x = \cos t$

$y = \sin t \times x$

$y = x \sin t$

Now $\sin^2 t + \cos^2 t = 1$

So $\sin^2 t + x^2 = 1$

$\Rightarrow \sin^2 t = 1 - x^2$

$\Rightarrow \sin t = \sqrt{1 - x^2}$

So $y = x\sqrt{1 - x^2}$ or $y^2 = x^2 (1 - x^2)$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise E, Question 3

Question:

A curve has parametric equations $x = \sin t$, $y = \cos 2t$, $0 \leq t < 2\pi$.

(a) Find the cartesian equation of the curve.
The curve cuts the x-axis at $(a, 0)$ and $(b, 0)$.

(b) Find the value of a and b.

Solution:

(a) $x = \sin t$, $y = \cos 2t$
As $\cos 2t = 1 - 2 \sin^2 t$
$y = 1 - 2x^2$

(b) Substitute $y = 0$ into $y = 1 - 2x^2$
$0 = 1 - 2x^2$
$2x^2 = 1$
$x^2 = \frac{1}{2}$
$x = \pm \sqrt{\frac{1}{2}} = \pm \frac{\sqrt{1}}{\sqrt{2}} = \pm \frac{1}{\sqrt{2}}$
$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
So the curve meets the x-axis at $\left(\frac{\sqrt{2}}{2} , 0 \right)$ and $\left(- \frac{\sqrt{2}}{2} , 0 \right)$

Solutionbank

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Coordinate geometry in the (x, y) plane

Exercise E, Question 4

Question:

A curve has parametric equations $x = \frac{1}{1 + t}$, $y = \frac{1}{(1 + t) (1 - t)}$, $t \neq \pm 1$.

Express t in terms of x. Hence show that the cartesian equation of the curve is $y = \frac{x^2}{2x - 1}$.

Solution:

(1) $x = \frac{1}{1 + t}$

$x \times \left(1 + t \right) = \frac{1}{(1 + t)} \times \left(1 + t \right)$ Multiply each side by $(1 + t)$

$x (1 + t) = 1$ Simplify

$\frac{x (1 + t)}{x} = \frac{1}{x}$ Divide each side by x

$1 + t = \frac{1}{x}$ Simplify

So $t = \frac{1}{x} - 1$

Substitute $t = \frac{1}{x} - 1$ into $y = \frac{1}{(1 + t) (1 - t)}$

$$y = \frac{1}{(1 + \frac{1}{x} - 1) [1 - (\frac{1}{x} - 1)]}$$

$$= \frac{1}{\frac{1}{x} (1 - \frac{1}{x} + 1)}$$

$$= \frac{1}{\frac{1}{x} (2 - \frac{1}{x})}$$

$$= \frac{1}{\frac{1}{x}\left(\frac{2x}{x} - \frac{1}{x}\right)}$$

$$= \frac{1}{\frac{1}{x}\left(\frac{2x-1}{x}\right)}$$

$$= \frac{1}{\left(\frac{2x-1}{x^2}\right)}$$

$$= \frac{x^2}{2x-1} \qquad \left(\text{Remember } \frac{1}{\left(\frac{a}{b}\right)} = \frac{b}{a} \right)$$

So the cartesian equation of the curve is $y = \frac{x^2}{2x-1}$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise E, Question 5

Question:

A circle has parametric equations $x = 4 \sin t - 3$, $y = 4 \cos t + 5$.

(a) Find the cartesian equation of the circle.

(b) Draw a sketch of the circle.

(c) Find the exact coordinates of the points of intersection of the circle with the y-axis.

Solution:

(a) $x = 4 \sin t - 3$, $y = 4 \cos t + 5$

$$4 \sin t = x + 3$$

$$\sin t = \frac{x+3}{4}$$

and

$$4 \cos t = y - 5$$

$$\cos t = \frac{y-5}{4}$$

As $\sin^2 t + \cos^2 t = 1$

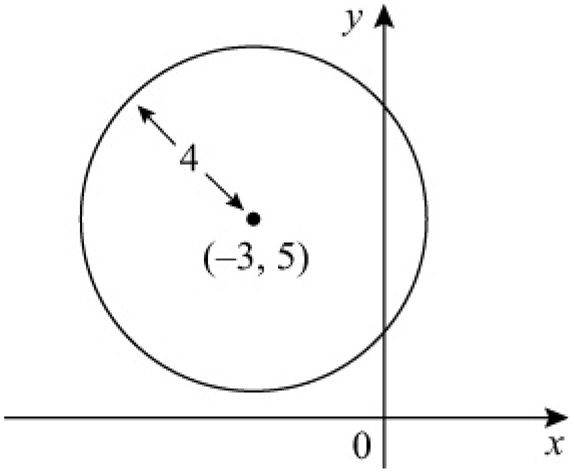
$$\left(\frac{x+3}{4} \right)^2 + \left(\frac{y-5}{4} \right)^2 = 1$$

$$\frac{(x+3)^2}{4^2} + \frac{(y-5)^2}{4^2} = 1$$

$$\frac{(x+3)^2}{4^2} \times 4^2 + \frac{(y-5)^2}{4^2} \times 4^2 = 1 \times 4^2$$

$$(x+3)^2 + (y-5)^2 = 4^2 \quad \text{or} \quad (x+3)^2 + (y-5)^2 = 16$$

(b) The circle $(x+3)^2 + (y-5)^2 = 4^2$ has centre $(-3, 5)$ and radius 4.



(c) Substitute $x = 0$ into $(x + 3)^2 + (y - 5)^2 = 4^2$

$(0 + 3)^2 + (y - 5)^2 = 4^2$

$3^2 + (y - 5)^2 = 4^2$

$9 + (y - 5)^2 = 16$

$(y - 5)^2 = 7$

$y - 5 = \pm \sqrt{7}$

$y = 5 \pm \sqrt{7}$

So the circle meets the y-axis at $(0 , 5 + \sqrt{7})$ and $(0 , 5 - \sqrt{7})$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise E, Question 6

Question:

Find the cartesian equation of the line with parametric equations $x = \frac{2-3t}{1+t}$, $y = \frac{3+2t}{1+t}$, $t \neq -1$.

Solution:

$$x = \frac{2-3t}{1+t}$$

$$x\left(1+t\right) = \frac{2-3t}{(1+t)} \times \left(1+t\right)$$

$$x(1+t) = 2-3t$$
$$x+xt = 2-3t$$
$$x+xt+3t = 2$$
$$xt+3t = 2-x$$
$$t(x+3) = 2-x$$

$$t\frac{(x+3)}{(x+3)} = \frac{2-x}{x+3}$$

$$t = \frac{2-x}{x+3}$$

Substitute $t = \frac{2-x}{x+3}$ into $y = \frac{3+2t}{1+t}$

$$y = \frac{3+2\left(\frac{2-x}{x+3}\right)}{1+\left(\frac{2-x}{x+3}\right)}$$

$$= \frac{3+2\left(\frac{2-x}{x+3}\right)}{1+\left(\frac{2-x}{x+3}\right)} \times \frac{(x+3)}{(x+3)}$$

$$= \frac{3 \times \left(x + 3 \right) + 2 \left(\frac{2 - x}{x + 3} \right) \times \left(x + 3 \right)}{1 \times \left(x + 3 \right) + \left(\frac{2 - x}{x + 3} \right) \times \left(x + 3 \right)}$$

$$= \frac{3 \left(x + 3 \right) + 2 \left(2 - x \right)}{\left(x + 3 \right) + \left(2 - x \right)}$$

$$= \frac{3x + 9 + 4 - 2x}{x + 3 + 2 - x}$$

$$= \frac{x + 13}{5}$$

So $y = \frac{x}{5} + \frac{13}{5}$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise E, Question 7

Question:

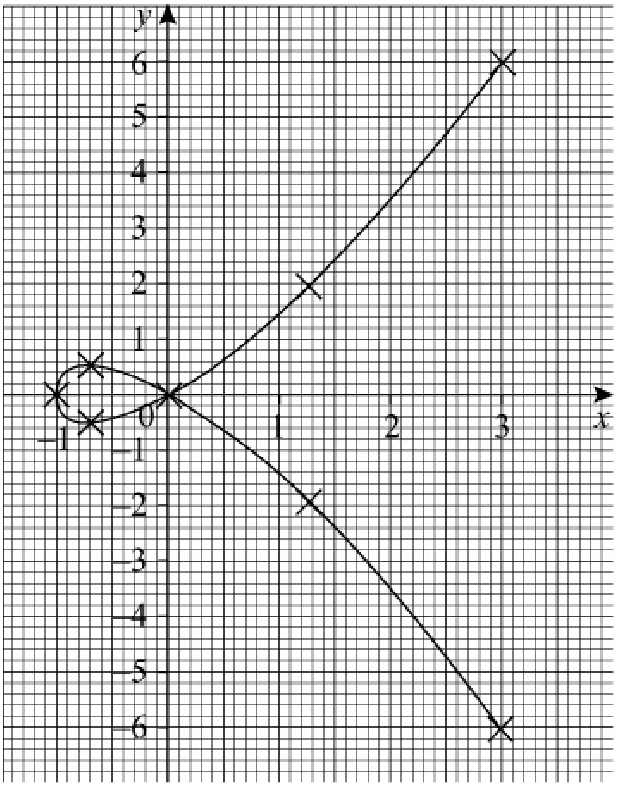
A curve has parametric equations $x = t^2 - 1$, $y = t - t^3$, where t is a parameter.

(a) Draw a graph of the curve for $-2 \leq t \leq 2$.

(b) Find the area of the finite region enclosed by the loop of the curve.

Solution:

t	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
$x = t^2 - 1$	3	1.25	0	−0.75	−1	−0.75	0	1.25	3
$y = t - t^3$	6	1.875	0	−0.375	0	0.375	0	−1.875	−6



(b) $A = 2\int_{-1}^{0} y\,dx = 2\int_{0}^{1} y\frac{dx}{dt}dt$, When $x = -1$, $t^2 - 1 = -1$, So $t = 0$

When $x = 0$, $t^2 - 1 = 0$, So $t = 1$

$$\frac{dx}{dt} = 2t$$

$$\text{So } y\frac{dx}{dt} = \left(t - t^3\right) \times 2t = 2t^2 - 2t^4$$

$$\text{Therefore } A = 2\int_{0}^{1} 2t^2 - 2t^4 dt$$

$$= 2\left[\frac{2}{3}t^3 - \frac{2}{5}t^5\right]_0^1$$

$$= 2\left(\left[\frac{2}{3}(1)^3 - \frac{2}{5}(1)^5\right] - \left[\frac{2}{3}(0)^3 - \frac{2}{5}(0)^5\right]\right)$$

$$= 2\left[\left(\frac{2}{3} - \frac{2}{5}\right) - 0\right]$$

$$= 2 \times \frac{4}{15}$$

$$= \frac{8}{15}$$

So the area of the loop is $\frac{8}{15}$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise E, Question 8

Question:

A curve has parametric equations $x = t^2 - 2$, $y = 2t$, where $-2 \leq t \leq 2$.

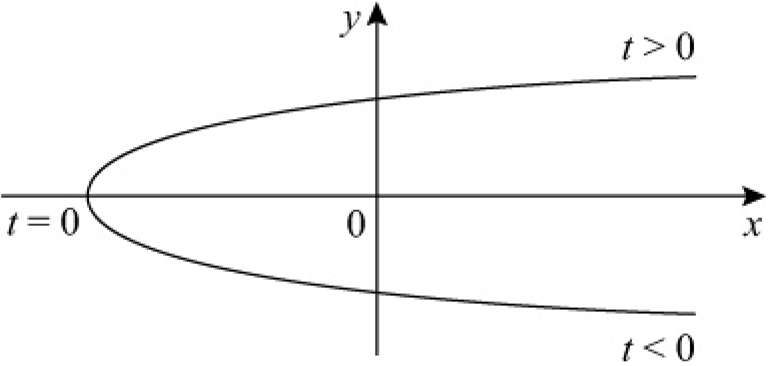
(a) Draw a graph of the curve.

(b) Indicate on your graph where
  (i) $t = 0$
  (ii) $t > 0$
  (iii) $t < 0$

(c) Calculate the area of the finite region enclosed by the curve and the y-axis.

Solution:

(a)



(b) (i) When $t = 0$, $y = 2(0) = 0$.
This is where the curve meets the x-axis.
(ii) When $t > 0$, $y > 0$.
This is where the curve is above the x-axis.
(iii) When $t < 0$, $y < 0$.
This is where the curve is below the x-axis.

(c) $A = 2\int_{-2}^{0} y\,dx = 2\int_{0}^{\sqrt{2}} y\frac{dx}{dt}dt$, When $x = -2$, $t^2 - 2 = -2$, so $t = 0$

When $x = 0$, $t^2 - 2 = 0$, so $t = \sqrt{2}$

$$\frac{dx}{dt} = 2t$$

So $y \frac{dx}{dt} = 2t \times 2t = 4t^2$

Therefore $A = 2 \int_0^{\sqrt{2}} 4t^2 dt$

$$= 2 \left[\frac{4}{3} t^3 \right]_0^{\sqrt{2}}$$

$$= 2 \left[\frac{4}{3} (\sqrt{2})^3 - \frac{4}{3} (0)^3 \right]$$

$$= 2 \times \frac{4}{3} (\sqrt{2})^3$$

$$= \frac{8}{3} (\sqrt{2})^3$$

$$= \frac{16}{3} \sqrt{2} , \qquad \text{As } (\sqrt{2})^3 = (\sqrt{2} \times \sqrt{2}) \times \sqrt{2} = 2 \sqrt{2}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Coordinate geometry in the (x, y) plane
Exercise E, Question 9

Question:

Find the area of the finite region bounded by the curve with parametric equations $x = t^3$, $y = \frac{4}{t}$, $t \neq 0$, the x-axis and the lines $x = 1$ and $x = 8$.

Solution:

(1) When $x = 1$, $t^3 = 1$, so $t = \sqrt[3]{1} = 1$
When $x = 8$, $t^3 = 8$, so $t = \sqrt[3]{8} = 2$

(2) $A = \int_1^8 y\mathrm{d}x = \int_1^2 y \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t$

(3) $\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$

So $y \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{4}{t} \times 3t^2 = 12t$

Therefore $A = \int_1^2 12t\mathrm{d}t$
$= [\, 6t^2 \,]_1^2$
$= 6(2)^2 - 6(1)^2$
$= 24 - 6$
$= 18$

Solutionbank
Edexcel AS and A Level Modular Mathematics

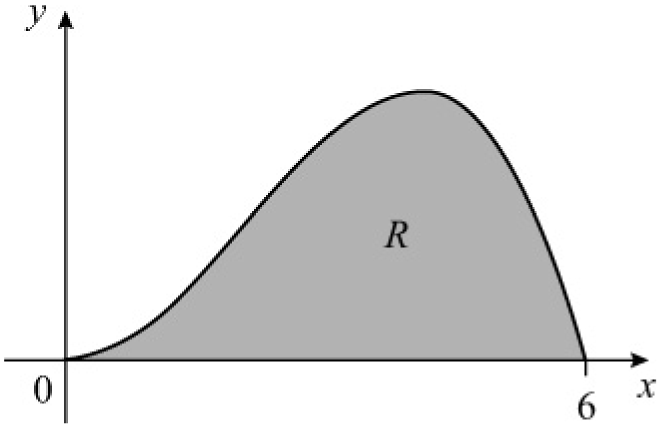
Coordinate geometry in the (x, y) plane
Exercise E, Question 10

Question:

The diagram shows a sketch of the curve with parametric equations $x = 3\sqrt{t}$, $y = t(4 - t)$, where $0 \le t \le 4$. The region R is bounded by the curve and the x-axis.

(a) Show that $y\frac{\mathrm{d}x}{\mathrm{d}t} = 6t^{\frac{1}{2}} - \frac{3}{2}t^{\frac{3}{2}}$.

(b) Find the area of R.



Solution:

(a) $x = 3\sqrt{t} = 3t^{\frac{1}{2}}$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2} \times 3t^{\frac{1}{2} - 1} = \frac{3}{2}t^{-\frac{1}{2}}$$

$$y\frac{\mathrm{d}x}{\mathrm{d}t} = t\left(4 - t\right) \times \frac{3}{2}t^{-\frac{1}{2}}$$

$$= \left(4t - t^2\right) \times \frac{3}{2}t^{-\frac{1}{2}}$$

$$= 4t \times \frac{3}{2}t^{-\frac{1}{2}} - t^2 \times \frac{3}{2}t^{-\frac{1}{2}}$$

$$= 6t^{1 - \frac{1}{2}} - \frac{3}{2}t^{2 - \frac{1}{2}}$$

$$= 6t^{\frac{1}{2}} - \frac{3}{2}t^{\frac{3}{2}}$$

(b) $A = \int_0^4 y \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t$

$$= \int_0^4 6t^{\frac{1}{2}} - \frac{3}{2}t^{\frac{3}{2}} \mathrm{d}t$$

$$= \left[\frac{6t^{\frac{3}{2}}}{\left(\frac{3}{2} \right)} - \frac{\frac{3}{2}t^{\frac{5}{2}}}{\left(\frac{5}{2} \right)} \right]_0^4$$

$$= \left[4t^{\frac{3}{2}} - \frac{3}{5}t^{\frac{5}{2}} \right]_0^4$$

$$= \left[4(4)^{\frac{3}{2}} - \frac{3}{5}(4)^{\frac{5}{2}} \right] - \left[4(0)^{\frac{3}{2}} - \frac{3}{5}(0)^{\frac{5}{2}} \right]$$

$$= \left(4 \times 8 - \frac{3}{5} \times 32 \right) - 0$$

$$= 32 - 19\frac{1}{5}$$

$$= 12\frac{4}{5}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

The binomial expansion

Exercise A, Question 1

Question:

Find the binomial expansion of the following up to and including the terms in x^3. State the range values of x for which these expansions are valid.

(a) $(1 + 2x)^{3}$

(b) $\frac{1}{1 - x}$

(c) $\sqrt{(1 + x)}$

(d) $\frac{1}{(1 + 2x)^{3}}$

(e) $\sqrt[3]{(1 - 3x)}$

(f) $(1 - 10x)^{\frac{3}{2}}$

(g) $\left(1 + \frac{x}{4} \right)^{- 4}$

(h) $\frac{1}{(1 + 2x^2)}$

Solution:

(a) $(1 + 2x)^{3}$ Use expansion with $n = 3$ and x replaced with $2x$

$$= 1 + 3 \left(2x \right) + \frac{3 \times 2 \times (2x)^{2}}{2 !} + \frac{3 \times 2 \times 1 \times (2x)^{3}}{3 !} + \frac{3 \times 2 \times 1 \times 0 \times (2x)^{4}}{4 !} + \ldots$$

$= 1 + 6x + 12x^2 + 8x^3 + 0x^4$ All terms after $0x^4$ will also be zero

$= 1 + 6x + 12x^2 + 8x^3$

Expansion is finite and exact. Valid for all values of x.

(b) $\frac{1}{1-x}$ Write in index form

$= (1-x)^{-1}$ Use expansion with $n = -1$ and x replaced with $-x$

$= 1 + \left(-1\right)\left(-x\right) + \frac{(-1)(-2)(-x)^2}{2!} +$

$\frac{(-1)(-2)(-3)(-x)^3}{3!} + \ldots$

$= 1 + 1x + 1x^2 + 1x^3 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots$

Expansion is infinite. Valid when $|-x| < 1 \Rightarrow |x| < 1$.

(c) $\sqrt{1+x}$ Write in index form

$= (1+x)^{\frac{1}{2}}$ Use expansion with $n = \frac{1}{2}$ and x replaced with x

$= 1 + \left(\frac{1}{2}\right)\left(x\right) + \frac{(\frac{1}{2})(-\frac{1}{2})(x)^2}{2!} +$

$\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(x)^3}{3!} + \ldots$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$

Expansion is infinite. Valid when $|x| < 1$.

(d) $\frac{1}{(1+2x)^3}$ Write in index form

$= (1+2x)^{-3}$ Use expansion with $n = -3$ and x replaced with $2x$

$= 1 + \left(-3\right)\left(2x\right) + \frac{(-3)(-4)(2x)^2}{2!} +$

$\frac{(-3)(-4)(-5)(2x)^3}{3!} + \ldots$

$= 1 - 6x + 24x^2 - 80x^3 + \ldots$

Expansion is infinite. Valid when $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$.

(e) $\sqrt[3]{(1-3x)}$ Write in index form

$= (1-3x)^{\frac{1}{3}}$ Use expansion with $n = \frac{1}{3}$ and x replaced with $-3x$

$$= 1 + \left(\frac{1}{3}\right)\left(-3x\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-3x\right)^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-3x\right)^3}{3!} + \ldots$$

$$= 1 - x - x^2 - \frac{5}{3}x^3 + \ldots$$

Expansion is infinite. Valid when $|-3x| < 1 \Rightarrow |x| < \frac{1}{3}$.

(f) $(1-10x)^{\frac{3}{2}}$ Use expansion with $n = \frac{3}{2}$ and x replaced with $-10x$

$$= 1 + \left(\frac{3}{2}\right)\left(-10x\right) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-10x\right)^2}{2!} + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-10x\right)^3}{3!} + \ldots$$

$$= 1 - 15x + \frac{3}{8} \times 100x^2 - \frac{1}{16} \times \left(-1000x^3\right) + \ldots$$

$$= 1 - 15x + \frac{75}{2}x^2 + \frac{125}{2}x^3 + \ldots$$

Expansion is infinite. Valid when $|-10x| < 1 \Rightarrow |x| < \frac{1}{10}$.

(g) $\left(1 + \frac{x}{4} \right)^{-4}$ Use expansion with $n = -4$ and x replaced with $\frac{x}{4}$

$$= 1 + \left(-4 \right) \left(\frac{x}{4} \right) + \frac{(-4)(-5)}{2!} \left(\frac{x}{4} \right)^2 + \frac{(-4)(-5)(-6)}{3!} \left(\frac{x}{4} \right)^3 + \ldots$$

$$= 1 - x + 10 \times \frac{x^2}{16} - 20 \times \frac{x^3}{64} + \ldots$$

$$= 1 - x + \frac{5}{8}x^2 - \frac{5}{16}x^3 + \ldots$$

Expansion is infinite. Valid when $\left| \frac{x}{4} \right| < 1 \Rightarrow |x| < 4$.

(h) $\frac{1}{1 + 2x^2}$ Write in index form

$= (1 + 2x^2)^{-1}$ Use expansion with $n = -1$ and x replaced with $2x^2$

$$= 1 + \left(-1 \right) \left(2x^2 \right) + \frac{(-1)(-2)(2x^2)^2}{2!} + \ldots$$

$$= 1 - 2x^2 + \ldots$$

Expansion is infinite. Valid when $|2x^2| < 1 \Rightarrow |x| < \frac{1}{\sqrt{2}}$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

The binomial expansion

Exercise A, Question 2

Question:

By first writing $\frac{(1 + x)}{(1 - 2x)}$ as $(1 + x) (1 - 2x)^{-1}$ show that the cubic approximation to $\frac{(1 + x)}{(1 - 2x)}$ is $1 + 3x + 6x^2 + 12x^3$. State the range of values of x for which this expansion is valid.

Solution:

$\frac{1 + x}{1 - 2x} = \left(1 + x \right) (1 - 2x)^{-1}$ Expand $(1 - 2x)^{-1}$ using binomial expansion

$$= \left(1 + x \right) \left[1 + \left(-1 \right) \left(-2x \right) + \frac{(-1) (-2) (-2x)^2}{2 !} + \frac{(-1) (-2) (-3) (-2x)^3}{3 !} + \ldots \right]$$

$= (1 + x) (1 + 2x + 4x^2 + 8x^3 + \ldots)$ Multiply out

$= 1 + 2x + 4x^2 + 8x^3 + \ldots + x + 2x^2 + 4x^3 + 8x^4 + \ldots$ Add like terms

$= 1 + 3x + 6x^2 + 12x^3 + \ldots$

$(1 - 2x)^{-1}$ is only valid when $| -2x | < 1 \Rightarrow | x | < \frac{1}{2}$

So expansion of $\frac{1 + x}{1 - 2x}$ is only valid when $| x | < \frac{1}{2}$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

The binomial expansion

Exercise A, Question 3

Question:

Find the binomial expansion of $\sqrt{(1+3x)}$ in ascending powers of x up to and including the term in x^3. By substituting $x = 0.01$ in the expansion, find an approximation to $\sqrt{103}$. By comparing it with the exact value, comment on the accuracy of your approximation.

Solution:

$$\sqrt{(1+3x)} = (1+3x)^{\frac{1}{2}}$$

$$= 1 + \left(\frac{1}{2}\right)\left(3x\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(3x)^2}{2\,!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(3x)^3}{3\,!} + \ldots$$

$$= 1 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{27}{16}x^3 + \ldots$$

This expansion is valid if $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

Substitute $x = 0.01$ (OK, as $|x| < \frac{1}{3}$) into both sides to give

$$\sqrt{1 + 3 \times 0.01} \simeq 1 + \frac{3}{2} \times 0.01 - \frac{9}{8} \times 0.01^2 + \frac{27}{16} \times 0.01^3$$

$$\sqrt{1.03} \simeq 1 + 0.015 - 0.0001125 + 0.0000016875$$

$$\sqrt{\frac{103}{100}} \simeq 1.014889188 \qquad \left(\sqrt{\frac{103}{100}} = \frac{\sqrt{103}}{\sqrt{100}} = \frac{\sqrt{103}}{10}\right)$$

$$\frac{\sqrt{103}}{10} \simeq 1.014889188 \qquad (\times 10)$$

$$\sqrt{103} \simeq 10.14889188$$

Using a calculator

$\sqrt{103} = 10.14889157$

Hence approximation correct to 6 d.p.

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The binomial expansion
Exercise A, Question 4

Question:

In the expansion of $(1 + ax) ^{- \frac{1}{2}}$ the coefficient of x^2 is 24. Find possible values of the constant a and the corresponding term in x^3.

Solution:

$$(1 + ax) ^{- \frac{1}{2}} = 1 + \left(- \frac{1}{2} \right) \left(ax \right) + \frac{(- \frac{1}{2}) (- \frac{3}{2}) (ax) ^2}{2 !} + \frac{(- \frac{1}{2}) (- \frac{3}{2}) (- \frac{5}{2}) (ax) ^3}{3 !} + \ldots$$

$$= 1 - \frac{1}{2}ax + \frac{3}{8}a^2x^2 - \frac{5}{16}a^3x^3 + \ldots$$

This expansion is valid if $| ax | < 1 \Rightarrow | x | < \frac{1}{a}$.

If coefficient of x^2 is 24 then

$\frac{3}{8}a^2 = 24$

$a^2 = 64$

$a = \pm 8$

Term in x^3 is

$- \frac{5}{16}a^3x^3 = - \frac{5}{16} (\pm 8) ^3x^3 = \pm 160x^3$

If $a = 8$, term in x^3 is $- 160x^3$

If $a = - 8$, term in x^3 is $+ 160x^3$

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The binomial expansion

Exercise A, Question 5

Question:

Show that if x is small, the expression $\sqrt{\left(\frac{1+x}{1-x}\right)}$ is approximated by $1 + x + \frac{1}{2}x^2$.

Solution:

$$\sqrt{\frac{1+x}{1-x}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

$$= (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$ Expand using the binomial expansion

$$= \left[1 + \left(\frac{1}{2}\right)(x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x)^2}{2!} + \ldots\right]\left[1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!} + \ldots\right]$$

$$= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots\right)$$

$$= 1\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots\right) + \frac{1}{2}x\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots\right) - \frac{1}{8}x^2\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2 + \ldots$$ Add like terms

$$= 1 + x + \frac{1}{2}x^2 + \ldots$$

Hence $\sqrt{\frac{1+x}{1-x}} \simeq 1 + x + \frac{1}{2}x^2$

If terms larger than or equal to x^3 are ignored.

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The binomial expansion
Exercise A, Question 6

Question:

Find the first four terms in the expansion of $(1 - 3x) ^{\frac{3}{2}}$. By substituting in a suitable value of x, find an approximation to $97 ^{\frac{3}{2}}$.

Solution:

$$(1 - 3x) ^{\frac{3}{2}} = 1 + \left(\frac{3}{2} \right) (- 3x) + \frac{\left(\frac{3}{2} \right) \left(\frac{1}{2} \right) (- 3x) ^{2}}{2 !} + \frac{\left(\frac{3}{2} \right) \left(\frac{1}{2} \right) \left(- \frac{1}{2} \right) (- 3x) ^{3}}{3 !} + \dots$$

$$= 1 - \frac{9x}{2} + \frac{27x^{2}}{8} + \frac{27x^{3}}{16} + \dots$$

Expansion is valid if $| - 3x | < 1 \Rightarrow | x | < \frac{1}{3}$.

Substitute $x = 0.01$ into both sides of expansion to give

$$(1 - 3 \times 0.01) ^{\frac{3}{2}} = 1 - \frac{9 \times 0.01}{2} + \frac{27 \times (0.01) ^{2}}{8} + \frac{27 \times (0.01) ^{3}}{16} + \dots$$

$$(0.97) ^{\frac{3}{2}} \simeq 1 - 0.045 + 0.0003375 + 0.000001687$$

$$(0.97) ^{\frac{3}{2}} \simeq 0.955339187$$

$$\left(\frac{97}{100} \right) ^{\frac{3}{2}} \simeq 0.955339187 , \qquad \left[\left(\frac{97}{100} \right) ^{\frac{3}{2}} = \frac{97 ^{\frac{3}{2}}}{100 ^{\frac{3}{2}}} = \frac{97 ^{\frac{3}{2}}}{(\sqrt{100}) ^{3}} = \frac{97 ^{\frac{3}{2}}}{1000} \right]$$

$$\frac{97 ^{\frac{3}{2}}}{1000} \simeq 0.955339187 \qquad \left(\times 1000 \right)$$

$$97 ^{\frac{3}{2}} \simeq 955.339187$$

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The binomial expansion
Exercise B, Question 1

Question:

Find the binomial expansions of the following in ascending powers of x as far as the term in x^3. State the range of values of x for which the expansions are valid.

(a) $\sqrt{(4 + 2x)}$

(b) $\frac{1}{2 + x}$

(c) $\frac{1}{(4 - x) ^{2}}$

(d) $\sqrt{(9 + x)}$

(e) $\frac{1}{\sqrt{(2 + x)}}$

(f) $\frac{5}{3 + 2x}$

(g) $\frac{1 + x}{2 + x}$

(h) $\sqrt{\left(\frac{2 + x}{1 - x} \right)}$

Solution:

(a) $\sqrt{(4 + 2x)}$ Write in index form.

$= (4 + 2x) ^{\frac{1}{2}}$ Take out a factor of 4

$= \left[4 \left(1 + \frac{2x}{4} \right) \right] ^{\frac{1}{2}}$ Remember to put the 4 to the power $\frac{1}{2}$

$= 4 ^{\frac{1}{2}} \left(1 + \frac{x}{2} \right) ^{\frac{1}{2}}$ $4 ^{\frac{1}{2}} = 2$

$= 2 \left(1 + \frac{x}{2} \right) ^{\frac{1}{2}}$ Use the binomial expansion with $n = \frac{1}{2}$ and $x = \frac{x}{2}$

$$= 2 \left[1 + \left(\frac{1}{2} \right) \left(\frac{x}{2} \right) + \frac{(\frac{1}{2}) (- \frac{1}{2}) (\frac{x}{2})^{2}}{2 !} + \right.$$

$$\left. \frac{(\frac{1}{2}) (- \frac{1}{2}) (- \frac{3}{2}) (\frac{x}{2})^{3}}{3 !} + \ldots \right]$$

$= 2 \left(1 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{128} + \ldots \right)$ Multiply by the 2

$= 2 + \frac{x}{2} - \frac{x^2}{16} + \frac{x^3}{64}$

Valid if $\left| \frac{x}{2} \right| < 1 \Rightarrow | x | < 2$

(b) $\frac{1}{2 + x}$ Write in index form

$= (2 + x)^{- 1}$ Take out a factor of 2

$= \left[2 \left(1 + \frac{x}{2} \right) \right]^{- 1}$ Remember to put 2 to the power $- 1$

$= 2^{- 1} \left(1 + \frac{x}{2} \right)^{- 1}$, $2^{- 1} = \frac{1}{2}$. Use the binomial expansion with $n = - 1$ and $x = \frac{x}{2}$

$$= \frac{1}{2} \left[1 + \left(- 1 \right) \left(\frac{x}{2} \right) + \frac{(- 1) (- 2)}{2 !} \left(\frac{x}{2} \right) 2 + \right.$$

$$\left. \frac{(- 1) (- 2) (- 3)}{3 !} \left(\frac{x}{2} \right) 3 + \ldots \right]$$

$= \frac{1}{2}\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots\right)$ Multiply by the $\frac{1}{2}$

$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$

Valid if $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(c) $\frac{1}{(4 - x)^2}$ Write in index form

$= (4 - x)^{-2}$ Take 4 out as a factor

$= \left[4\left(1 - \frac{x}{4}\right)\right]^{-2}$

$= 4^{-2}\left(1 - \frac{x}{4}\right)^{-2}$, $4^{-2} = \frac{1}{16}$. Use the binomial expansion with $n = -2$ and $x = \frac{x}{4}$

$$= \frac{1}{16}\left[1 + \left(-2\right)\left(-\frac{x}{4}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{x}{4}\right)^3 + \ldots\right]$$

$= \frac{1}{16}\left(1 + \frac{x}{2} + \frac{3x^2}{16} + \frac{x^3}{16} + \ldots\right)$ Multiply by $\frac{1}{16}$

$= \frac{1}{16} + \frac{x}{32} + \frac{3x^2}{256} + \frac{x^3}{256}$

Valid for $\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$

(d) $\sqrt{9 + x}$ Write in index form

$= (9 + x)^{\frac{1}{2}}$ Take 9 out as a factor

$= \left[9\left(1 + \frac{x}{9}\right)\right]^{\frac{1}{2}}$

$= 9^{\frac{1}{2}}\left(1 + \frac{x}{9}\right)^{\frac{1}{2}}$, $9^{\frac{1}{2}} = 3$. Use binomial expansion with $n = \frac{1}{2}$ and $x = \frac{x}{9}$

$$= 3\left[1 + \left(\frac{1}{2}\right)\left(\frac{x}{9}\right) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\left(\frac{x}{9}\right)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}\left(\frac{x}{9}\right)^3 + \ldots\right]$$

$$= 3\left(1 + \frac{x}{18} - \frac{x^2}{648} + \frac{x^3}{11664} + \ldots\right)$$ Multiply by 3

$$= 3 + \frac{x}{6} - \frac{x^2}{216} + \frac{x^3}{3888}$$

Valid for $\left|\frac{x}{9}\right| < 1 \Rightarrow |x| < 9$

(e) $\frac{1}{\sqrt{2+x}}$ Write in index form

$= (2+x)^{-\frac{1}{2}}$ Take out a factor of 2

$= \left[2\left(1+\frac{x}{2}\right)\right]^{-\frac{1}{2}}$

$= 2^{-\frac{1}{2}}\left(1+\frac{x}{2}\right)^{-\frac{1}{2}}$, $2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$. Use binomial expansion with $n = -\frac{1}{2}$ and $x = \frac{x}{2}$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\frac{x}{2}\right)^2 + \right.$$

$$\left.\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3\,!}\left(\frac{x}{2}\right)^{3}+\quad\ldots\right]$$

$$=\frac{1}{\sqrt{2}}\left(1-\frac{x}{4}+\frac{3x^{2}}{32}-\frac{5x^{3}}{128}+\quad\ldots\right)$$ Multiply by $\frac{1}{\sqrt{2}}$

$$=\frac{1}{\sqrt{2}}-\frac{x}{4\sqrt{2}}+\frac{3x^{2}}{32\sqrt{2}}-\frac{5x^{3}}{128\sqrt{2}}+\quad\ldots$$ Rationalise surds

$$=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}x}{8}+\frac{3\sqrt{2}x^{2}}{64}-\frac{52x^{3}}{256}$$

Valid if $\left|\frac{x}{2}\right|<1\Rightarrow|x|<2$

(f) $\frac{5}{3+2x}$ Write in index form

$=5(3+2x)^{-1}$ Take out a factor of 3

$$=5\left[3\left(1+\frac{2x}{3}\right)\right]^{-1}$$

$=5\times3^{-1}\left(1+\frac{2x}{3}\right)^{-1}$, $3^{-1}=\frac{1}{3}$. Use binomial expansion with $n=-1$ and $x=\frac{2x}{3}$

$$=\frac{5}{3}\left[1+\left(-1\right)\left(\frac{2x}{3}\right)+\frac{(-1)(-2)}{2\,!}\left(\frac{2x}{3}\right)^{2}+\right.$$

$$\left.\frac{(-1)(-2)(-3)}{3\,!}\left(\frac{2x}{3}\right)^{3}+\quad\ldots\right]$$

$$=\frac{5}{3}\left(1-\frac{2x}{3}+\frac{4x^{2}}{9}-\frac{8x^{3}}{27}+\quad\ldots\right)$$ Multiply by $\frac{5}{3}$

$$=\frac{5}{3}-\frac{10x}{9}+\frac{20x^{2}}{27}-\frac{40x^{3}}{81}$$

Valid if $\left|\frac{2x}{3}\right|<1\Rightarrow|x|<\frac{3}{2}$

(g) $\frac{1+x}{2+x}$ Write $\frac{1}{2+x}$ in index form

$= (1+x)(2+x)^{-1}$ Take out a factor of 2

$= \left(1+x\right)\left[2\left(1+\frac{x}{2}\right)\right]^{-1}$

$= \left(1+x\right)2^{-1}\left(1+\frac{x}{2}\right)^{-1}$ Expand $\left(1+\frac{x}{2}\right)^{-1}$ using the binomial expansion

$= \left(1+x\right)\frac{1}{2}\left[1+\left(-1\right)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right]$

$= \left(1+x\right)\frac{1}{2}\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)$ Multiply $\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)$ by $\frac{1}{2}$

$= \left(1+x\right)\left(\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots\right)$ Multiply your answer by $(1+x)$

$= 1\left(\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots\right)+x\left(\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots\right)$

$= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{8}+\ldots$ Collect like terms

$= \frac{1}{2}+\frac{1}{4}x-\frac{1}{8}x^2+\frac{1}{16}x^3$

Valid if $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(h) $\sqrt{\frac{2+x}{1-x}}$

$= (2+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ Put both in index form

$= 2^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ Expand both using the binomial expansion

$$= \sqrt{2}\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\left(\frac{x}{2}\right)^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}\left(\frac{x}{2}\right)^3+\ldots\right]\left[1+\left(-\frac{1}{2}\right)\left(-x\right)+\frac{(-\frac{1}{2})(-\frac{3}{2})(-x)^2}{2!}+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-x)^3}{3!}+\ldots\right]$$

$$= \sqrt{2}\left(1+\frac{1}{4}x-\frac{1}{32}x^2+\frac{1}{128}x^3+\ldots\right)\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots\right)$$ Multiply out

$$= \sqrt{2}\left[1\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots\right)+\frac{1}{4}x\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots\right)-\frac{1}{32}x^2\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots\right)+\frac{1}{128}x^3\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots\right)\ldots\right]$$

$$= \sqrt{2}\left(1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{1}{4}x+\frac{1}{8}x^2+\frac{3}{32}x^3-\frac{1}{32}x^2-\frac{1}{64}x^3+\frac{1}{128}x^3+\ldots\right)$$ Collect like terms

$$= \sqrt{2}\left(1+\frac{3}{4}x+\frac{15}{32}x^2+\frac{51}{128}x^3+\ldots\right)$$ Multiply by $\sqrt{2}$

$$= \sqrt{2}+\frac{3\sqrt{2}}{4}x+\frac{15\sqrt{2}}{32}x^2+\frac{51\sqrt{2}}{128}x^3$$

Valid if $\left|\frac{x}{2}\right| < 1$ and $|-x| < 1 \Rightarrow |x| < 1$ for both to be valid

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The binomial expansion
Exercise B, Question 2

Question:

Prove that if x is sufficiently small, $\frac{3+2x-x^2}{4-x}$ may be approximated by $\frac{3}{4}+\frac{11}{16}x-\frac{5}{64}x^2$. What does 'sufficiently small' mean in this question?

Solution:

$\frac{3+2x-x^2}{4-x} \equiv \left(3+2x-x^2\right)\left(4-x\right)^{-1}$ Write $\frac{1}{4-x}$ as $\left(4-x\right)^{-1}$

$=\left(3+2x-x^2\right)\left[4\left(1-\frac{x}{4}\right)\right]^{-1}$ Take out a factor of 4

$=\left(3+2x-x^2\right)\frac{1}{4}\left(1-\frac{x}{4}\right)^{-1}$ Expand $\left(1-\frac{x}{4}\right)^{-1}$ using the binomial expansion

$=\left(3+2x-x^2\right)\frac{1}{4}\left[1+\left(-1\right)\left(-\frac{x}{4}\right)+\frac{\left(-1\right)\left(-2\right)}{2!}\left(-\frac{x}{4}\right)^2+\ldots\right]$ Ignore terms higher than x^2

$=\left(3+2x-x^2\right)\frac{1}{4}\left(1+\frac{x}{4}+\frac{x^2}{16}+\ldots\right)$ Multiply expansion by $\frac{1}{4}$

$=\left(3+2x-x^2\right)\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)$ Multiply result by $\left(3+2x-x^2\right)$

$=3\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)+2x\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)-x^2\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)$

$= \frac{3}{4} + \frac{3}{16}x + \frac{3}{64}x^2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{4}x^2 + \ldots$ Ignore any terms bigger than x^2

$= \frac{3}{4} + \frac{11}{16}x - \frac{5}{64}x^2$

Expansion is valid if $\left| \frac{-x}{4} \right| < 1 \Rightarrow |x| < 4$

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The binomial expansion

Exercise B, Question 3

Question:

Find the first four terms in the expansion of $\sqrt{(4-x)}$. By substituting $x = \frac{1}{9}$, find a fraction that is an approximation to $\sqrt{35}$. By comparing this to the exact value, state the degree of accuracy of your approximation.

Solution:

$$\sqrt{(4-x)} = (4-x)^{\frac{1}{2}}$$

$$= \left[4\left(1-\frac{x}{4}\right)\right]^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}}\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$$

$$= 2\left[1+\left(\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3+\ldots\right]$$

$$= 2\left(1-\frac{x}{8}-\frac{x^2}{128}-\frac{x^3}{1024}+\ldots\right)$$

$$= 2-\frac{x}{4}-\frac{x^2}{64}-\frac{x^3}{512}+\ldots$$

Valid for $\left|-\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$

Substitute $x = \frac{1}{9}$ into both sides of the expansion:

$$\sqrt{\left(4-\frac{1}{9}\right)} \simeq 2-\frac{\frac{1}{9}}{4}-\frac{\left(\frac{1}{9}\right)^2}{64}-\frac{\left(\frac{1}{9}\right)^3}{512}$$

$$\sqrt{\frac{35}{9}} \simeq 2-\frac{1}{36}-\frac{1}{5184}-\frac{1}{373248}$$

$$\frac{\sqrt{35}}{3} \simeq \frac{736055}{373248}$$

$$\sqrt{35} \simeq 3\times\frac{736055}{373248}=\frac{736055}{124416}=5.916079 \mid 925$$

By calculator $\sqrt{35} = 5.916079 | 783$
Fraction accurate to 6 decimal places

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The binomial expansion
Exercise B, Question 4

Question:

The expansion of $(a + bx)^{-2}$ may be approximated by $\frac{1}{4} + \frac{1}{4}x + cx^2$. Find the values of the constants a, b and c.

Solution:

$$(a + bx)^{-2} = \left[a \left(1 + \frac{bx}{a} \right) \right]^{-2} \quad \text{Take out a factor of } a$$

$$= a^{-2} \left(1 + \frac{bx}{a} \right)^{-2}$$

$$= \frac{1}{a^2} \left(1 + \frac{bx}{a} \right)^{-2}$$

$$= \frac{1}{a^2} \left[1 + (-2) \left(\frac{bx}{a} \right) + \frac{(-2) (-3)}{2 !} \left(\frac{bx}{a} \right)^2 + \ldots \right]$$

$$= \frac{1}{a^2} - \frac{2bx}{a^3} + \frac{3b^2x^2}{a^4} + \ldots$$

Compare this to $\frac{1}{4} + \frac{1}{4}x + cx^2$

Comparing constant terms: $\frac{1}{a^2} = \frac{1}{4}$

$\Rightarrow \quad a^2 = 4 \quad (\surd)$

$\Rightarrow \quad a = \pm 2$

Comparing terms in x: $\frac{-2b}{a^3} = \frac{1}{4}$

$\Rightarrow \quad b = \frac{a^3}{-8}$ Substitute $a = \pm 2$

$\Rightarrow \quad b = \frac{(\pm 2)^3}{-8}$

$\Rightarrow \quad b = \pm 1$

Compare terms in x^2: $c = \frac{3b^2}{a^4}$ Substitute $a^4 = 16$, $b^2 = 1$

$\Rightarrow \quad c = \frac{3 \times 1}{16}$

$$\Rightarrow \quad c = \frac{3}{16}$$

Hence $a = \pm 2$, $b \pm 1$, $c = \frac{3}{16}$

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The binomial expansion

Exercise C, Question 1

Question:

(a) Express $\frac{8x+4}{(1-x)(2+x)}$ as partial fractions.

(b) Hence or otherwise expand $\frac{8x+4}{(1-x)(2+x)}$ in ascending powers of x as far as the term in x^2.

(c) State the set of values of x for which the expansion is valid.

Solution:

(a) Let $\frac{8x+4}{(1-x)(2+x)} \equiv \frac{A}{(1-x)} + \frac{B}{(2+x)} \equiv \frac{A(2+x)+B(1-x)}{(1-x)(2+x)}$

Set the numerators equal: $8x+4 \equiv A(2+x)+B(1-x)$

Substitute $x=1$: $8 \times 1 + 4 = A \times 3 + B \times 0$

$\Rightarrow \quad 12 = 3A$

$\Rightarrow \quad A = 4$

Substitute $x = -2$: $8 \times (-2) + 4 = A \times 0 + B \times 3$

$\Rightarrow \quad -12 = 3B$

$\Rightarrow \quad B = -4$

Hence $\frac{8x+4}{(1-x)(2+x)} \equiv \frac{4}{(1-x)} - \frac{4}{(2+x)}$

(b) $\frac{4}{(1-x)} = 4(1-x)^{-1}$

$$= 4\left[1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2!} + \ldots\right]$$

$$= 4(1 + x + x^2 + \ldots)$$

$$= 4 + 4x + 4x^2 + \ldots$$

$\frac{4}{(2+x)} = 4(2+x)^{-1}$

$$= 4 \left[2 \left(1 + \frac{x}{2} \right) \right]^{-1}$$

$$= 4 \times 2^{-1} \left(1 + \frac{x}{2} \right)^{-1}$$

$$= 4 \times \frac{1}{2} \times \left[1 + \left(-1 \right) \left(\frac{x}{2} \right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{2} \right)^{2} + \ldots \right]$$

$$= 2 \left(1 - \frac{x}{2} + \frac{x^2}{4} + \ldots \right)$$

$$= 2 - x + \frac{1}{2}x^2 + \ldots$$

Therefore

$$\frac{8x + 4}{(1 - x)(2 + x)} \equiv \frac{4}{(1 - x)} - \frac{4}{(2 + x)}$$

$$= \left(4 + 4x + 4x^2 + \ldots \right) - \left(2 - x + \frac{1}{2}x^2 + \ldots \right)$$

$$= 2 + 5x + \frac{7x^2}{2}$$

(c) $\frac{4}{(1 - x)}$ is valid for $|x| < 1$

$\frac{4}{(2 + x)}$ is valid for $|x| < 2$

Both are valid when $|x| < 1$.

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The binomial expansion

Exercise C, Question 2

Question:

(a) Express $\frac{-2x}{(2+x)^2}$ as a partial fraction.

(b) Hence prove that $\frac{-2x}{(2+x)^2}$ can be expressed in the form $0-\frac{1}{2}x+Bx^2+Cx^3$ where constants B and C are to be determined.

(c) State the set of values of x for which the expansion is valid.

Solution:

(a) Let $\frac{-2x}{(2+x)^2} \equiv \frac{A}{(2+x)} + \frac{B}{(2+x)^2} \equiv \frac{A(2+x)+B}{(2+x)^2}$

Set the numerators equal: $-2x \equiv A(2+x)+B$

Substitute $x=-2$: $4=A\times 0+B \Rightarrow B=4$

Equate terms in x: $-2=A \Rightarrow A=-2$

Hence $\frac{-2x}{(2+x)^2} \equiv \frac{-2}{(2+x)} + \frac{4}{(2+x)^2}$

(b) $\frac{-2}{2+x} = -2(2+x)^{-1}$

$$= -2\left[2\left(1+\frac{x}{2}\right)\right]^{-1}$$

$$= -2\times 2^{-1}\times\left(1+\frac{x}{2}\right)^{-1}$$

$$= -1\times\left[1+\left(-1\right)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right]$$

$$= -1\times\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)$$

$$= -1 + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} + \dots$$

$$\frac{4}{(2+x)^2} = 4(2+x)^{-2}$$

$$= 4\left[2\left(1+\frac{x}{2}\right)\right]^{-2}$$

$$= 4 \times 2^{-2} \times \left(1+\frac{x}{2}\right)^{-2}$$

$$= 1 \times \left[1 + \left(-2\right)\left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3 + \dots\right]$$

$$= 1 \times \left(1 - x + \frac{3x^2}{4} - \frac{x^3}{2} + \dots\right)$$

$$= 1 - x + \frac{3x^2}{4} - \frac{x^3}{2} + \dots$$

Hence

$$\frac{-2x}{(2+x)^2} \equiv \frac{-2}{(2+x)} + \frac{4}{(2+x)^2}$$

$$= -1 + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} + 1 - x + \frac{3x^2}{4} - \frac{x^3}{2} + \dots$$

$$= 0 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{3}{8}x^3$$

Hence $B = \frac{1}{2}$, (coefficient of x^2) and $C = -\frac{3}{8}$, (coefficients of x^3)

(c) $\frac{-2}{(2+x)}$ is valid for $|x| < 2$

$\frac{4}{(2+x)^2}$ is valid for $|x| < 2$

Hence whole expression is valid $|x| < 2$.

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The binomial expansion
Exercise C, Question 3

Question:

(a) Express $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ as a partial fraction.

(b) Hence or otherwise expand $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ in ascending powers of x as far as
the term in x^3.

(c) State the set of values of x for which the expansion is valid.

Solution:

(a) Let $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{A}{(1+x)} + \frac{B}{(1-x)} + \frac{C}{(2+x)}$

$$\equiv \frac{A(1-x)(2+x)+B(1+x)(2+x)+C(1+x)(1-x)}{(1+x)(1-x)(2+x)}$$

Set the numerators equal:

$$6+7x+5x^2 \equiv A\left(1-x\right)\left(2+x\right)+B\left(1+x\right)\left(2+x\right)+C\left(1+x\right)\left(1-x\right)$$

Substitute $x=1$: $6+7+5=A\times 0+B\times 2\times 3+C\times 0$

$\Rightarrow \quad 18=6B$

$\Rightarrow \quad B=3$

Substitute $x=-1$: $6-7+5=A\times 2\times 1+B\times 0+C\times 0$

$\Rightarrow \quad 4=2A$

$\Rightarrow \quad A=2$

Substitute $x=-2$: $6-14+20=A\times 0+B\times 0+C\times(-1)\times 3$

$\Rightarrow \quad 12=-3C$

$\Rightarrow \quad C=-4$

Hence $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{2}{(1+x)} + \frac{3}{(1-x)} - \frac{4}{(2+x)}$

(b) $\frac{2}{1+x} = 2(1+x)^{-1}$

$$= 2\left[1 + \left(-1\right)\left(x\right) + \frac{(-1)(-2)(x)^2}{2!} + \frac{(-1)(-2)(-3)(x)^3}{3!} + \ldots\right]$$

$$= 2(1 - x + x^2 - x^3 + \ldots)$$

$$\simeq 2 - 2x + 2x^2 - 2x^3 \quad \text{Valid for } |x| < 1$$

$\frac{3}{1-x} = 3(1-x)^{-1}$

$$= 3\left[1 + \left(-1\right)\left(-x\right) + \frac{(-1)(-2)(-x)^2}{2!} + \frac{(-1)(-2)(-3)(-x)^3}{3!} + \ldots\right]$$

$$= 3(1 + x + x^2 + x^3 + \ldots)$$

$$\simeq 3 + 3x + 3x^2 + 3x^3 \quad \text{Valid for } |x| < 1$$

$\frac{4}{2+x} = 4(2+x)^{-1}$

$$= 4\left[2\left(1 + \frac{x}{2}\right)\right]^{-1}$$

$$= 4 \times 2^{-1} \times \left(1 + \frac{x}{2}\right)^{-1}$$

$$= 2\left[1 + \left(-1\right)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right]$$

$$= 2\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots\right)$$

$$\simeq 2 - x + \frac{x^2}{2} - \frac{x^3}{4} \quad \text{Valid for } |x| < 2$$

Hence

$$\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{2}{(1+x)} + \frac{3}{(1-x)} - \frac{4}{(2+x)}$$

$$= \left(2-2x+2x^2-2x^3\right) + \left(3+3x+3x^2+3x^3\right) - \left(2-x+\frac{x^2}{2}-\frac{x^3}{4}\right)$$

$$= 2+3-2-2x+3x+x+2x^2+3x^2-\frac{x^2}{2}-2x^3+3x^3+\frac{x^3}{4}$$

$$= 3+2x+\frac{9}{2}x^2+\frac{5}{4}x^3$$

(c) All expansions are valid when $|x| < 1$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

The binomial expansion

Exercise D, Question 1

Question:

Find binomial expansions of the following in ascending powers of x as far as the term in x^3. State the set of values of x for which the expansion is valid.

(a) $(1 - 4x)^{3}$

(b) $\sqrt{(16 + x)}$

(c) $\frac{1}{(1 - 2x)}$

(d) $\frac{4}{2 + 3x}$

(e) $\frac{4}{\sqrt{(4 - x)}}$

(f) $\frac{1 + x}{1 + 3x}$

(g) $\left(\frac{1 + x}{1 - x} \right)^{2}$

(h) $\frac{x - 3}{(1 - x) (1 - 2x)}$

Solution:

(a) $(1 - 4x)^{3}$ Use binomial expansion with $n = 3$ and $x = - 4x$

$= 1 + \left(3 \right) \left(- 4x \right) + \frac{(3) (2) (- 4x)^{2}}{2 !} +$

$\frac{(3) (2) (1) (- 4x)^{3}}{3 !}$ As $n = 3$ expansion is finite

and exact

$= 1 - 12x + 48x^{2} - 64x^{3}$ Valid for all x

(b) $\sqrt{16 + x}$ Write in index form

$= (16 + x)^{\frac{1}{2}}$ Take out a factor of 16

$$= \left[16 \left(1 + \frac{x}{16} \right) \right]^{\frac{1}{2}}$$

$$= 16^{\frac{1}{2}} \left(1 + \frac{x}{16} \right)^{\frac{1}{2}}$$ Use binomial expansion with $n = \frac{1}{2}$ and $x = \frac{x}{16}$

$$= 4 \left[1 + \frac{1}{2} \left(\frac{x}{16} \right) + \frac{\left(\frac{1}{2} \right) \left(- \frac{1}{2} \right)}{2\,!} \left(\frac{x}{16} \right)^2 + \frac{\left(\frac{1}{2} \right) \left(- \frac{1}{2} \right) \left(- \frac{3}{2} \right)}{3\,!} \left(\frac{x}{16} \right)^3 + \ldots \right]$$

$$= 4 \left(1 + \frac{x}{32} - \frac{x^2}{2048} + \frac{x^3}{65536} + \ldots \right)$$ Multiply by 4

$$= 4 + \frac{x}{8} - \frac{x^2}{512} + \frac{x^3}{16384} + \ldots$$

Valid for $\left| \frac{x}{16} \right| < 1 \Rightarrow |x| < 16$

(c) $\frac{1}{1 - 2x}$ Write in index form

$= (1 - 2x)^{-1}$ Use binomial expansion with $n = -1$ and $x = -2x$

$$= 1 + \left(-1 \right) \left(-2x \right) + \frac{(-1)(-2)(-2x)^2}{2\,!} + \frac{(-1)(-2)(-3)(-2x)^3}{3\,!} + \ldots$$

$$= 1 + 2x + 4x^2 + 8x^3 + \ldots$$

Valid for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

(d) $\frac{4}{2 + 3x}$ Write in index form

$= 4(2 + 3x)^{-1}$ Take out a factor of 2

$$= 4 \left[2 \left(1 + \frac{3x}{2} \right) \right]^{-1}$$

$= 4 \times 2^{-1} \times \left(1 + \frac{3x}{2} \right)^{-1}$ Use binomial expansion with $n = -1$ and

$x = \frac{3x}{2}$

$$= 2 \left[1 + \left(-1 \right) \left(\frac{3x}{2} \right) + \frac{(-1)(-2)}{2!} \left(\frac{3x}{2} \right)^2 + \frac{(-1)(-2)(-3)}{3!} \left(\frac{3x}{2} \right)^3 + \ldots \right]$$

$$= 2 \left(1 - \frac{3x}{2} + \frac{9x^2}{4} - \frac{27x^3}{8} + \ldots \right)$$ Multiply by 2

$$= 2 - 3x + \frac{9x^2}{2} - \frac{27x^3}{4} + \ldots$$

Valid for $\left| \frac{3x}{2} \right| < 1 \Rightarrow |x| < \frac{2}{3}$

(e) $\frac{4}{\sqrt{4-x}} = 4 \left(\sqrt{4-x} \right)^{-1}$ Write in index form

$= 4 (4 - x)^{-\frac{1}{2}}$ Take out a factor of 4

$= 4 \left[4 \left(1 - \frac{x}{4} \right) \right]^{-\frac{1}{2}}$

$= 4 \times 4^{-\frac{1}{2}} \left(1 - \frac{x}{4} \right)^{-\frac{1}{2}}$ Use binomial expansion with $n = -\frac{1}{2}$ and

$x = -\frac{x}{4}$

$$= 4^{\frac{1}{2}} \left[1 + \left(-\frac{1}{2} \right) \left(-\frac{x}{4} \right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} \left(-\frac{x}{4} \right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} \left(-\frac{x}{4} \right)^3 + \ldots \right]$$

$$= 2 \left(1 + \frac{x}{8} + \frac{3}{128}x^2 + \frac{5}{1024}x^3 + \ldots \right)$$ Multiply by 2

$= 2 + \frac{x}{4} + \frac{3}{64}x^2 + \frac{5}{512}x^3 + \ldots$

Valid $\left| -\frac{x}{4} \right| < 1 \Rightarrow |x| < 4$

(f) $\frac{1+x}{1+3x} = \left(1+x\right)\left(1+3x\right)^{-1}$ Write $\frac{1}{1+3x}$ in index form then expand

$= \left(1+x\right)\left[1 + \left(-1\right)\left(3x\right) + \frac{(-1)(-2)(3x)^2}{2!} + \frac{(-1)(-2)(-3)(3x)^3}{3!} + \ldots\right]$

$= (1+x)(1 - 3x + 9x^2 - 27x^3 + \ldots)$ Multiply out

$= 1 - 3x + 9x^2 - 27x^3 + x - 3x^2 + 9x^3 + \ldots$ Collect like terms

$= 1 - 2x + 6x^2 - 18x^3 + \ldots$

Valid for $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

(g) $\left(\frac{1+x}{1-x}\right)^2 = \frac{(1+x)^2}{(1-x)^2}$ Write in index form

$= (1+x)^2(1-x)^{-2}$ Expand $(1-x)^{-2}$ using binomial expansion

$= \left(1 + 2x + x^2\right)\left[1 + \left(-2\right)\left(-x\right) + \frac{(-2)(-3)(-x)^2}{2!} + \frac{(-2)(-3)(-4)(-x)^3}{3!} + \ldots\right]$

$= (1 + 2x + x^2)(1 + 2x + 3x^2 + 4x^3 + \ldots)$ Multiply out brackets

$= 1 + 2x + 3x^2 + 4x^3 + 2x + 4x^2 + 6x^3 + x^2 + 2x^3 + \ldots$ Collect like terms

$= 1 + 4x + 8x^2 + 12x^3 + \ldots$

Valid for $|x| < 1$

(h) Let $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{A}{(1-x)} + \frac{B}{(1-2x)}$ Put in partial fraction form

$\equiv \frac{A(1-2x) + B(1-x)}{(1-x)(1-2x)}$ Add fractions.

Set the numerators equal: $x - 3 \equiv A(1-2x) + B(1-x)$

Substitute $x = 1$: $1 - 3 = A \times -1 + B \times 0$

$\Rightarrow -2 = -1A$

$\Rightarrow \quad A = 2$

Substitute $x = \frac{1}{2}$: $\quad \frac{1}{2} - 3 = A \times 0 + B \times \frac{1}{2}$

$\Rightarrow \quad -2\frac{1}{2} = \frac{1}{2}B$

$\Rightarrow \quad B = -5$

Hence $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{2}{(1-x)} - \frac{5}{(1-2x)}$

$$\frac{2}{(1-x)} = 2(1-x)^{-1}$$

$$= 2\left[1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2!} + \frac{(-1)(-2)(-3)(-x)^3}{3!} + \ldots\right]$$

$$= 2(1 + x + x^2 + x^3 + \ldots)$$

$$\simeq 2 + 2x + 2x^2 + 2x^3$$

$$\frac{5}{(1-2x)} = 5(1-2x)^{-1}$$

$$= 5\left[1 + (-1)(-2x) + \frac{(-1)(-2)(-2x)^2}{2!} + \frac{(-1)(-2)(-3)(-2x)^3}{3!} + \ldots\right]$$

$$= 5(1 + 2x + 4x^2 + 8x^3 + \ldots)$$

$$\simeq 5 + 10x + 20x^2 + 40x^3$$

Hence $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{2}{(1-x)} - \frac{5}{(1-2x)}$

$$\simeq (2 + 2x + 2x^2 + 2x^3) - (5 + 10x + 20x^2 + 40x^3)$$

$$\simeq -3 - 8x - 18x^2 - 38x^3$$

$\frac{2}{1-x}$ is valid for $|x| < 1$

$\frac{5}{1-2x}$ is valid for $|2x| < 1 \quad \Rightarrow \quad |x| < \frac{1}{2}$

Both are valid when $|x| < \frac{1}{2}$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

The binomial expansion

Exercise D, Question 2

Question:

Find the first four terms of the expansion in ascending powers of x of:

$$\left(1 - \frac{1}{2}x \right)^{\frac{1}{2}}, \ \left| x \right| < 2$$

and simplify each coefficient. (adapted)

Solution:

$$\left(1 - \frac{1}{2}x \right)^{\frac{1}{2}} = 1 + \left(\frac{1}{2} \right) \left(- \frac{1}{2}x \right) + \frac{\left(\frac{1}{2} \right) \left(- \frac{1}{2} \right) \left(- \frac{1}{2}x \right)^{2}}{2\,!} + \frac{\left(\frac{1}{2} \right) \left(- \frac{1}{2} \right) \left(- \frac{3}{2} \right) \left(- \frac{1}{2}x \right)^{3}}{3\,!} + \ldots$$

$$= 1 - \frac{1}{4}x + \left(- \frac{1}{8} \right) \times \left(\frac{1}{4}x^{2} \right) + \left(\frac{1}{16} \right) \times \left(- \frac{1}{8}x^{3} \right) + \ldots$$

$$= 1 - \frac{1}{4}x - \frac{1}{32}x^{2} - \frac{1}{128}x^{3}$$

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The binomial expansion

Exercise D, Question 3

Question:

Show that if x is sufficiently small then $\frac{3}{\sqrt{(4+x)}}$ can be approximated by $\frac{3}{2}-\frac{3}{16}x+\frac{9}{256}x^2$.

Solution:

$\frac{3}{\sqrt{4+x}} = 3(\sqrt{4+x})^{-1}$ Write in index form

$= 3(4+x)^{-\frac{1}{2}}$ Take out a factor of 4

$= 3\left[4\left(1+\frac{x}{4}\right)\right]^{-\frac{1}{2}}$

$= 3 \times 4^{-\frac{1}{2}} \times \left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$ $4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{2}$

$$= \frac{3}{2} \times \left[1+\left(-\frac{1}{2}\right)\left(\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2!}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{4}\right)^3}{3!}+\ldots\right]$$

$= \frac{3}{2}\left(1-\frac{x}{8}+\frac{3}{128}x^2+\ldots\right)$ Multiply by $\frac{3}{2}$

$= \frac{3}{2}-\frac{3}{16}x+\frac{9}{256}x^2+\ldots$

$= \frac{3}{2}-\frac{3}{16}x+\frac{9}{256}x^2$ If terms higher than x^2 are ignored

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The binomial expansion

Exercise D, Question 4

Question:

Given that $| x | < 4$, find, in ascending powers of x up to and including the term in x^3, the series expansion of:

(a) $(4 - x) ^{\frac{1}{2}}$

(b) $(4 - x) ^{\frac{1}{2}} (1 + 2x)$ Ⓔ (adapted)

Solution:

(a) $(4 - x) ^{\frac{1}{2}}$ Take out a factor of 4

$$= \left[4 \left(1 - \frac{x}{4} \right) \right] ^{\frac{1}{2}}$$

$$= 4 ^{\frac{1}{2}} \left(1 - \frac{x}{4} \right) ^{\frac{1}{2}}$$ Use binomial expansion with $n = \frac{1}{2}$ and $x = - \frac{x}{4}$

$$= 2 \left[1 + \left(\frac{1}{2} \right) \left(- \frac{x}{4} \right) + \frac{(\frac{1}{2}) (- \frac{1}{2}) (- \frac{x}{4}) ^{2}}{2 !} + \frac{(\frac{1}{2}) (- \frac{1}{2}) (- \frac{3}{2}) (- \frac{x}{4}) ^{3}}{3 !} + \ldots \right]$$

$$= 2 \left(1 - \frac{x}{8} - \frac{x^2}{128} - \frac{x^3}{1024} + \ldots \right)$$ Multiply by 2

$$= 2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512} + \ldots$$

(b) $\left(4 - x \right)^{\frac{1}{2}} \left(1 + 2x \right)$ Use answer from part (a)

$= \left(2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512} + \ldots \right) \left(1 + 2x \right)$ Multiply out brackets

$= 2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512} + \ldots + 4x - \frac{x^2}{2} - \frac{x^3}{32} + \ldots$ Collect like terms

$= 2 + \frac{15}{4}x - \frac{33}{64}x^2 - \frac{17}{512}x^3 + \ldots$

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The binomial expansion
Exercise D, Question 5

Question:

(a) Find the first four terms of the expansion, in ascending powers of x, of

$(2 + 3x)^{-1}, \ | x | < \frac{2}{3}$

(b) Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of:

$\frac{1 + x}{2 + 3x}, \ | x | < \frac{2}{3}$

Solution:

(a) $(2 + 3x)^{-1}$ Take out factor of 2

$= \left[2 \left(1 + \frac{3x}{2} \right) \right]^{-1}$

$= 2^{-1} \left(1 + \frac{3x}{2} \right)^{-1}$ Use binomial expansion with $n = -1$ and $x = \frac{3x}{2}$

$$= \frac{1}{2} \left[1 + \left(-1 \right) \left(\frac{3x}{2} \right) + \frac{(-1) (-2)}{2 !} \left(\frac{3x}{2} \right)^2 + \frac{(-1) (-2) (-3)}{3 !} \left(\frac{3x}{2} \right)^3 + \ldots \right]$$

$= \frac{1}{2} \left(1 - \frac{3}{2}x + \frac{9}{4}x^2 - \frac{27}{8}x^3 + \ldots \right)$ Multiply by $\frac{1}{2}$

$= \frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \ldots$

Valid for $\left| \frac{3x}{2} \right| < 1 \Rightarrow | x | < \frac{2}{3}$

(b) $\frac{1 + x}{2 + 3x}$ Put in index form

$= (1 + x) (2 + 3x)^{-1}$ Use expansion from part (a)

$$= \left(1 + x \right) \left(\frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \ldots \right)$$ Multiply out

$$= \frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \frac{1}{2}x - \frac{3}{4}x^2 + \frac{9}{8}x^3 + \ldots$$ Collect like terms

$$= \frac{1}{2} - \frac{1}{4}x + \frac{3}{8}x^2 - \frac{9}{16}x^3 + \ldots$$

Valid for $\left| \frac{3x}{2} \right| < 1 \Rightarrow | x | < \frac{2}{3}$

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The binomial expansion
Exercise D, Question 6

Question:

Find, in ascending powers of x up to and including the term in x^3, the series expansion of $(4+x)^{-\frac{1}{2}}$, giving your coefficients in their simplest form. E

Solution:

$$(4+x)^{-\frac{1}{2}} = \left[4\left(1+\frac{x}{4}\right)\right]^{-\frac{1}{2}}$$ Take out factor of 4

$$= 4^{-\frac{1}{2}}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}} \qquad 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{2}$$

$$= \frac{1}{2}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$$ Use binomial expansion with $n = -\frac{1}{2}$ and $x = \frac{x}{4}$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{4}\right)^3}{3!} + \ldots\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{8}x + \frac{3}{128}x^2 - \frac{5}{1024}x^3 + \ldots\right)$$

$$\simeq \frac{1}{2} - \frac{1}{16}x + \frac{3}{256}x^2 - \frac{5}{2048}x^3$$

Valid for $\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$

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Edexcel AS and A Level Modular Mathematics

The binomial expansion

Exercise D, Question 7

Question:

$f(x) = (1 + 3x)^{-1}, \ |x| < \frac{1}{3}.$

(a) Expand f(x) in ascending powers of x up to and including the term in x^3.

(b) Hence show that, for small x:

$\frac{1 + x}{1 + 3x} \approx 1 - 2x + 6x^2 - 18x^3.$

(c) Taking a suitable value for x, which should be stated, use the series expansion

in part (b) to find an approximate value for $\frac{101}{103}$, giving your answer to 5 decimal places.

Solution:

(a) $(1 + 3x)^{-1}$ Use binomial expansion with $n = -1$ and $x = 3x$

$= 1 + \left(-1\right)\left(3x\right) + \frac{(-1)(-2)(3x)^2}{2!} + \frac{(-1)(-2)(-3)(3x)^3}{3!} + \ldots$

$= 1 - 3x + 9x^2 - 27x^3 + \ldots$

(b) $\frac{1 + x}{1 + 3x} = \left(1 + x\right)(1 + 3x)^{-1}$ Use expansion from part (a)

$= (1 + x)(1 - 3x + 9x^2 - 27x^3 + \ldots)$ Multiply out

$= 1 - 3x + 9x^2 - 27x^3 + x - 3x^2 + 9x^3 + \ldots$ Collect like terms

$= 1 - 2x + 6x^2 - 18x^3 + \ldots$ Ignore terms greater than x^3

Hence $\frac{1 + x}{1 + 3x} \approx 1 - 2x + 6x^2 - 18x^3$

(c) Substitute $x = 0.01$ into both sides of the above

$$\frac{1 + 0.01}{1 + 3 \times 0.01} - \quad 1 - 2 \times 0.01 + 6 \times 0.01^2 - 18 \times 0.01^3$$

$$\frac{1.01}{1.03} \simeq \quad 1 - 0.02 + 0.0006 - 0.000018 \; , \qquad \left[\frac{1.01}{1.03} = \frac{101}{103} \right]$$

$$\frac{101}{103} \simeq \quad 0.980582 \qquad \text{Round to 5 d.p.}$$

$$\frac{101}{103} \simeq \quad 0.98058 \qquad (5 \text{ d.p.})$$

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Edexcel AS and A Level Modular Mathematics

The binomial expansion

Exercise D, Question 8

Question:

Obtain the first four non-zero terms in the expansion, in ascending powers of x, of the function f(x) where $f(x) = \frac{1}{\sqrt{(1+3x^2)}}$, $3x^2 < 1$.

Solution:

$$f(x) = \frac{1}{\sqrt{1+3x^2}} = \left(\sqrt{1+3x^2}\right)^{-1}$$

$$= (1+3x^2)^{-\frac{1}{2}}$$ Use binomial expansion with $n = -\frac{1}{2}$ and $x = 3x^2$

$$= 1 + \left(-\frac{1}{2}\right)(3x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(3x^2)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(3x^2)^3}{3!} + \ldots$$

$$\simeq 1 - \frac{3x^2}{2} + \frac{27x^4}{8} - \frac{135x^6}{16}$$

Valid for $|3x^2| < 1$

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The binomial expansion
Exercise D, Question 9

Question:

Give the binomial expansion of $(1+x)^{\frac{1}{2}}$ up to and including the term in x^3.

By substituting $x=\frac{1}{4}$, find the fraction that is an approximation to $\sqrt{5}$.

Solution:

Using binomial expansion

$$(1+x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)(x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x)^2}{2!}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x)^3}{3!}+\ldots$$

$$\simeq 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$$

Expansion is valid if $|x|<1$.

Substituting $x=\frac{1}{4}$ in both sides of expansion gives

$$\left(1+\frac{1}{4}\right)^{\frac{1}{2}} \simeq 1+\frac{1}{2}\times\frac{1}{4}-\frac{1}{8}\times\left(\frac{1}{4}\right)^2+\frac{1}{16}\times\left(\frac{1}{4}\right)^3$$

$$\left(\frac{5}{4}\right)^{\frac{1}{2}} \simeq 1+\frac{1}{8}-\frac{1}{128}+\frac{1}{1024} \qquad \left[\left(\frac{5}{4}\right)^{\frac{1}{2}}=\sqrt{\frac{5}{4}}\right]$$

$$\sqrt{\frac{5}{4}} \simeq \frac{1145}{1024} \qquad \left[\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{\sqrt{4}}=\frac{\sqrt{5}}{2}\right]$$

$$\frac{\sqrt{5}}{2} \simeq \frac{1145}{1024}$$ Multiply both sides by 2

$$\sqrt{5} \simeq \frac{1145}{512}$$

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The binomial expansion
Exercise D, Question 10

Question:

When $(1 + ax)^{n}$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 27 respectively.

(a) Find the values of a and n.

(b) Find the coefficient of x^3.

(c) State the values of x for which the expansion is valid.

Solution:

(a) Using binomial expansion

$$(1 + ax)^{n} = 1 + n \left(ax \right) + \frac{n (n - 1) (ax)^{2}}{2 !} + \frac{n (n - 1) (n - 2) (ax)^{3}}{3 !} + \ldots$$

If coefficient of x is -6 then $na = -6$ ①

If coefficient of x^2 is 27 then $\frac{n (n - 1) a^{2}}{2} = 27$ ②

From ① $a = \frac{-6}{n}$. Substitute in ②:

$$\frac{n (n - 1)}{2} \left(\frac{-6}{n} \right)^{2} = 27$$

$$\frac{n (n - 1)}{2} \times \frac{36}{n^{2}} = 27$$

$$\frac{(n - 1) 18}{n} = 27$$

$$(n - 1) 18 = 27n$$
$$18n - 18 = 27n$$
$$-18 = 9n$$
$$n = -2$$

Substitute $n = -2$ back in ①: $-2a = -6 \Rightarrow a = 3$

(b) Coefficient of x^3 is

$$\frac{n\left(n-1\right)\left(n-2\right)a^3}{3\,!} = \frac{\left(-2\right)\times\left(-3\right)\times\left(-4\right)\times 3^3}{3\times 2\times 1} = -108$$

(c) $\left(1+3x\right)^{-2}$ is valid if $\left|3x\right| < 1 \Rightarrow \left|x\right| < \frac{1}{3}$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 11

Question:

(a) Express $\frac{9x^2+26x+20}{(1+x)(2+x)^2}$ as a partial fraction.

(b) Hence or otherwise show that the expansion of $\frac{9x^2+26x+20}{(1+x)(2+x)^2}$ in ascending powers of x can be approximated to $5-\frac{7x}{2}+Bx^2+Cx^3$ where B and C are constants to be found.

(c) State the set of values of x for which this expansion is valid.

Solution:

(a) Let $\frac{9x^2+26x+20}{(1+x)(2+x)^2} \equiv \frac{A}{(1+x)}+\frac{B}{(2+x)}+\frac{C}{(2+x)^2}$

$$\Rightarrow \quad \frac{9x^2+26x+20}{(1+x)(2+x)^2} \equiv \frac{A(2+x)^2+B(1+x)(2+x)+C(1+x)}{(1+x)(2+x)^2}$$

Set the numerators equal:

$9x^2+26x+20 \equiv A(2+x)^2+B(1+x)(2+x)+C(1+x)$

Substitute $x=-2$: $\quad 36-52+20=A\times 0+B\times 0+C\times(-1)$

$\Rightarrow \quad 4=-1C$

$\Rightarrow \quad C=-4$

Substitute $x=-1$: $\quad 9-26+20=A\times 1+B\times 0+C\times 0$

$\Rightarrow \quad 3=1A$

$\Rightarrow \quad A=3$

Equate terms in x^2: $\quad 9=A+B$

$\Rightarrow \quad 9=3+B$

$\Rightarrow \quad B=6$

Hence $\frac{9x^2+26x+20}{(1+x)(2+x)^2} \equiv \frac{3}{(1+x)}+\frac{6}{(2+x)}-\frac{4}{(2+x)^2}$

(b) Using binomial expansion

$$\frac{3}{(1+x)} = 3(1+x)^{-1}$$

$$= 3\left[1 + (-1)(x) + \frac{(-1)(-2)(x)^2}{2!} + \frac{(-1)(-2)(-3)(x)^3}{3!} + \ldots\right]$$

$$= 3(1 - x + x^2 - x^3 + \ldots)$$

$$= 3 - 3x + 3x^2 - 3x^3 + \ldots$$

$$\frac{6}{(2+x)} = 6(2+x)^{-1}$$

$$= 6\left[2\left(1 + \frac{x}{2}\right)\right]^{-1}$$

$$= 6 \times 2^{-1}\left(1 + \frac{x}{2}\right)^{-1}$$

$$= 6 \times \frac{1}{2}\left[1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right]$$

$$= 3\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots\right)$$

$$= 3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \ldots$$

$$\frac{4}{(2+x)^2} = 4(2+x)^{-2}$$

$$= 4\left[2\left(1 + \frac{x}{2}\right)\right]^{-2}$$

$$= 4 \times 2^{-2} \times \left(1 + \frac{x}{2}\right)^{-2}$$

$$= 4 \times \frac{1}{4} \times \left[1 + (-2)\left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right]$$

$$= 1 \times \left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \ldots\right)$$

$$= 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \ldots$$

Hence

$$\frac{9x^2 + 26x + 20}{(1+x)(2+x)^2} \equiv \frac{3}{(1+x)} + \frac{6}{(2+x)} - \frac{4}{(2+x)^2}$$

$$\simeq \left(3 - 3x + 3x^2 - 3x^3\right) + \left(3 - \frac{3x}{2} + \frac{3}{4}x^2 - \frac{3}{8}x^3\right) - \left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3\right)$$

$$\simeq 3 - 3x + 3x^2 - 3x^3 + 3 - \frac{3x}{2} + \frac{3}{4}x^2 - \frac{3}{8}x^3 - 1 + x - \frac{3}{4}x^2 + \frac{1}{2}x^3$$

$$- \quad 5 - \frac{7x}{2} + 3x^2 - \frac{23}{8}x^3$$

Hence $B = 3$ and $C = \frac{-23}{8}$

(c) $\frac{3}{(1+x)}$ is valid if $|x| < 1$

$\frac{6}{(2+x)}$ is valid if $|x| < 2$

$\frac{4}{(2+x)^2}$ is valid if $|x| < 2$

Therefore, they *all* become valid if $|x| < 1$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise A, Question 1

Question:

Find $\frac{dy}{dx}$ for each of the following, leaving your answer in terms of the parameter t:

(a) $x = 2t, y = t^2 - 3t + 2$

(b) $x = 3t^2, y = 2t^3$

(c) $x = t + 3t^2, y = 4t$

(d) $x = t^2 - 2, y = 3t^5$

(e) $x = \frac{2}{t}, y = 3t^2 - 2$

(f) $x = \frac{1}{2t - 1}, y = \frac{t^2}{2t - 1}$

(g) $x = \frac{2t}{1 + t^2}, y = \frac{1 - t^2}{1 + t^2}$

(h) $x = t^2 e^t, y = 2t$

(i) $x = 4 \sin 3t, y = 3 \cos 3t$

(j) $x = 2 + \sin t, y = 3 - 4 \cos t$

(k) $x = \sec t, y = \tan t$

(l) $x = 2t - \sin 2t, y = 1 - \cos 2t$

Solution:

(a) $x = 2t, y = t^2 - 3t + 2$

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = 2t - 3$$

Using the chain rule

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2t-3}{2}$$

(b) $x = 3t^2, y = 2t^3$

$$\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2$$

Using the chain rule

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{6t^2}{6t} = t$$

(c) $x = t + 3t^2, y = 4t$

$$\frac{dx}{dt} = 1 + 6t, \frac{dy}{dt} = 4$$

$$\therefore \frac{dy}{dx} = \frac{4}{1+6t}$$ (from the chain rule)

(d) $x = t^2 - 2, y = 3t^5$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 15t^4$$

$$\therefore \frac{dy}{dx} = \frac{15t^4}{2t} = \frac{15t^3}{2}$$ (from the chain rule)

(e) $x = \frac{2}{t}, y = 3t^2 - 2$

$$\frac{dx}{dt} = -2t^{-2}, \frac{dy}{dt} = 6t$$

$$\therefore \frac{dy}{dx} = \frac{6t}{-2t^{-2}} = -3t^3$$ (from the chain rule)

(f) $x = \frac{1}{2t-1}, y = \frac{t^2}{2t-1}$

As $x = (2t-1)^{-1}, \frac{dx}{dt} = -2(2t-1)^{-2}$ (from the chain rule)

Use the quotient rule to give

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(2t-1)(2t) - t^2(2)}{(2t-1)^2} = \frac{2t^2-2t}{(2t-1)^2} = \frac{2t(t-1)}{(2t-1)^2}$$

Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)}$

$$= \frac{2t(t-1)}{(2t-1)^2} \div -2(2t-1)^{-2}$$

$$= \frac{2t(t-1)}{(2t-1)^2} \div \frac{-2}{(2t-1)^2}$$

$$= \frac{2t(t-1)}{(2t-1)^2} \times \frac{(2t-1)^2}{-2}$$

$$= -t(t-1) \text{ or } t(1-t)$$

(g) $x = \frac{2t}{1+t^2}, y = \frac{1-t^2}{1+t^2}$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(1+t^2)2 - 2t(2t)}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2}$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)}$$

$$= \frac{-4t}{(1+t^2)^2} \div \frac{2-2t^2}{(1+t^2)^2}$$

$$= \frac{-4t}{2(1-t^2)}$$

$$= -\frac{2t}{(1-t^2)} \text{ or } \frac{2t}{t^2-1}$$

(h) $x = t^2\mathrm{e}^t, y = 2t$

$\frac{\mathrm{d}x}{\mathrm{d}t} = t^2\mathrm{e}^t + \mathrm{e}^t 2t$ (from the product rule) and $\frac{\mathrm{d}y}{\mathrm{d}t} = 2$

$$\therefore \frac{dy}{dx} = \frac{2}{t^2e^t + 2te^t} = \frac{2}{te^t(t+2)} \quad \text{(from the chain rule)}$$

(i) $x = 4\sin 3t, y = 3\cos 3t$

$$\frac{dx}{dt} = 12\cos 3t, \frac{dy}{dt} = -9\sin 3t$$

$$\therefore \frac{dy}{dx} = \frac{-9\sin 3t}{12\cos 3t} = -\frac{3}{4}\tan 3t \quad \text{(from the chain rule)}$$

(j) $x = 2 + \sin t, y = 3 - 4\cos t$

$$\frac{dx}{dt} = \cos t, \frac{dy}{dt} = 4\sin t$$

$$\therefore \frac{dy}{dx} = \frac{4\sin t}{\cos t} = 4\tan t \quad \text{(from the chain rule)}$$

(k) $x = \sec t, y = \tan t$

$$\frac{dx}{dt} = \sec t\tan t, \frac{dy}{dt} = \sec^2 t$$

Hence $\frac{dy}{dx} = \frac{\sec^2 t}{\sec t\tan t}$

$$= \frac{\sec t}{\tan t}$$

$$= \frac{1}{\cos t} \times \frac{\cos t}{\sin t}$$

$$= \frac{1}{\sin t}$$

$$= \operatorname{cosec} t$$

(l) $x = 2t - \sin 2t, y = 1 - \cos 2t$

$$\frac{dx}{dt} = 2 - 2\cos 2t, \frac{dy}{dt} = 2\sin 2t$$

Hence $\frac{dy}{dx} = \frac{2\sin 2t}{2 - 2\cos 2t}$

$$= \frac{2 \times 2\sin t\cos t}{2 - 2(1 - 2\sin^2 t)} \quad \text{(using double angle formulae)}$$

$$= \frac{\sin t\cos t}{\sin^2 t}$$

$$= \frac{\cos t}{\sin t}$$

$$= \cot t$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 2

Question:

(a) Find the equation of the tangent to the curve with parametric equations $x = 3t - 2\sin t$, $y = t^2 + t\cos t$, at the point P, where $t = \frac{\pi}{2}$.

(b) Find the equation of the tangent to the curve with parametric equations $x = 9 - t^2$, $y = t^2 + 6t$, at the point P, where $t = 2$.

Solution:

(a) $x = 3t - 2\sin t$, $y = t^2 + t\cos t$

$$\frac{dx}{dt} = 3 - 2\cos t, \quad \frac{dy}{dt} = 2t + \left(-t\sin t + \cos t \right)$$

$$\therefore \frac{dy}{dx} = \frac{2t - t\sin t + \cos t}{3 - 2\cos t}$$

When $t = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{\left(\pi - \frac{\pi}{2} \right)}{3} = \frac{\pi}{6}$

$\therefore$ the tangent has gradient $\frac{\pi}{6}$.

When $t = \frac{\pi}{2}$, $x = \frac{3\pi}{2} - 2$ and $y = \frac{\pi^2}{4}$

$\therefore$ the tangent passes through the point $\left(\frac{3\pi}{2} - 2 , \frac{\pi^2}{4} \right)$

The equation of the tangent is

$$y - \frac{\pi^2}{4} = \frac{\pi}{6}\left[x - \left(\frac{3\pi}{2} - 2 \right) \right]$$

$$\therefore y - \frac{\pi^2}{4} = \frac{\pi}{6}x - \frac{\pi^2}{4} + \frac{\pi}{3}$$

i.e. $y = \frac{\pi}{6}x + \frac{\pi}{3}$

(b) $x = 9 - t^2$, $y = t^2 + 6t$

$$\frac{dx}{dt} = -2t, \frac{dy}{dt} = 2t + 6$$

$$\therefore \frac{dy}{dx} = \frac{2t+6}{-2t}$$

At the point where $t = 2$, $\frac{dy}{dx} = \frac{10}{-4} = \frac{-5}{2}$

Also at $t = 2$, $x = 5$ and $y = 16$.

$\therefore$ the tangent has equation

$$y - 16 = \frac{-5}{2}\left(x - 5\right)$$

$$\therefore 2y - 32 = -5x + 25$$

i.e. $2y + 5x = 57$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise A, Question 3

Question:

(a) Find the equation of the normal to the curve with parametric equations $x = \mathrm{e}^{t}$, $y = \mathrm{e}^{t} + \mathrm{e}^{-t}$, at the point P, where $t = 0$.

(b) Find the equation of the normal to the curve with parametric equations $x = 1 - \cos 2t$, $y = \sin 2t$, at the point P, where $t = \frac{\pi}{6}$.

Solution:

(a) $x = \mathrm{e}^{t}, y = \mathrm{e}^{t} + \mathrm{e}^{-t}$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^{t} \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{e}^{t} - \mathrm{e}^{-t}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^{t} - \mathrm{e}^{-t}}{\mathrm{e}^{t}}$$

When $t = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\therefore$ gradient of curve is 0

$\therefore$ normal is parallel to the y-axis.

When $t = 0$, $x = 1$ and $y = 2$

$\therefore$ equation of the normal is $x = 1$

(b) $x = 1 - \cos 2t, y = \sin 2t$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2 \sin 2t \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 2 \cos 2t$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 \cos 2t}{2 \sin 2t} = \cot 2t$$

When $t = \frac{\pi}{6}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\tan \frac{\pi}{3}} = \frac{1}{\sqrt{3}}$

$\therefore$ gradient of the normal is $-\sqrt{3}$

When $t = \frac{\pi}{6}$, $x = 1 - \cos \frac{\pi}{3} = \frac{1}{2}$ and $y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore$ equation of the normal is

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3}\left(x - \frac{1}{2}\right)$$

i.e. $y - \frac{\sqrt{3}}{2} = -\sqrt{3}x + \frac{\sqrt{3}}{2}$

$\therefore y + \sqrt{3}x = \sqrt{3}$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 4

Question:

Find the points of zero gradient on the curve with parametric equations $x = \frac{t}{1-t}$, $y = \frac{t^2}{1-t}$, $t \neq 1$.

You do not need to establish whether they are maximum or minimum points.

Solution:

$$x = \frac{t}{1-t},\ y = \frac{t^2}{1-t}$$

Use the quotient rule to give

$$\frac{dx}{dt} = \frac{(1-t) \times 1 - t(-1)}{(1-t)^2} = \frac{1}{(1-t)^2}$$

and

$$\frac{dy}{dt} = \frac{(1-t)2t - t^2(-1)}{(1-t)^2} = \frac{2t-t^2}{(1-t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2t-t^2}{(1-t)^2} \div \frac{1}{(1-t)^2} = t\left(2-t\right)$$

When $\frac{dy}{dx} = 0$, $t = 0$ or 2

When $t = 0$ then $x = 0$, $y = 0$
When $t = 2$ then $x = -2$, $y = -4$
$\therefore (0, 0)$ and $(-2, -4)$ are the points of zero gradient.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise B, Question 1

Question:

Find an expression in terms of x and y for $\frac{dy}{dx}$, given that:

(a) $x^2 + y^3 = 2$

(b) $x^2 + 5y^2 = 14$

(c) $x^2 + 6x - 8y + 5y^2 = 13$

(d) $y^3 + 3x^2y - 4x = 0$

(e) $3y^2 - 2y + 2xy = x^3$

(f) $x = \frac{2y}{x^2 - y}$

(g) $(x - y)^4 = x + y + 5$

(h) $e^x y = x e^y$

(i) $\sqrt{(xy)} + x + y^2 = 0$

Solution:

(a) $x^2 + y^3 = 2$

Differentiate with respect to x:

$$2x + 3y^2 \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{3y^2}$$

(b) $x^2 + 5y^2 = 14$

$$2x + 10y \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{10y} = -\frac{x}{5y}$$

(c) $x^2 + 6x - 8y + 5y^2 = 13$

$$2x + 6 - 8\frac{dy}{dx} + 10y\frac{dy}{dx} = 0$$

$$2x + 6 = \left(8 - 10y \right) \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2x + 6}{8 - 10y} = \frac{x + 3}{4 - 5y}$$

(d) $y^3 + 3x^2y - 4x = 0$

Differentiate with respect to x:

$$3y^2\frac{dy}{dx} + \left(3x^2\frac{dy}{dx} + y \times 6x \right) - 4 = 0$$

$$\frac{dy}{dx}\left(3y^2 + 3x^2 \right) = 4 - 6xy$$

$$\therefore \frac{dy}{dx} = \frac{4 - 6xy}{3 (x^2 + y^2)}$$

(e) $3y^2 - 2y + 2xy - x^3 = 0$

$$6y\frac{dy}{dx} - 2\frac{dy}{dx} + \left(2x\frac{dy}{dx} + y \times 2 \right) - 3x^2 = 0$$

$$\frac{dy}{dx}\left(6y - 2 + 2x \right) = 3x^2 - 2y$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - 2y}{2x + 6y - 2}$$

(f) $x = \frac{2y}{x^2 - y}$

$\therefore x^3 - xy = 2y$

i.e. $x^3 - xy - 2y = 0$

Differentiate with respect to x:

$$3x^2 - \left(x\frac{dy}{dx} + y \times 1 \right) - 2\frac{dy}{dx} = 0$$

$$3x^2 - y = \frac{dy}{dx}\left(x + 2 \right)$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - y}{x + 2}$$

(g) $\left(x - y \right)^{4} = x + y + 5$

Differentiate with respect to x:

$4 \left(x - y \right)^{3} \left(1 - \frac{dy}{dx} \right) = 1 + \frac{dy}{dx}$ (The chain rule was used to differentiate the first term.)

$$\therefore 4 \left(x - y \right)^{3} - 1 = \frac{dy}{dx} \left[1 + 4 \left(x - y \right)^{3} \right]$$

$$\therefore \frac{dy}{dx} = \frac{4 \left(x - y \right)^{3} - 1}{1 + 4 \left(x - y \right)^{3}}$$

(h) $e^{x}y = xe^{y}$

Differentiate with respect to x:

$$e^{x} \frac{dy}{dx} + ye^{x} = xe^{y} \frac{dy}{dx} + e^{y} \times 1$$

$$e^{x} \frac{dy}{dx} - xe^{y} \frac{dy}{dx} = e^{y} - ye^{x}$$

$$\frac{dy}{dx} \left(e^{x} - xe^{y} \right) = e^{y} - ye^{x}$$

$$\therefore \frac{dy}{dx} = \frac{e^{y} - ye^{x}}{e^{x} - xe^{y}}$$

(i) $\sqrt{xy} + x + y^{2} = 0$

Differentiate with respect to x:

$$\frac{1}{2} \left(xy \right)^{-\frac{1}{2}} \left(x \frac{dy}{dx} + y \times 1 \right) + 1 + 2y \frac{dy}{dx} = 0$$

Multiply both sides by $2\sqrt{xy}$:

$$\left(x \frac{dy}{dx} + y \right) + 2\sqrt{xy} + 4y\sqrt{xy} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} \left(x + 4y\sqrt{xy} \right) = - \left(2\sqrt{xy} + y \right)$$

$$\therefore \frac{dy}{dx} = \frac{- \left(2\sqrt{xy} + y \right)}{x + 4y\sqrt{xy}} .$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise B, Question 2

Question:

Find the equation of the tangent to the curve with implicit equation $x^2 + 3xy^2 - y^3 = 9$ at the point (2, 1).

Solution:

$x^2 + 3xy^2 - y^3 = 9$

Differentiate with respect to x:

$$2x + \left[3x \left(2y \frac{dy}{dx} \right) + y^2 \times 3 \right] - 3y^2 \frac{dy}{dx} = 0$$

When $x = 2$ and $y = 1$

$$4 + \left(12 \frac{dy}{dx} + 3 \right) - 3 \frac{dy}{dx} = 0$$

$$\therefore 9 \frac{dy}{dx} = -7$$

i.e. $\frac{dy}{dx} = \frac{-7}{9}$

$\therefore$ the gradient of the tangent at (2, 1) is $\frac{-7}{9}$.

The equation of the tangent is

$$\left(y - 1 \right) = \frac{-7}{9} \left(x - 2 \right)$$

$$\therefore 9y - 9 = -7x + 14$$

$$\therefore 9y + 7x = 23$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise B, Question 3

Question:

Find the equation of the normal to the curve with implicit equation $(x + y)^3 = x^2 + y$ at the point (1, 0).

Solution:

$(x + y)^3 = x^2 + y$

Differentiate with respect to x:

$$3 (x + y)^2 \left(1 + \frac{dy}{dx} \right) = 2x + \frac{dy}{dx}$$

At the point (1, 0), $x = 1$ and $y = 0$

$$\therefore 3 \left(1 + \frac{dy}{dx} \right) = 2 + \frac{dy}{dx}$$

$$\therefore 2 \frac{dy}{dx} = -1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{2}$$

$\therefore$ The gradient of the normal at (1, 0) is 2.

$\therefore$ the equation of the normal is

$y - 0 = 2 (x - 1)$

i.e. $y = 2x - 2$

Solutionbank

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Differentiation

Exercise B, Question 4

Question:

Find the coordinates of the points of zero gradient on the curve with implicit equation $x^2 + 4y^2 - 6x - 16y + 21 = 0$.

Solution:

$x^2 + 4y^2 - 6x - 16y + 21 = 0 \quad ①$

Differentiate with respect to x:

$$2x + 8y\frac{dy}{dx} - 6 - 16\frac{dy}{dx} = 0$$

$$8y\frac{dy}{dx} - 16\frac{dy}{dx} = 6 - 2x$$

$$\left(8y - 16\right)\frac{dy}{dx} = 6 - 2x$$

$$\therefore \frac{dy}{dx} = \frac{6 - 2x}{8y - 16}$$

For zero gradient $\frac{dy}{dx} = 0 \quad \Rightarrow \quad 6 - 2x = 0 \quad \Rightarrow \quad x = 3$

Substitute $x = 3$ into ① to give

$9 + 4y^2 - 18 - 16y + 21 = 0$

$\Rightarrow \quad 4y^2 - 16y + 12 = 0 \; [\div 4]$

$\Rightarrow \quad y^2 - 4y + 3 = 0$

$\Rightarrow \quad (y - 1)(y - 3) = 0$

$\Rightarrow \quad y = 1$ or 3

$\therefore$ the coordinates of the points of zero gradient are (3, 1) and (3, 3).

Solutionbank

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Differentiation

Exercise C, Question 1

Question:

Find $\frac{dy}{dx}$ for each of the following:

(a) $y = 3^x$

(b) $y = \left(\frac{1}{2}\right)^x$

(c) $y = xa^x$

(d) $y = \frac{2^x}{x}$

Solution:

(a) $y = 3^x$

$$\frac{dy}{dx} = 3^x \ln 3$$

(b) $y = \left(\frac{1}{2}\right)^x$

$$\frac{dy}{dx} = \left(\frac{1}{2}\right)^x \ln \frac{1}{2}$$

(c) $y = xa^x$

Use the product rule to give

$$\frac{dy}{dx} = xa^x \ln a + a^x \times 1 = a^x \left(x \ln a + 1 \right)$$

(d) $y = \frac{2^x}{x}$

Use the quotient rule to give

$$\frac{dy}{dx} = \frac{x \times 2^x \ln 2 - 2^x \times 1}{x^2} = \frac{2^x (x \ln 2 - 1)}{x^2}$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise C, Question 2

Question:

Find the equation of the tangent to the curve $y = 2^{x} + 2^{-x}$ at the point $\left(2 , 4 \frac{1}{4} \right)$.

Solution:

$y = 2^{x} + 2^{-x}$

$\frac{dy}{dx} = 2^{x} \ln 2 - 2^{-x} \ln 2$

When $x = 2$, $\frac{dy}{dx} = 4 \ln 2 - \frac{1}{4} \ln 2 = \frac{15}{4} \ln 2$

$\therefore$ the equation of the tangent at $\left(2 , 4 \frac{1}{4} \right)$ is

$y - 4 \frac{1}{4} = \frac{15}{4} \ln 2 \left(x - 2 \right)$

$\therefore 4y = (15 \ln 2) x + (17 - 30 \ln 2)$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise C, Question 3

Question:

A particular radioactive isotope has an activity R millicuries at time t days given by the equation $R = 200 \left(0.9 \right)^{t}$. Find the value of $\frac{dR}{dt}$, when $t = 8$.

Solution:

$R = 200 \left(0.9 \right)^{t}$

$\frac{dR}{dt} = 200 \times \ln 0.9 \times \left(0.9 \right)^{t}$

Substitute $t = 8$ to give

$\frac{dR}{dt} = -9.07$ (3 s.f.)

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise C, Question 4

Question:

The population of Cambridge was 37 000 in 1900 and was about 109 000 in 2000. Find an equation of the form $P = P_0 k^t$ to model this data, where t is measured as years since 1900. Evaluate $\frac{dP}{dt}$ in the year 2000. What does this value represent?

Solution:

$P = P_0 k^t$

When $t = 0$, $P = 37\ 000$

$\therefore 37\ 000 = P_0 \times k^0 = P_0 \times 1$

$\therefore P_0 = 37\ 000$

$\therefore P = 37\ 000 \left(k \right)^t$

When $t = 100$, $P = 109\ 000$

$\therefore 109\ 000 = 37\ 000 \left(k \right)^{100}$

$\therefore k^{100} = \frac{109\ 000}{37\ 000}$

$\therefore k = \sqrt[100]{\frac{109}{37}} \approx 1.01$

$\frac{dP}{dt} = 37\ 000 k^t \ln k$

When $t = 100$

$\frac{dP}{dt} = 37\ 000 \times \left(\frac{109}{37} \right) \times \ln k = 1000 \times 109 \times \frac{1}{100} \ln \frac{109}{37}$

$= 1178$ people per year

Rate of increase of the population during the year 2000.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise D, Question 1

Question:

Given that $V = \frac{1}{3}\pi r^3$ and that $\frac{\mathrm{d}V}{\mathrm{d}t} = 8$, find $\frac{\mathrm{d}r}{\mathrm{d}t}$ when $r = 3$.

Solution:

$$V = \frac{1}{3}\pi r^3$$

$$\therefore \frac{\mathrm{d}V}{\mathrm{d}r} = \pi r^2$$

Using the chain rule

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$\therefore 8 = \pi r^2 \times \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$\therefore \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{8}{\pi r^2}$$

When $r = 3$, $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{8}{9\pi}$

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 2

Question:

Given that $A = \frac{1}{4}\pi r^2$ and that $\frac{\mathrm{d}r}{\mathrm{d}t} = 6$, find $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $r = 2$.

Solution:

$$A = \frac{1}{4}\pi r^2$$

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{1}{2}\pi r$$

Using the chain rule

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2}\pi r \times 6 = 3\pi r$$

When $r = 2$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 6\pi$

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Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise D, Question 3

Question:

Given that $y = x\mathrm{e}^{x}$ and that $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$, find $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $x = 2$.

Solution:

$y = x\mathrm{e}^{x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^{x} + \mathrm{e}^{x} \times 1$

Using the chain rule

$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^{x} \left(x + 1 \right) \times 5$

When $x = 2$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 15\mathrm{e}^{2}$

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Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise D, Question 4

Question:

Given that $r = 1 + 3\cos\theta$ and that $\frac{d\theta}{dt} = 3$, find $\frac{dr}{dt}$ when $\theta = \frac{\pi}{6}$.

Solution:

$r = 1 + 3\cos\theta$

$\frac{dr}{d\theta} = -3\sin\theta$

Using the chain rule

$\frac{dr}{dt} = \frac{dr}{d\theta} \times \frac{d\theta}{dt} = -3\sin\theta \times 3 = -9\sin\theta$

When $\theta = \frac{\pi}{6}$, $\frac{dr}{dt} = \frac{-9}{2}$

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Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise E, Question 1

Question:

In a study of the water loss of picked leaves the mass *M* grams of a single leaf was measured at times *t* days after the leaf was picked. It was found that the rate of loss of mass was proportional to the mass *M* of the leaf.
Write down a differential equation for the rate of change of mass of the leaf.

Solution:

$\frac{\mathrm{d}M}{\mathrm{d}t}$ represents rate of change of mass.

$\therefore \frac{\mathrm{d}M}{\mathrm{d}t} \propto -M$, as rate of *loss* indicates a negative quantity.

$\therefore \frac{\mathrm{d}M}{\mathrm{d}t} = -kM$, where k is the positive constant of proportionality.

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise E, Question 2

Question:

A curve C has equation $y = \mathrm{f}(x)$, $y > 0$. At any point P on the curve, the gradient of C is proportional to the product of the x and the y coordinates of P.

The point A with coordinates (4, 2) is on C and the gradient of C at A is $\frac{1}{2}$.

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{16}$.

Solution:

The gradient of the curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} \propto xy$ (which is the *product* of x and y)

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = kxy$, where k is a constant of proportion.

When $x = 4$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$

$\therefore \frac{1}{2} = k \times 4 \times 2$

$\therefore k = \frac{1}{16}$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{16}$

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Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise E, Question 3

Question:

Liquid is pouring into a container at a constant rate of 30 $\text{cm}^3 \ \text{s}^{-1}$. At time t seconds liquid is leaking from the container at a rate of $\frac{2}{15}V \ \ \text{cm}^3\text{s}^{-1}$, where $V \ \ \text{cm}^3$ is the volume of liquid in the container at that time.

Show that $-15\frac{dV}{dt} = 2V - 450$

Solution:

Let the rate of increase of the volume of liquid be $\frac{dV}{dt}$.

Then $\frac{dV}{dt} = 30 - \frac{2}{15}V$

Multiply both sides by -15:

$-15\frac{dV}{dt} = 2V - 450$

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Differentiation

Exercise E, Question 4

Question:

An electrically charged body loses its charge Q coulombs at a rate, measured in coulombs per second, proportional to the charge Q.
Write down a differential equation in terms of Q and t where t is the time in seconds since the body started to lose its charge.

Solution:

The rate of change of the charge is $\frac{\mathrm{d}Q}{\mathrm{d}t}$.

$\therefore \frac{\mathrm{d}Q}{\mathrm{d}t} \propto -Q$, as the body is *losing* charge the negative sign is required.

$\therefore \frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ$, where k is the positive constant of proportion.

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Differentiation

Exercise E, Question 5

Question:

The ice on a pond has a thickness x mm at a time t hours after the start of freezing. The rate of increase of x is inversely proportional to the square of x. Write down a differential equation in terms of x and t.

Solution:

The rate of increase of x is $\frac{dx}{dt}$.

$\therefore \frac{dx}{dt} \propto \frac{1}{x^2}$, as there is an *inverse* proportion.

$\therefore \frac{dx}{dt} = \frac{k}{x^2}$, where k is the constant of proportion.

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Differentiation
Exercise E, Question 6

Question:

In another pond the amount of pondweed (P) grows at a rate proportional to the amount of pondweed already present in the pond. Pondweed is also removed by fish eating it at a constant rate of Q per unit of time.
Write down a differential equation relating P and t, where t is the time which has elapsed since the start of the observation.

Solution:

The rate of increase of pondweed is $\frac{dP}{dt}$.

This is proportional to P.

$$\therefore \ \frac{dP}{dt} \propto P$$

$$\therefore \ \frac{dP}{dt} = kP, \text{ where } k \text{ is a constant.}$$

But also pondweed is removed at a rate Q

$$\therefore \ \frac{dP}{dt} = kP - Q$$

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Differentiation

Exercise E, Question 7

Question:

A circular patch of oil on the surface of some water has radius *r* and the radius increases over time at a rate inversely proportional to the radius.
Write down a differential equation relating *r* and *t*, where *t* is the time which has elapsed since the start of the observation.

Solution:

The rate of increase of the radius is $\frac{\mathrm{d}r}{\mathrm{d}t}$.

$\therefore \ \frac{\mathrm{d}r}{\mathrm{d}t} \propto \frac{1}{r}$, as it is *inversely* proportional to the radius.

$\therefore \ \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r}$, where *k* is the constant of proportion.

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Differentiation
Exercise E, Question 8

Question:

A metal bar is heated to a certain temperature, then allowed to cool down and it is noted that, at time t, the rate of loss of temperature is proportional to the difference in temperature between the metal bar, θ, and the temperature of its surroundings θ_0.

Write down a differential equation relating θ and t.

Solution:

The rate of change of temperature is $\frac{d\theta}{dt}$.

$\therefore \frac{d\theta}{dt} \propto - \left(\theta - \theta_0 \right)$ The rate of *loss* indicates the negative sign.

$\therefore \frac{d\theta}{dt} = - k \left(\theta - \theta_0 \right)$, where k is the positive constant of proportion.

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Differentiation
Exercise E, Question 9

Question:

Fluid flows out of a cylindrical tank with constant cross section. At time t minutes, $t > 0$, the volume of fluid remaining in the tank is $V \text{m}^3$. The rate at which the fluid flows in $\text{m}^3 \text{ min}^{-1}$ is proportional to the square root of V. Show that the depth h metres of fluid in the tank satisfies the differential equation $\frac{\text{d}h}{\text{d}t} = -k\sqrt{h}$, where k is a positive constant.

Solution:

Let the rate of flow of fluid be $\frac{-\text{d}V}{\text{d}t}$, as fluid is flowing *out* of the tank, and the volume left in the tank is decreasing.

$$\therefore \frac{-\text{d}V}{\text{d}t} \propto \sqrt{V}$$

$\therefore \frac{\text{d}V}{\text{d}t} = -k'\sqrt{V}$, where k' is a positive constant.

But $V = Ah$, where A is the constant cross section.

$$\therefore \frac{\text{d}V}{\text{d}h} = A$$

Use the chain rule to find $\frac{\text{d}h}{\text{d}t}$:

$$\frac{\text{d}V}{\text{d}t} = \frac{\text{d}V}{\text{d}h} \times \frac{\text{d}h}{\text{d}t}$$

$$\therefore \frac{\text{d}h}{\text{d}t} = \frac{\text{d}V}{\text{d}t} \div \frac{\text{d}V}{\text{d}h} = \frac{-k'\sqrt{V}}{A}$$

But $V = Ah$,

$\therefore \frac{\text{d}h}{\text{d}t} = \frac{-k\sqrt{Ah}}{A} = \left(\frac{-k'}{\sqrt{A}}\right)\sqrt{h} = -k\sqrt{h}$, where $\frac{k'}{\sqrt{A}}$ is a positive constant.

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Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise E, Question 10

Question:

At time t seconds the surface area of a cube is A cm^2 and the volume is Vcm^3. The surface area of the cube is expanding at a constant rate 2 cm^{2}s^{-1}.

Show that $\frac{dV}{dt} = \frac{1}{2}V^{\frac{1}{3}}$.

Solution:

Rate of expansion of surface area is $\frac{dA}{dt}$.

Need $\frac{dV}{dt}$ so use the chain rule.

$$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

As $\frac{dA}{dt} = 2$, $\frac{dV}{dt} = 2\frac{dV}{dA}$ or $2 \div \left(\frac{dA}{dV}\right)$ ①

Let the cube have edge of length x cm.

Then $V = x^3$ and $A = 6x^2$.

Eliminate x to give $A = 6V^{\frac{2}{3}}$

$$\therefore \frac{dA}{dV} = 4V^{\frac{-1}{3}}$$

From ① $\frac{dV}{dt} = \frac{2}{4V^{-\frac{1}{3}}} = \frac{2V^{\frac{1}{3}}}{4} = \frac{1}{2}V^{\frac{1}{3}}$

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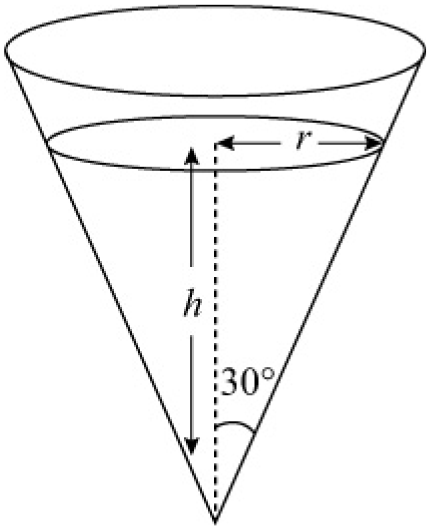
Differentiation

Exercise E, Question 11

Question:

An inverted conical funnel is full of salt. The salt is allowed to leave by a small hole in the vertex. It leaves at a constant rate of 6 $\text{cm}^3\ \text{s}^{-1}$.
Given that the angle of the cone between the slanting edge and the vertical is 30 degrees, show that the volume of the salt is $\frac{1}{9}\pi h^3$, where h is the height of salt at time t seconds.
Show that the rate of change of the height of the salt in the funnel is inversely proportional to h^2. Write down the differential equation relating h and t.

Solution:



Use $V = \frac{1}{3}\pi r^2 h$

As $\tan 30° = \frac{r}{h}$

$\therefore r = h \tan 30° = \frac{h}{\sqrt{3}}$

$\therefore V = \frac{1}{3}\pi \left(\frac{h^2}{3} \right) \times h = \frac{1}{9}\pi h^3$ ①

It is given that $\frac{dV}{dt} = -6$.

To find $\frac{dh}{dt}$ use the chain rule:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}h}{\mathrm{d}V} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}h}$$

From ① $\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi h^2$

$$\therefore \frac{\mathrm{d}h}{\mathrm{d}t} = -6 \div \frac{1}{3}\pi h^2$$

$$\therefore \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{-18}{\pi h^2}$$

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Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise F, Question 1

Question:

The curve C is given by the equations

$x = 4t - 3, y = \frac{8}{t^2}, t > 0$

where t is a parameter.
At A, $t = 2$. The line l is the normal to C at A.

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Hence find an equation of l. Ⓔ

Solution:

(a) $x = 4t - 3, y = \frac{8}{t^2} = 8t^{-2}$

$\therefore \frac{dx}{dt} = 4$ and $\frac{dy}{dt} = -16t^{-3}$

$\therefore \frac{dy}{dx} = \frac{-16t^{-3}}{4} = \frac{-4}{t^3}$

(b) When $t = 2$ the curve has gradient $\frac{-4}{8} = -\frac{1}{2}$.

$\therefore$ the normal has gradient 2.

Also the point A has coordinates (5, 2)

$\therefore$ the equation of the normal is

$y - 2 = 2 (x - 5)$

i.e. $y = 2x - 8$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise F, Question 2

Question:

The curve *C* is given by the equations $x = 2t$, $y = t^2$, where *t* is a parameter. Find an equation of the normal to *C* at the point *P* on *C* where $t = 3$.

Solution:

$x = 2t, y = t^2$

$\frac{dx}{dt} = 2, \frac{dy}{dt} = 2t$

$\therefore \frac{dy}{dx} = \frac{2t}{2} = t$

When $t = 3$ the gradient of the curve is 3.

$\therefore$ the gradient of the normal is $-\frac{1}{3}$.

Also at the point *P* where $t = 3$, the coordinates are (6, 9).

$\therefore$ the equation of the normal is

$y - 9 = -\frac{1}{3}\left(x - 6\right)$

i.e. $3y - 27 = -x + 6$

$\therefore 3y + x = 33$

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Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise F, Question 3

Question:

The curve C has parametric equations

$x = t^3, y = t^2, t > 0$

Find an equation of the tangent to C at A (1, 1). E

Solution:

$x = t^3, y = t^2$

$\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 2t$

$\therefore \frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t}$

At the point (1, 1) the value of t is 1.

$\therefore$ the gradient of the curve is $\frac{2}{3}$, which is also the gradient of the tangent.

$\therefore$ the equation of the tangent is

$y - 1 = \frac{2}{3}\left(x - 1 \right)$

i.e. $y = \frac{2}{3}x + \frac{1}{3}$

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise F, Question 4

Question:

A curve C is given by the equations
$x = 2\cos t + \sin 2t,\ y = \cos t - 2\sin 2t,\ 0 < t < \pi$
where t is a parameter.

(a) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of t.

(b) Find the value of $\frac{dy}{dx}$ at the point P on C where $t = \frac{\pi}{4}$.

(c) Find an equation of the normal to the curve at P. E

Solution:

(a) $x = 2\cos t + \sin 2t,\ y = \cos t - 2\sin 2t$

$$\frac{dx}{dt} = -2\sin t + 2\cos 2t,\ \frac{dy}{dt} = -\sin t - 4\cos 2t$$

(b) $\therefore\ \frac{dy}{dx} = \frac{-\sin t - 4\cos 2t}{-2\sin t + 2\cos 2t}$

When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{\frac{-1}{\sqrt{2}} - 0}{\frac{-2}{\sqrt{2}} + 0} = \frac{1}{2}$

(c) $\therefore$ the gradient of the normal at the point P, where $t = \frac{\pi}{4}$, is -2.

The coordinates of P are found by substituting $t = \frac{\pi}{4}$ into the parametric equations, to give

$x = \frac{2}{\sqrt{2}} + 1,\ y = \frac{1}{\sqrt{2}} - 2$

$\therefore$ the equation of the normal is

$$y - \left(\frac{1}{\sqrt{2}} - 2\right) = -2\left[x - \left(\frac{2}{\sqrt{2}} + 1\right)\right]$$

i.e. $y - \frac{1}{\sqrt{2}} + 2 = -2x + \frac{4}{\sqrt{2}} + 2$

$\therefore y + 2x = \frac{5\sqrt{2}}{2}$

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise F, Question 5

Question:

A curve is given by $x = 2t + 3$, $y = t^3 - 4t$, where t is a parameter. The point A has parameter $t = -1$ and the line l is the tangent to C at A. The line l also cuts the curve at B.

(a) Show that an equation for l is $2y + x = 7$.

(b) Find the value of t at B. Ⓔ

Solution:

(a) $x = 2t + 3$, $y = t^3 - 4t$

At point A, $t = -1$.

$\therefore$ the coordinates of the point A are (1, 3)

$$\frac{dx}{dt} = 2 \text{ and } \frac{dy}{dt} = 3t^2 - 4$$

$$\therefore \frac{dy}{dx} = \frac{3t^2 - 4}{2}$$

At the point A, $\frac{dy}{dx} = -\frac{1}{2}$

$\therefore$ the gradient of the tangent at A is $-\frac{1}{2}$.

$\therefore$ the equation of the tangent at A is

$$y - 3 = -\frac{1}{2}\left(x - 1\right)$$

i.e. $2y - 6 = -x + 1$

$\therefore 2y + x = 7$

(b) This line cuts the curve at the point B.

$\therefore 2\left(t^3 - 4t\right) + \left(2t + 3\right) = 7$ gives the values of t at A and B.

i.e. $2t^3 - 6t - 4 = 0$

At A, $t = -1$

$\therefore \left(t + 1\right)$ is a root of this equation

$$2t^3 - 6t - 4 = \left(t + 1 \right) \left(2t^2 - 2t - 4 \right) = \left(t + 1 \right) \left(t + 1 \right) \left(2t - 4 \right) = 2 (t + 1) ^2 \left(t - 2 \right)$$

So when the line meets the curve, $t = -1$ (repeated root because the line touches the curve) or $t = 2$.

$\therefore$ at the point B, $t = 2$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise F, Question 6

Question:

A Pancho car has value £V at time t years. A model for V assumes that the rate of decrease of V at time t is proportional to V. Form an appropriate differential equation for V.

Solution:

$\frac{\mathrm{d}V}{\mathrm{d}t}$ is the rate of change of V.

$\frac{\mathrm{d}V}{\mathrm{d}t} \propto -V$, as a decrease indicates a negative quantity.

$\therefore \frac{\mathrm{d}V}{\mathrm{d}t} = -kV$, where k is a positive constant of proportionality.

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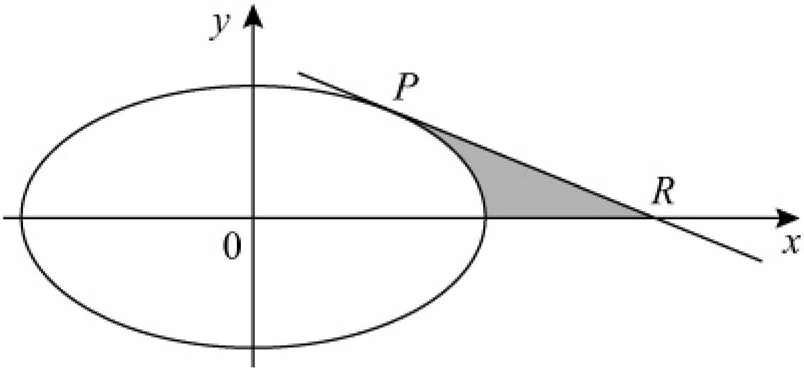
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise F, Question 7

Question:

The curve shown has parametric equations
$x = 5\cos\theta, y = 4\sin\theta, 0 \leq \theta < 2\pi$



(a) Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$.

(b) Find an equation of the tangent to the curve at the point P.

(c) Find the coordinates of the point R where this tangent meets the x-axis. **E**

Solution:

(a) $x = 5\cos\theta, y = 4\sin\theta$

$$\frac{dx}{d\theta} = -5\sin\theta \text{ and } \frac{dy}{d\theta} = 4\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{-4\cos\theta}{5\sin\theta}$$

At the point P, where $\theta = \frac{\pi}{4}, \frac{dy}{dx} = \frac{-4}{5}$.

(b) At the point P, $x = \frac{5}{\sqrt{2}}$ and $y = \frac{4}{\sqrt{2}}$.

$\therefore$ the equation of the tangent at P is

$$y - \frac{4}{\sqrt{2}} = \frac{-4}{5}\left(x - \frac{5}{\sqrt{2}}\right)$$

i.e. $y - \frac{4}{\sqrt{2}} = \frac{-4}{5}x + \frac{4}{\sqrt{2}}$

$\therefore y = \frac{-4}{5}x + \frac{8}{\sqrt{2}}$

Multiply equation by 5 and rationalise the denominator of the surd:
$5y + 4x = 20\sqrt{2}$

(c) The tangent meets the x-axis when $y = 0$.
$\therefore x = 5\sqrt{2}$ and R has coordinates $(5\sqrt{2}, 0)$.

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise F, Question 8

Question:

The curve C has parametric equations

$x = 4\cos 2t,\ y = 3\sin t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$

A is the point $\left(2,\ 1\frac{1}{2}\right)$, and lies on C.

(a) Find the value of t at the point A.

(b) Find $\frac{dy}{dx}$ in terms of t.

(c) Show that an equation of the normal to C at A is $6y - 16x + 23 = 0$.
The normal at A cuts C again at the point B.

(d) Find the y-coordinate of the point B.

Ⓔ

Solution:

(a) $x = 4\cos 2t$ and $y = 3\sin t$

A is the point $\left(2,\ 1\frac{1}{2}\right)$ and so

$4\cos 2t = 2$ and $3\sin t = 1\frac{1}{2}$

$\therefore \cos 2t = \frac{1}{2}$ and $\sin t = \frac{1}{2}$

As $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $t = \frac{\pi}{6}$ at the point A.

(b) $\frac{dx}{dt} = -8\sin 2t$ and $\frac{dy}{dt} = 3\cos t$

$$\therefore \frac{dy}{dx} = \frac{3\cos t}{-8\sin 2t}$$

$$= \frac{-3\cos t}{16\sin t\cos t} \qquad \text{(using the double angle formula)}$$

$$= \frac{-3}{16 \sin t}$$

$$= \frac{-3}{16} \text{cosec } t$$

(c) When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{-3}{8}$

$\therefore$ the gradient of the normal at the point A is $\frac{8}{3}$.

$\therefore$ the equation of the normal is

$$y - 1\frac{1}{2} = \frac{8}{3}\left(x - 2 \right)$$

Multiply equation by 6:

$6y - 9 = 16x - 32$

$\therefore 6y - 16x + 23 = 0$

(d) The normal cuts the curve when

$6 (3 \sin t) - 16 (4 \cos 2t) + 23 = 0$

$\therefore 18 \sin t - 64 \cos 2t + 23 = 0.$

$\therefore 18 \sin t - 64 (1 - 2 \sin^2 t) + 23 = 0$ (using the double angle formula)

$\therefore 128 \sin^2 t + 18 \sin t - 41 = 0$

But $\sin t = \frac{1}{2}$ is one solution of this equation, as point A lies on the line and on the curve.

$\therefore 128 \sin^2 t + 18 \sin t - 41 = (2 \sin t - 1) (64 \sin t + 41)$

$\therefore (2 \sin t - 1) (64 \sin t + 41) = 0$

$\therefore$ at point B, $\sin t = \frac{-41}{64}$

$\therefore$ the y coordinate of point B is $\frac{-123}{64}$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

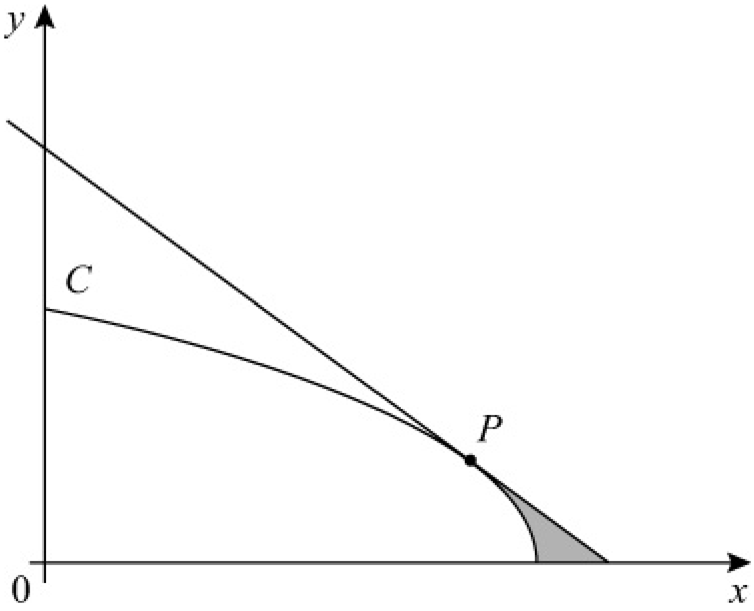
Differentiation
Exercise F, Question 9

Question:

The diagram shows the curve C with parametric equations

$x = a \ \sin^2 t, y = a \cos t, 0 \le t \le \frac{1}{2}\pi$

where a is a positive constant. The point P lies on C and has coordinates $\left(\frac{3}{4}a \ , \ \frac{1}{2}a \right)$.



(a) Find $\frac{dy}{dx}$, giving your answer in terms of t.

(b) Find an equation of the tangent at P to C.

E

Solution:

(a) $x = a \ \sin^2 t, y = a \cos t$

$$\frac{dx}{dt} = 2a \sin t \cos t \text{ and } \frac{dy}{dt} = -a \sin t$$

$$\therefore \frac{dy}{dx} = \frac{-a \sin t}{2a \sin t \cos t} = \frac{-1}{2 \cos t} = \frac{-1}{2} \sec \ t$$

(b) P is the point $\left(\frac{3}{4}a , \frac{1}{2}a \right)$ and lies on the curve.

$\therefore a \ \sin^2 t = \frac{3}{4}a$ and $a \cos t = \frac{1}{2}a$

$\therefore \sin t = \pm \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$ and $0 \leq t \leq \frac{1}{2}\pi$

$\therefore t = \frac{\pi}{3}$

$\therefore$ the gradient of the curve at point P is $- \frac{1}{2} \sec \frac{\pi}{3} = - 1$.

The equation of the tangent at P is

$y - \frac{1}{2}a = - 1 \left(x - \frac{3}{4}a \right)$

$\therefore y + x = \frac{1}{2}a + \frac{3}{4}a$

Multiply equation by 4 to give $4y + 4x = 5a$

Solutionbank
Edexcel AS and A Level Modular Mathematics

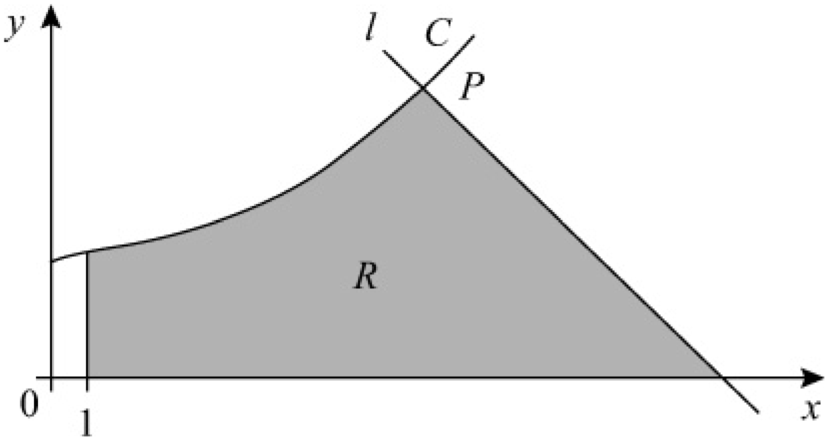
Differentiation
Exercise F, Question 10

Question:

This graph shows part of the curve *C* with parametric equations

$x = (t + 1)^2, y = \frac{1}{2}t^3 + 3, t \geq -1$

P is the point on the curve where $t = 2$. The line *l* is the normal to *C* at *P*.
Find the equation of *l*.



E

Solution:

$x = (t + 1)^2, y = \frac{1}{2}t^3 + 3$

$\frac{dx}{dt} = 2 \left(t + 1 \right)$ and $\frac{dy}{dt} = \frac{3}{2}t^2$

$$\therefore \frac{dy}{dx} = \frac{(\frac{3}{2}t^2)}{2 (t + 1)} = \frac{3t^2}{4 (t + 1)}$$

When $t = 2$, $\frac{dy}{dx} = \frac{3 \times 4}{4 \times 3} = 1$

The gradient of the normal at the point *P* where $t = 2$, is -1.
The coordinates of *P* are (9, 7).
∴ the equation of the normal is
$y - 7 = -1 (x - 9)$
i.e. $y - 7 = -x + 9$

$\therefore y + x = 16$

Solutionbank
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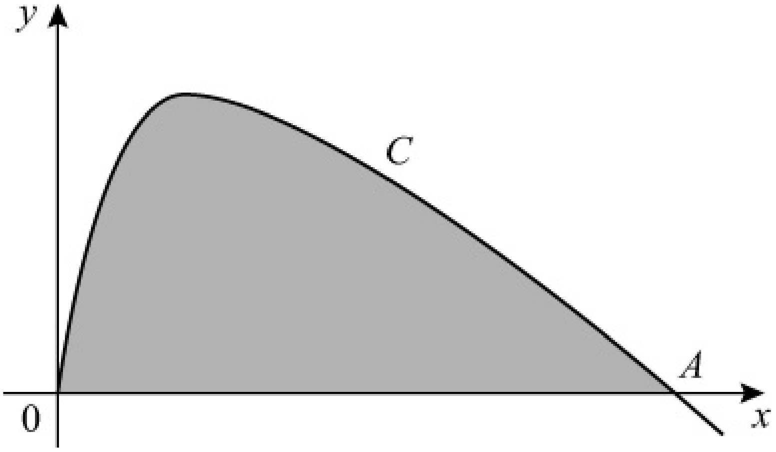
Differentiation
Exercise F, Question 11

Question:

The diagram shows part of the curve C with parametric equations
$x = t^2, y = \sin 2t, t \geq 0$
The point A is an intersection of C with the x-axis.



(a) Find, in terms of π, the x-coordinate of A.

(b) Find $\frac{dy}{dx}$ in terms of t, $t > 0$.

(c) Show that an equation of the tangent to C at A is $4x + 2\pi y = \pi^2$.

E

Solution:

(a) $x = t^2$ and $y = \sin 2t$
At the point A, $y = 0$.

$\therefore \sin 2t = 0$

$\therefore 2t = \pi$

$\therefore t = \frac{\pi}{2}$

The point A is $\left(\frac{\pi^2}{4} , \quad 0 \right)$

(b) $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2\cos 2t$

$\therefore \frac{dy}{dx} = \frac{\cos 2t}{t}$

(c) At point A, $\frac{dy}{dx} = \frac{-1}{\left(\frac{\pi}{2}\right)} = \frac{-2}{\pi}$

$\therefore$ the gradient of the tangent at A is $\frac{-2}{\pi}$.

$\therefore$ the equation of the tangent at A is

$$y - 0 = \frac{-2}{\pi}\left(x - \frac{\pi^2}{4}\right)$$

i.e. $y = \frac{-2x}{\pi} + \frac{\pi}{2}$

Multiply equation by 2π to give

$2\pi y + 4x = \pi^2$

Solutionbank

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Differentiation

Exercise F, Question 12

Question:

Find the gradient of the curve with equation
$5x^2 + 5y^2 - 6xy = 13$
at the point (1, 2).

E

Solution:

$5x^2 + 5y^2 - 6xy = 13$

Differentiate implicitly with respect to x:

$$10x + 10y \frac{dy}{dx} - \left(6x \frac{dy}{dx} + 6y \right) = 0$$

$$\therefore \frac{dy}{dx} \left(10y - 6x \right) + 10x - 6y = 0$$

At the point (1, 2)

$$\frac{dy}{dx} \left(14 \right) + 10 - 12 = 0$$

$$\therefore \frac{dy}{dx} = \frac{2}{14} = \frac{1}{7}$$

Solutionbank

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Differentiation
Exercise F, Question 13

Question:

Given that $e^{2x} + e^{2y} = xy$, find $\frac{dy}{dx}$ in terms of x and y.

Solution:

$e^{2x} + e^{2y} = xy$
Differentiate with respect to x:

$$2e^{2x} + 2e^{2y}\frac{dy}{dx} = x\frac{dy}{dx} + y \times 1$$

$$\therefore 2e^{2y}\frac{dy}{dx} - x\frac{dy}{dx} = y - 2e^{2x}$$

$$\therefore \frac{dy}{dx}\left(2e^{2y} - x\right) = y - 2e^{2x}$$

$$\therefore \frac{dy}{dx} = \frac{y - 2e^{2x}}{2e^{2y} - x}$$

Solutionbank

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Differentiation

Exercise F, Question 14

Question:

Find the coordinates of the turning points on the curve $y^3 + 3xy^2 - x^3 = 3$.

Solution:

$y^3 + 3xy^2 - x^3 = 3$

Differentiate with respect to x:

$$3y^2 \frac{dy}{dx} + \left(3x \times 2y \frac{dy}{dx} + y^2 \times 3 \right) - 3x^2 = 0$$

$$\therefore \frac{dy}{dx} \left(3y^2 + 6xy \right) = 3x^2 - 3y^2$$

$$\therefore \frac{dy}{dx} = \frac{3 (x^2 - y^2)}{3y (y + 2x)} = \frac{x^2 - y^2}{y (y + 2x)}$$

When $\frac{dy}{dx} = 0$, $x^2 = y^2$, i.e. $x = \pm y$

When $x = + y$, $y^3 + 3y^3 - y^3 = 3 \Rightarrow 3y^3 = 3 \Rightarrow y = 1$ and $x = 1$

When $x = - y$, $y^3 - 3y^3 + y^3 = 3 \Rightarrow - y^3 = 3 \Rightarrow y = \sqrt[3]{(- 3)}$ and $x = - \sqrt[3]{(- 3)}$

$\therefore$ the coordinates are (1, 1) and $\left(- \sqrt[3]{(- 3)} , \sqrt[3]{(- 3)} \right)$.

Solutionbank

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Differentiation
Exercise F, Question 15

Question:

Given that $y\left(x+y\right)=3$, evaluate $\frac{dy}{dx}$ when $y=1$.

Solution:

$y\left(x+y\right)=3$

$\therefore yx+y^2=3$

Differentiate with respect to x:

$\left(y+x\frac{dy}{dx}\right)+2y\frac{dy}{dx}=0$ ①

When $y=1$, $1\left(x+1\right)=3$ (from original equation)

$\therefore x=2$

Substitute into ①:

$1+2\frac{dy}{dx}+2\frac{dy}{dx}=0$

$\therefore 4\frac{dy}{dx}=-1$

i.e. $\frac{dy}{dx}=\frac{-1}{4}$

Solutionbank

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Differentiation

Exercise F, Question 16

Question:

(a) If $(1+x)(2+y)=x^2+y^2$, find $\frac{dy}{dx}$ in terms of x and y.

(b) Find the gradient of the curve $(1+x)(2+y)=x^2+y^2$ at each of the two points where the curve meets the y-axis.

(c) Show also that there are two points at which the tangents to this curve are parallel to the y-axis.

Solution:

(a) $(1+x)(2+y)=x^2+y^2$

Differentiate with respect to x:

$$\left(1+x\right)\left(\frac{dy}{dx}\right)+\left(2+y\right)\left(1\right)=2x+2y\frac{dy}{dx}$$

$$\therefore \left(1+x-2y\right)\frac{dy}{dx}=2x-y-2$$

$$\therefore \frac{dy}{dx}=\frac{2x-y-2}{1+x-2y}$$

(b) When the curve meets the y-axis, $x=0$.

Put $x=0$ in original equation $(1+x)(2+y)=x^2+y^2$.

Then $2+y=y^2$

i.e. $y^2-y-2=0$

$\Rightarrow \quad (y-2)(y+1)=0$

$\therefore y=2$ or $y=-1$ when $x=0$

At (0, 2), $\frac{dy}{dx}=\frac{-4}{-3}=\frac{4}{3}$

At $(0,\ -1)$, $\frac{dy}{dx}=\frac{-1}{3}$

(c) When the tangent is parallel to the y-axis it has infinite gradient and as

$$\frac{dy}{dx} = \frac{2x - y - 2}{1 + x - 2y}$$

So $1 + x - 2y = 0$

Substitute $1 + x = 2y$ into the equation of the curve:

$$2y (2 + y) = (2y - 1) ^{2} + y^{2}$$

$$2y^{2} + 4y = 4y^{2} - 4y + 1 + y^{2}$$

$$3y^{2} - 8y + 1 = 0$$

$$y = \frac{8 \pm \sqrt{64 - 12}}{6} = \frac{4 \pm \sqrt{13}}{3}$$

When $y = \frac{4 + \sqrt{13}}{3}$, $x = \frac{5 + 2\sqrt{13}}{3}$

When $y = \frac{4 - \sqrt{13}}{3}$, $x = \frac{5 - 2\sqrt{13}}{3}$

$\therefore$ there are two points at which the tangents are parallel to the y-axis.

They are $\left(\frac{5 + 2\sqrt{13}}{3}, \frac{4 + \sqrt{13}}{3} \right)$ and $\left(\frac{5 - 2\sqrt{13}}{3}, \frac{4 - \sqrt{13}}{3} \right)$.

Solutionbank

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Differentiation

Exercise F, Question 17

Question:

A curve has equation $7x^2 + 48xy - 7y^2 + 75 = 0$. A and B are two distinct points on the curve and at each of these points the gradient of the curve is equal to $\frac{2}{11}$. Use implicit differentiation to show that $x + 2y = 0$ at the points A and B.

Solution:

$7x^2 + 48xy - 7y^2 + 75 = 0$

Differentiate with respect to x (implicit differentiation):

$$14x + \left(48x \frac{\mathrm{d}y}{\mathrm{d}x} + 48y \right) - 14y \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{11}$

$$\therefore 14x + 48x \times \frac{2}{11} + 48y - 14y \times \frac{2}{11} = 0$$

Multiply equation by 11,

then $154x + 96x + 528y - 28y = 0$

$\therefore 250x + 500y = 0$

i.e. $x + 2y = 0$, after division by 250.

Solutionbank

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Differentiation

Exercise F, Question 18

Question:

Given that $y = x^x$, $x > 0$, $y > 0$, by taking logarithms show that

$$\frac{dy}{dx} = x^x \left(1 + \ln x \right)$$

Solution:

$y = x^x$

Take natural logs of both sides:

$\ln y = \ln x^x$

$\therefore \ln y = x \ln x$ Property of lns

Differentiate with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \ln x \times 1$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

$$\therefore \frac{dy}{dx} = y \left(1 + \ln x \right)$$

But $y = x^x$

$$\therefore \frac{dy}{dx} = x^x \left(1 + \ln x \right)$$

Solutionbank

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Differentiation

Exercise F, Question 19

Question:

(a) Given that $x = 2^t$, by using logarithms prove that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2^t \ln 2$$

A curve C has parametric equations $x = 2^t$, $y = 3t^2$. The tangent to C at the point with coordinates (2, 3) cuts the x-axis at the point P.

(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

(c) Calculate the x-coordinate of P, giving your answer to 3 decimal places.

E

Solution:

(a) Given $x = 2^t$

Take natural logs of both sides:

$$\ln x = \ln 2^t = t \ln 2$$

Differentiate with respect to t:

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} = \ln 2$$

$$\therefore \frac{\mathrm{d}x}{\mathrm{d}t} = x \ln 2 = 2^t \ln 2$$

(b) $x = 2^t$, $y = 3t^2$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2^t \ln 2,\ \frac{\mathrm{d}y}{\mathrm{d}t} = 6t$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6t}{2^t \ln 2}$$

(c) At the point (2, 3), $t = 1$.

The gradient of the curve at (2, 3) is $\frac{6}{2 \ln 2}$.

$\therefore$ the equation of the tangent is

$$y - 3 = \frac{6}{2 \ln 2} \left(x - 2 \right)$$

i.e. $y = \frac{3}{\ln 2} x - \frac{6}{\ln 2} + 3$

The tangent meets the x-axis when $y = 0$.

$$\therefore \frac{3}{\ln 2} x = \frac{6}{\ln 2} - 3$$

$$\therefore x = 2 - \ln 2 = 1.307 \text{ (3 decimal places)}$$

Solutionbank

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Differentiation
Exercise F, Question 20

Question:

(a) Given that $a^x \equiv e^{kx}$, where a and k are constants, $a > 0$ and $x \in \mathbb{R}$, prove that $k = \ln a$.

(b) Hence, using the derivative of e^{kx}, prove that when $y = 2^x$

$$\frac{dy}{dx} = 2^x \ln 2.$$

(c) Hence deduce that the gradient of the curve with equation $y = 2^x$ at the point (2, 4) is ln 16.

Solution:

(a) $a^x = e^{kx}$

Take lns of both sides:

$\ln a^x = \ln e^{kx}$

i.e. $x \ln a = kx$

As this is true for all values of x, $k = \ln a$.

(b) Therefore, $2^x = e^{\ln 2 \times x}$

When $y = 2^x = e^{\ln 2 \times x}$

$$\frac{dy}{dx} = \ln 2 \ e^{\ln 2 \times x} = \ln 2 \times 2^x$$

(c) At the point (2, 4), $x = 2$.

$\therefore$ the gradient of the curve is

$$2^2 \ln 2$$
$$= 4 \ln 2$$
$$= \ln 2^4 \quad \text{(property of logs)}$$
$$= \ln 16$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise F, Question 21

Question:

A population P is growing at the rate of 9% each year and at time t years may be approximated by the formula

$P = P_0 (1.09) ^t, t \geq 0$

where P is regarded as a continuous function of t and P_0 is the starting population at time $t = 0$.

(a) Find an expression for t in terms of P and P_0.

(b) Find the time T years when the population has doubled from its value at $t = 0$,
giving your answer to 3 significant figures.

(c) Find, as a multiple of P_0, the rate of change of population $\frac{dP}{dt}$ at time $t = T$. E

Solution:

(a) $P = P_0 (1.09) ^t$

Take natural logs of both sides:

$\ln P = \ln [P_0 (1.09) ^t] = \ln P_0 + t \ln 1.09$

$\therefore t \ln 1.09 = \ln P - \ln P_0$

$$\Rightarrow \quad t = \frac{\ln P - \ln P_0}{\ln 1.09} \quad \text{or} \quad \frac{\ln \left(\frac{P}{P_0} \right)}{\ln 1.09}$$

(b) When $P = 2P_0$, $t = T$.

$\therefore T = \frac{\ln 2}{\ln 1.09} = 8.04$ (to 3 significant figures)

(c) $\frac{dP}{dt} = P_0 (1.09) ^t \ln 1.09$

When $t = T$, $P = 2P_0$ so $(1.09) ^T = 2$ and

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P_0 \times 2 \times \ln\ 1.09$$

$$= \ln\ \left(1.09^2 \right)\ \times P_0 = \ln \left(1.1881 \right)\ \times P_0$$

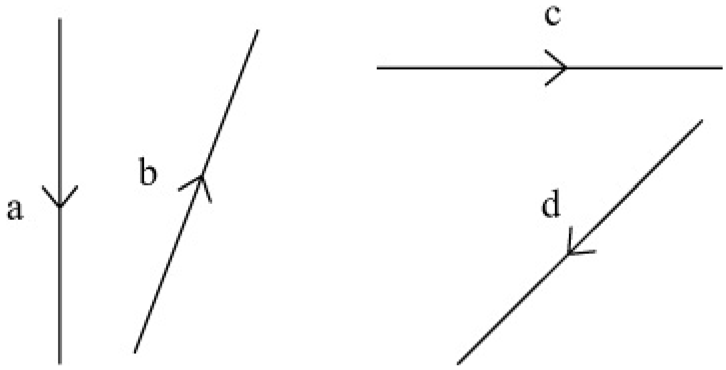
$= 0.172P_0$ (to 3 significant figures)

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

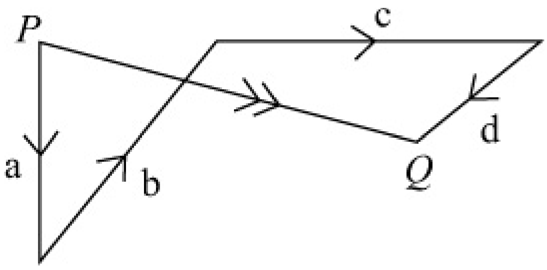
Vectors
Exercise A, Question 1

Question:

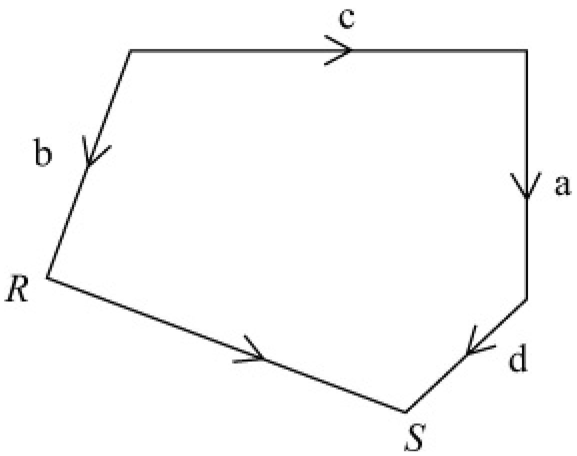
The diagram shows the vectors **a**, **b**, **c** and **d**. Draw a diagram to illustrate the vector addition a + b + c + d.



Solution:



a + b + c + d = PQ
(Vector goes from the start of **a** to the finish of **d**).
The vectors could be added in a different order,
e.g. b + c + a + d:



Here b + c + a + d = RS

(RS = PQ)

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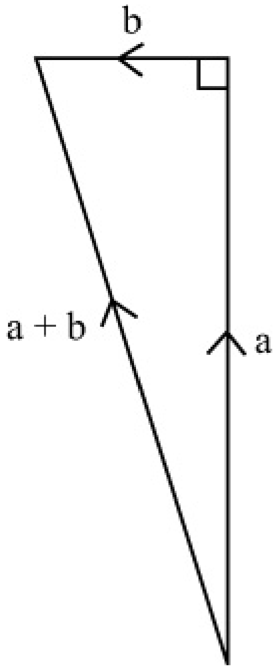
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise A, Question 2

Question:

The vector **a** is directed due north and $|a| = 24$. The vector **b** is directed due west and $|b| = 7$. Find $|a + b|$.

Solution:



$|a| = 24$
$|b| = 7$
$|a + b|^2 = 24^2 + 7^2 = 625$
$\therefore |a + b| = 25$

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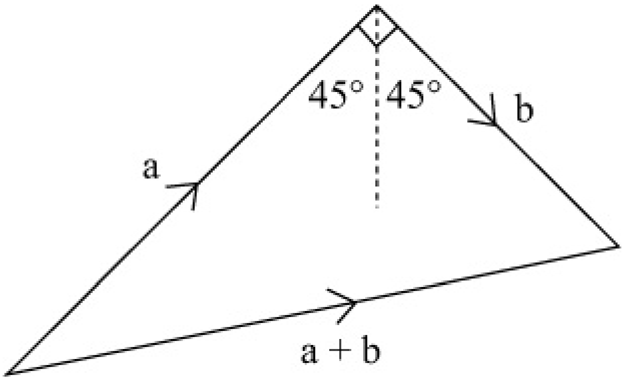
Vectors

Exercise A, Question 3

Question:

The vector **a** is directed north-east and $|a| = 20$. The vector **b** is directed south-east and $|b| = 13$. Find $|a + b|$.

Solution:



$|a| = 20$
$|b| = 13$
$|a + b|^2 = 20^2 + 13^2 = 569$
$|a + b| = \sqrt{569} = 23.9$ (3 s.f.)

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Vectors
Exercise A, Question 4

Question:

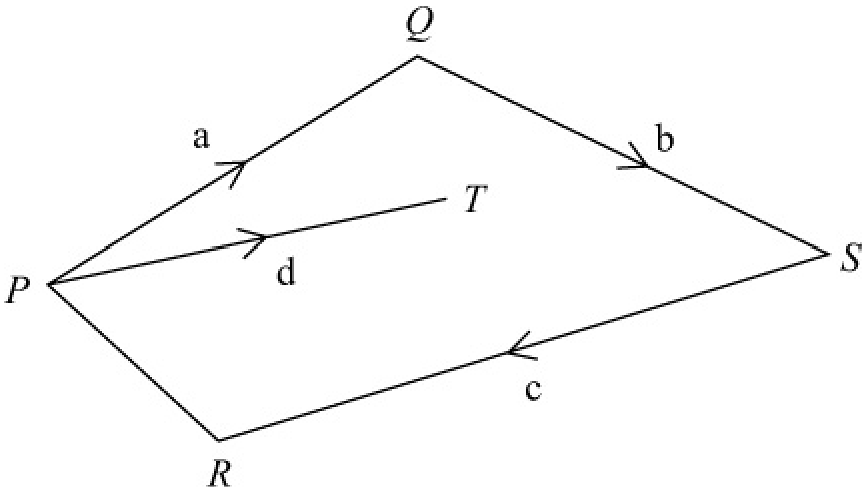
In the diagram, PQ = a, QS = b, SR = c and PT = d. Find in terms of **a**, **b**, **c** and **d**:

(a) QT

(b) PR

(c) TS

(d) TR



Solution:

(a) QT = QP + PT = − a + d

(b) PR = PQ + QS + SR = a + b + c

(c) TS = TP + PQ + QS = − d + a + b = a + b − d

(d) TR = TP + PR = − d + (a + b + c) = a + b + c − d

Solutionbank 4

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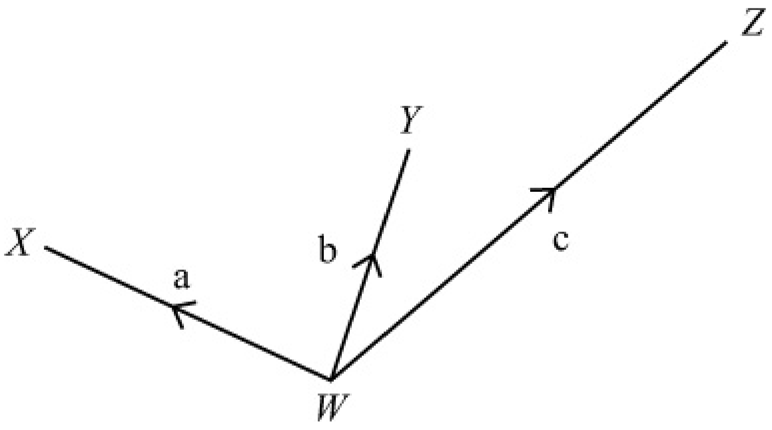
Vectors

Exercise A, Question 5

Question:

In the diagram, WX = a, WY = b and WZ = c. It is given that XY = YZ. Prove that a + c = 2b.
(2**b** is equivalent to b + b).



Solution:

XY = XW + WY = − a + b
YZ = YW + WZ = − b + c
Since XY = YZ,
− a + b = − b + c
b + b = a + c
a + c = 2b

Solutionbank 4
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Vectors
Exercise B, Question 1

Question:

In the triangle PQR, PQ = 2a and QR = 2b. The mid-point of *PR* is *M*.

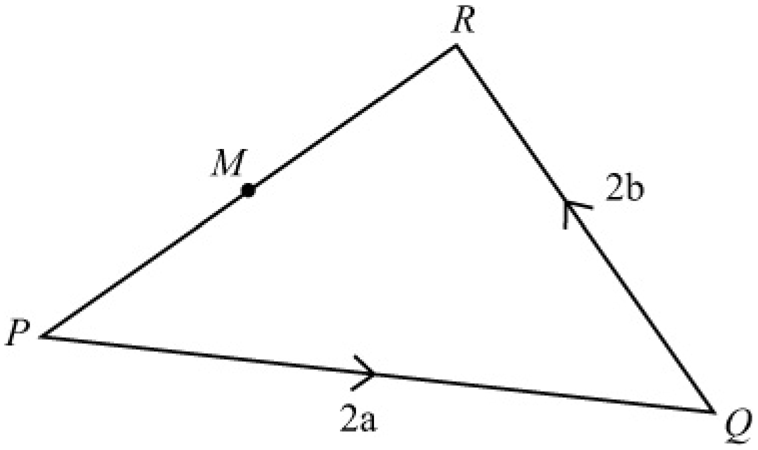
Find, in terms of **a** and **b**:

(a) PR

(b) PM

(c) QM.

Solution:



(a) PR = PQ + QR = 2a + 2b

(b) PM = $\frac{1}{2}$PR = $\frac{1}{2}\left(2a + 2b \right)$ = a + b

(c) QM = QP + PM = − 2a + a + b = − a + b

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Vectors
Exercise B, Question 2

Question:

ABCD is a trapezium with *AB* parallel to *DC* and DC = 3AB. *M* is the mid-point of *DC*, AB = a and BC = b.
Find, in terms of **a** and **b**:

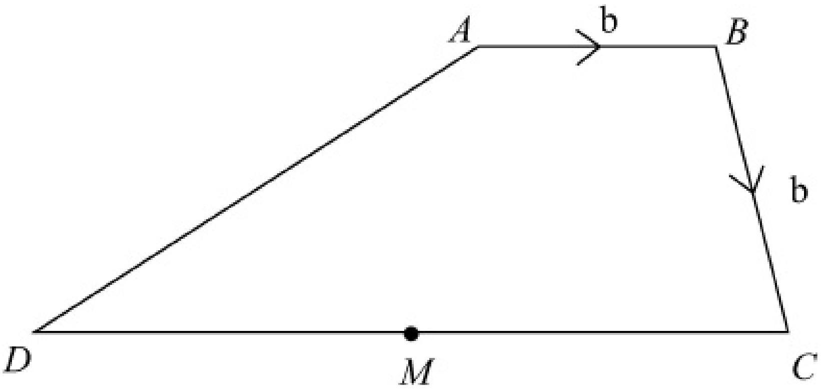
(a) AM

(b) BD

(c) MB

(d) DA.

Solution:



Since DC = 3AB, DC = 3a

Since *M* is the mid-point of *DC*, DM = MC = $\frac{3}{2}$a

(a) AM = AB + BC + CM = a + b − $\frac{3}{2}$a = − $\frac{1}{2}$a + b

(b) BD = BC + CD = b − 3a

(c) MB = MC + CB = $\frac{3}{2}$a − b

(d) DA = DC + CB + BA = 3a − b − a = 2a − b

Solutionbank 4

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Vectors

Exercise B, Question 3

Question:

In each part, find whether the given vector is parallel to a − 3b:

(a) 2a − 6b

(b) 4a − 12b

(c) a + 3b

(d) 3b − a

(e) 9b − 3a

(f) $\frac{1}{2}a - \frac{2}{3}b$

Solution:

(a) 2a − 6b = 2 (a − 3b)
Yes, parallel to a − 3b.

(b) 4a − 12b = 4 (a − 3b)
Yes, parallel to a − 3b.

(c) a + 3b is not parallel to a − 3b

(d) 3b − a = − 1 (a − 3b)
Yes, parallel to a − 3b.

(e) 9b − 3a = − 3 (a − 3b)
Yes, parallel to a − 3b.

(f) $\frac{1}{2}a - \frac{2}{3}b = \frac{1}{2}\left(a - \frac{4}{3}b \right)$

No, not parallel to a − 3b.

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise B, Question 4

Question:

The non-zero vectors **a** and **b** are not parallel. In each part, find the value of λ and the value of μ:

(a) $a + 3b = 2\lambda a - \mu b$

(b) $(\lambda + 2)a + (\mu - 1)b = 0$

(c) $4\lambda a - 5b - a + \mu b = 0$

(d) $(1 + \lambda)a + 2\lambda b = \mu a + 4\mu b$

(e) $(3\lambda + 5)a + b = 2\mu a + (\lambda - 3)b$

Solution:

(a) $a + 3b = 2\lambda a - \mu b$
$1 = 2\lambda$ and $3 = -\mu$
$\lambda = \frac{1}{2}$ and $\mu = -3$

(b) $(\lambda + 2)a + (\mu - 1)b = 0$
$\lambda + 2 = 0$ and $\mu - 1 = 0$
$\lambda = -2$ and $\mu = 1$

(c) $4\lambda a - 5b - a + \mu b = 0$
$4\lambda - 1 = 0$ and $-5 + \mu = 0$
$\lambda = \frac{1}{4}$ and $\mu = 5$

(d) $(1 + \lambda)a + 2\lambda b = \mu a + 4\mu b$
$1 + \lambda = \mu$ and $2\lambda = 4\mu$
Since $2\lambda = 4\mu$, $\lambda = 2\mu$
$1 + 2\mu = \mu$
$\mu = -1$ and $\lambda = -2$

(e) $(3\lambda + 5)a + b = 2\mu a + (\lambda - 3)b$
$3\lambda + 5 = 2\mu$ and $1 = \lambda - 3$
$\lambda = 4$ and $2\mu = 12 + 5$

$\lambda = 4$ and $\mu = 8\frac{1}{2}$

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise B, Question 5

Question:

In the diagram, OA = a, OB = b and *C* divides *AB* in the ratio 5:1.

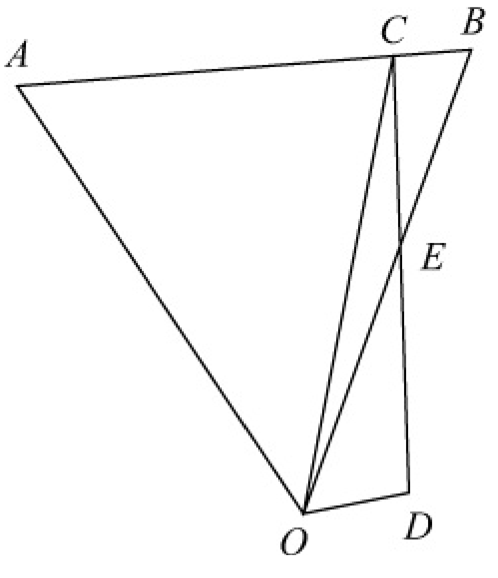
(a) Write down, in terms of **a** and **b**, expressions for AB, AC and OC.
Given that OE = λ b, where λ is a scalar:

(b) Write down, in terms of **a**, **b** and λ, an expression for CE.
Given that OD = μ (b − a) , where μ is a scalar:

(c) Write down, in terms of **a**, **b**, λ and μ, an expression for ED.
Given also that *E* is the mid-point of *CD*:

(d) Deduce the values of λ and μ.

Ⓔ



Solution:

(a) AB = AO + OB = − a + b

$$AC = \frac{5}{6}AB = \frac{5}{6}\left(-a+b \right)$$

$$OC = OA + AC = a + \frac{5}{6}\left(-a+b \right) = \frac{1}{6}a + \frac{5}{6}b$$

(b) OE = λ b:

$$CE = CO + OE = -\left(\frac{1}{6}a + \frac{5}{6}b\right) + \lambda b = -\frac{1}{6}a + \left(\lambda - \frac{5}{6}\right)b$$

(c) $OD = \mu(b - a)$:

$$ED = EO + OD = -\lambda b + \mu(b - a) = -\mu a + (\mu - \lambda)b$$

(d) If *E* is the mid-point of *CD*, CE = ED:

$$-\frac{1}{6}a + \left(\lambda - \frac{5}{6}\right)b = -\mu a + \left(\mu - \lambda\right)b$$

Since **a** and **b** are not parallel

$$-\frac{1}{6} = -\mu \quad \Rightarrow \quad \mu = \frac{1}{6}$$

and

$$\left(\lambda - \frac{5}{6}\right) = \left(\mu - \lambda\right)$$

$$\Rightarrow \quad 2\lambda = \mu + \frac{5}{6}$$

$$\Rightarrow \quad 2\lambda = 1$$

$$\Rightarrow \quad \lambda = \frac{1}{2}$$

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Vectors
Exercise B, Question 6

Question:

In the diagram OA = a, OB = b, 3OC = 2OA and 4OD = 7OB.
The line *DC* meets the line *AB* at *E*.

(a) Write down, in terms of **a** and **b**, expressions for
(i) AB
(ii) DC
Given that DE = λ DC and EB = μ AB where λ and μ are constants:

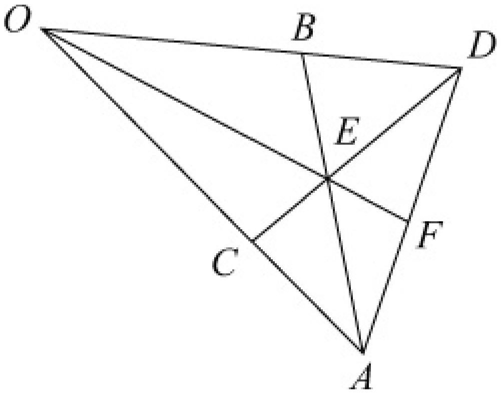
(b) Use $\triangle$ EBD to form an equation relating to **a**, **b**, λ and μ.
Hence:

(c) Show that $\lambda = \frac{9}{13}$.

(d) Find the exact value of μ.

(e) Express OE in terms of **a** and **b**.
The line OE produced meets the line *AD* at *F*.
Given that OF = *k*OE where *k* is a constant and that AF = $\frac{1}{10}\left(7b - 4a \right)$:

(f) Find the value of *k*. Ⓔ



Solution:

(a) OC = $\frac{2}{3}$OA = $\frac{2}{3}$a, OD = $\frac{7}{4}$OB = $\frac{7}{4}$b

(i) AB = AO + OB = − a + b

(ii) DC = DO + OC = $\frac{2}{3}$a − $\frac{7}{4}$b

(b) DE = λ DC and EB = μ AB.
From △ EBD, DE = DB + BE

Since OD = $\frac{7}{4}$b, BD = OD − OB = $\frac{7}{4}$b − b = $\frac{3}{4}$b

∴ DB = − $\frac{3}{4}$b

So λ DC = DB − μ AB

$$\lambda\left(\frac{2}{3}a-\frac{7}{4}b\right)=-\frac{3}{4}b-\mu\left(-a+b\right)$$

$$\left(\frac{2}{3}\lambda-\mu\right)a+\left(\frac{3}{4}+\mu-\frac{7}{4}\lambda\right)b=0$$

(c) So $\frac{2}{3}\lambda-\mu=0 \Rightarrow \mu=\frac{2}{3}\lambda$

and $\frac{3}{4}+\mu-\frac{7}{4}\lambda=0$

$$\Rightarrow \frac{3}{4}+\frac{2}{3}\lambda-\frac{7}{4}\lambda=0$$

$$\Rightarrow \frac{13}{12}\lambda=\frac{3}{4}$$

$$\Rightarrow \lambda=\frac{3}{4}\times\frac{12}{13}=\frac{9}{13}$$

(d) $\mu=\frac{2}{3}\lambda=\frac{2}{3}\times\frac{9}{13}=\frac{6}{13}$

(e) OE = OB + BE = OB − μ AB = b − $\frac{6}{13}\left(-a+b\right)=\frac{6}{13}$a + $\frac{7}{13}$b

(f) OF = *k*OE and AF = $\frac{7}{10}$b − $\frac{4}{10}$a.

OF = $\frac{6k}{13}$a + $\frac{7k}{13}$b

From △ OFA, OF = OA + AF

$$\frac{6k}{13}a+\frac{7k}{13}b=a+\left(\frac{7}{10}b-\frac{4}{10}a\right)=\frac{6}{10}a+\frac{7}{10}b$$

So $\frac{6k}{13}=\frac{6}{10}$ (and $\frac{7k}{13}=\frac{7}{10}$)

$$\Rightarrow \quad k = \frac{13}{10}$$

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise B, Question 7

Question:

In △ OAB, *P* is the mid-point of *AB* and *Q* is the point on *OP* such that OQ = $\frac{3}{4}$OP. Given that OA = a and OB = b, find, in terms of **a** and **b**:

(a) AB

(b) OP

(c) OQ

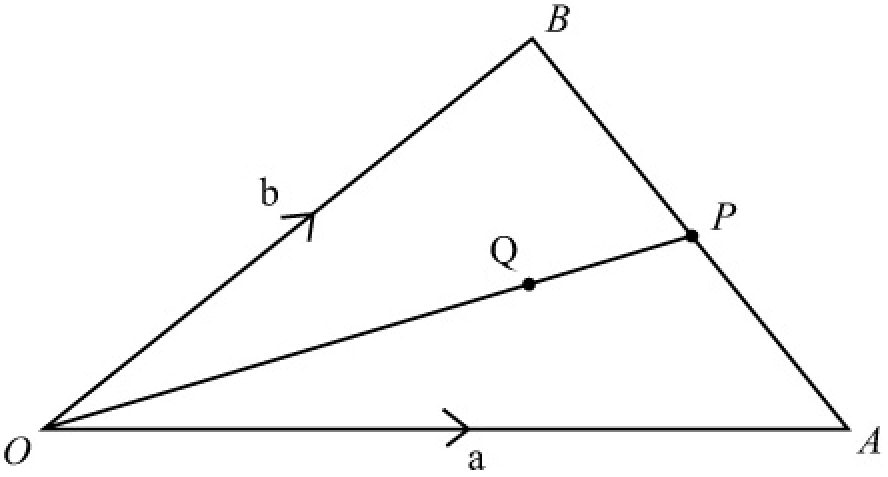
(d) AQ
The point *R* on *OB* is such that OR = kOB, where $0 < k < 1$.

(e) Find, in terms of **a**, **b** and *k*, the vector AR.
Given that *AQR* is a straight line:

(f) Find the ratio in which *Q* divides *AR* and the value of *k*.

E

Solution:



BP = PA and OQ = $\frac{3}{4}$OP

(a) AB = AO + OB = − a + b

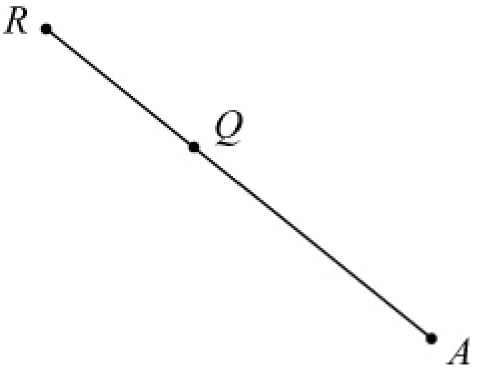
(b) $\text{OP} = \text{OA} + \text{AP} = \text{OA} + \frac{1}{2}\text{AB} = \text{a} + \frac{1}{2}\left(-\text{a} + \text{b} \right) = \frac{1}{2}\text{a} + \frac{1}{2}\text{b}$

(c) $\text{OQ} = \frac{3}{4}\text{OP} = \frac{3}{4}\left(\frac{1}{2}\text{a} + \frac{1}{2}\text{b} \right) = \frac{3}{8}\text{a} + \frac{3}{8}\text{b}$

(d) $\text{AQ} = \text{AO} + \text{OQ} = -\text{a} + \left(\frac{3}{8}\text{a} + \frac{3}{8}\text{b} \right) = -\frac{5}{8}\text{a} + \frac{3}{8}\text{b}$

(e) Given $\text{OR} = k\text{OB}$ $(0 < k < 1)$
In $\triangle$ OAR, $\text{AR} = \text{AO} + \text{OR} = -\text{a} + k\text{b}$

(f) Since *AQR* is a straight line, AR and AQ are parallel vectors.



Suppose $\text{AQ} = \lambda\ \text{AR}$

$-\frac{5}{8}\text{a} + \frac{3}{8}\text{b} = \lambda \left(-\text{a} + k\text{b} \right)$

So $-\frac{5}{8} = -\lambda \Rightarrow \lambda = \frac{5}{8}$

and $\frac{3}{8} = \lambda\text{k}$

$\Rightarrow k = \frac{3}{8\lambda}$

$\Rightarrow k = \frac{3}{5}$

Since $\text{AQ} = \frac{5}{8}\text{AR}, \text{RQ} = \frac{3}{8}\text{AR}$

So *Q* divides *AR* in the ratio 5:3.

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Vectors
Exercise B, Question 8

Question:

In the figure OE : EA = 1 : 2, AF : FB = 3 : 1 and OG : OB = 3 : 1. The vector OA = a and the vector OB = b.
Find, in terms of **a**, **b** or **a** and **b**, expressions for:

(a) OE

(b) OF

(c) EF

(d) BG

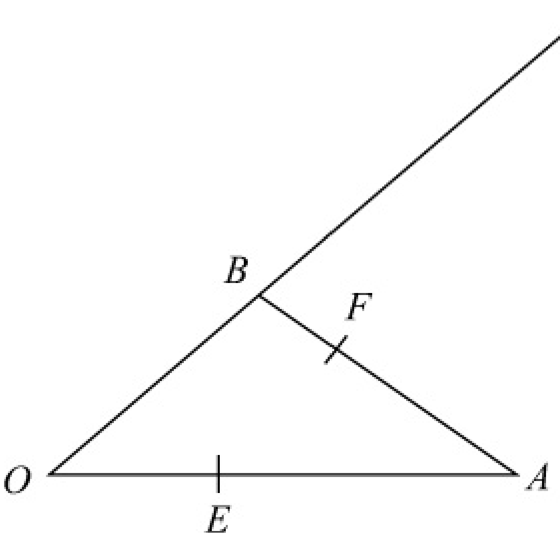
(e) FB

(f) FG

(g) Use your results in (c) and (f) to show that the points *E*, *F* and *G* are collinear and find the ratio EF : FG.

(h) Find EB and AG and hence prove that *EB* is parallel to *AG*.

E



Solution:

(a) $OE = \frac{1}{3}OA = \frac{1}{3}a$

(b) $OF = OA + AF = OA + \frac{3}{4}AB$

$= a + \frac{3}{4}\left(b - a \right)$

$= a + \frac{3}{4}b - \frac{3}{4}a$

$= \frac{1}{4}a + \frac{3}{4}b$

(c) $EF = EA + AF = \frac{2}{3}OA + \frac{3}{4}AB$

$= \frac{2}{3}a + \frac{3}{4}\left(b - a \right)$

$= \frac{2}{3}a + \frac{3}{4}b - \frac{3}{4}a$

$= - \frac{1}{12}a + \frac{3}{4}b$

(d) $BG = 2OB = 2b$

(e) $FB = \frac{1}{4}AB = \frac{1}{4}\left(b - a \right) = - \frac{1}{4}a + \frac{1}{4}b$

(f) $FG = FB + BG = - \frac{1}{4}a + \frac{1}{4}b + 2b = - \frac{1}{4}a + \frac{9}{4}b$

(g) $FG = - \frac{1}{4}a + \frac{9}{4}b = 3\left(- \frac{1}{12}a + \frac{3}{4}b \right) = 3EF$

So EF and FG are parallel vectors.
So *E*, *F* and *G* are collinear.
EF : FG = 1 : 3

(h) $EB = EO + OB = - \frac{1}{3}a + b$

$AG = AO + OG = - a + 3b = 3\left(- \frac{1}{3}a + b \right) = 3EB$

So *EB* is parallel to *AG*.

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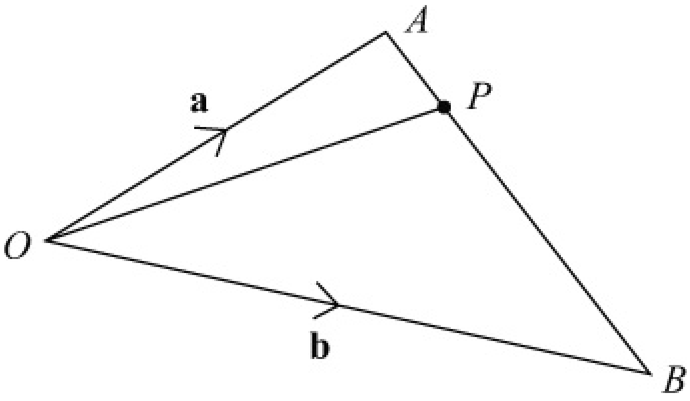
Vectors
Exercise C, Question 1

Question:

The points *A* and *B* have position vectors **a** and **b** respectively (referred to the origin *O*).
The point *P* divides *AB* in the ratio 1:5.
Find, in terms of **a** and **b**, the position vector of *P*.

Solution:



AP : PB = 1 : 5

So AP = $\frac{1}{6}$AB = $\frac{1}{6}\left(b - a\right)$

OP = OA + AP = a + $\frac{1}{6}\left(b - a\right)$

= a + $\frac{1}{6}$b − $\frac{1}{6}$a

= $\frac{5}{6}$a + $\frac{1}{6}$b

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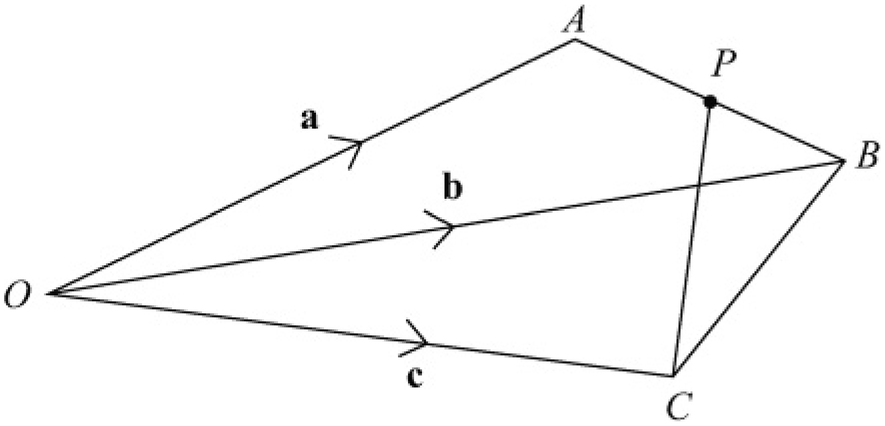
Vectors
Exercise C, Question 2

Question:

The points *A*, *B* and *C* have position vectors **a**, **b** and **c** respectively (referred to the origin *O*). The point *P* is the mid-point of *AB*.
Find, in terms of **a**, **b** and **c**, the vector PC.

Solution:



PC = PO + OC = − OP + OC

But OP = OA + AP = OA + $\frac{1}{2}$AB = a + $\frac{1}{2}\left(b - a\right)$ = $\frac{1}{2}$a + $\frac{1}{2}$b

So PC = $-\left(\frac{1}{2}a + \frac{1}{2}b\right)$ + c = $-\frac{1}{2}a - \frac{1}{2}b + c$

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Edexcel AS and A Level Modular Mathematics

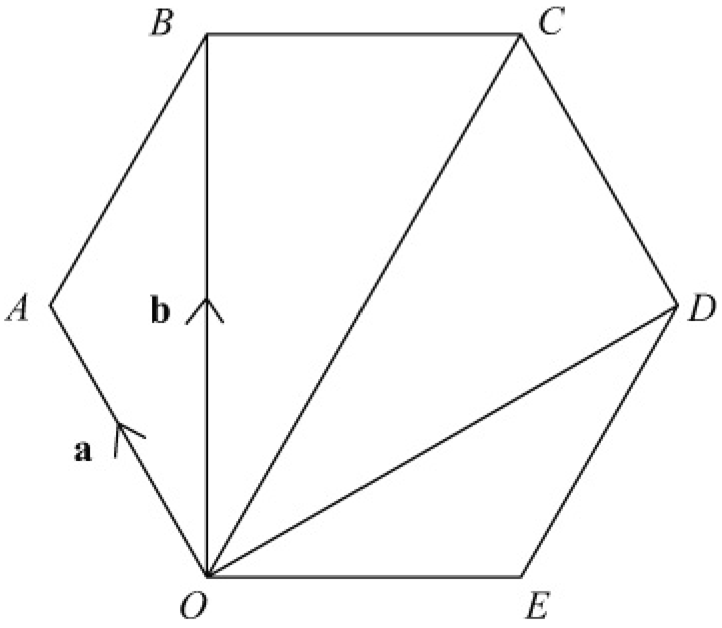
Vectors

Exercise C, Question 3

Question:

OABCDE is a regular hexagon. The points *A* and *B* have position vectors **a** and **b** respectively, referred to the origin *O*.
Find, in terms of **a** and **b**, the position vectors of *C*, *D* and *E*.

Solution:



OC = 2AB = 2 (b − a) = − 2a + 2b
OD = OC + CD = OC + AO = (− 2a + 2b) − a = − 3a + 2b
OE = OD + DE = OD + BA = (− 3a + 2b) + (a − b) = − 2a + b

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Edexcel AS and A Level Modular Mathematics

Vectors

Exercise D, Question 1

Question:

Given that a = 9i + 7j, b = 11i − 3j and c = − 8i − j, find:

(a) a + b + c

(b) 2a − b + c

(c) 2b + 2c − 3a
(Use column matrix notation in your working.)

Solution:

(a) a + b + c = $\begin{pmatrix} 9 \\ 7 \end{pmatrix} + \begin{pmatrix} 11 \\ -3 \end{pmatrix} + \begin{pmatrix} -8 \\ -1 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \end{pmatrix}$

(b) 2a − b + c = $2\begin{pmatrix} 9 \\ 7 \end{pmatrix} + \begin{pmatrix} -11 \\ 3 \end{pmatrix} + \begin{pmatrix} -8 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 18 \\ 14 \end{pmatrix} + \begin{pmatrix} -11 \\ 3 \end{pmatrix} + \begin{pmatrix} -8 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 16 \end{pmatrix}$

(c) 2b + 2c − 3a = $2\begin{pmatrix} 11 \\ -3 \end{pmatrix} + 2\begin{pmatrix} -8 \\ -1 \end{pmatrix} - 3\begin{pmatrix} 9 \\ 7 \end{pmatrix}$

$= \begin{pmatrix} 22 \\ -6 \end{pmatrix} + \begin{pmatrix} -16 \\ -2 \end{pmatrix} + \begin{pmatrix} -27 \\ -21 \end{pmatrix} = \begin{pmatrix} -21 \\ -29 \end{pmatrix}$

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Vectors
Exercise D, Question 2

Question:

The points *A*, *B* and *C* have coordinates $(3 , - 1)$, (4, 5) and $(- 2 , 6)$ respectively, and *O* is the origin.
Find, in terms of **i** and **j**:

(a) the position vectors of *A*, *B* and *C*

(b) AB

(c) AC
Find, in surd form:

(d) $| OC |$

(e) $| AB |$

(f) $| AC |$

Solution:

(a) $a = 3i - j$, $b = 4i + 5j$, $c = - 2i + 6j$

(b) $AB = b - a = (4i + 5j) - (3i - j)$
$= 4i + 5j - 3i + j$
$= i + 6j$

(c) $AC = c - a = (- 2i + 6j) - (3i - j)$
$= - 2i + 6j - 3i + j$
$= - 5i + 7j$

(d) $| OC | = | - 2i + 6j | = \sqrt{(- 2)^2 + 6^2} = \sqrt{40} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$

(e) $| AB | = | i + 6j | = \sqrt{1^2 + 6^2} = \sqrt{37}$

(f) $| AC | = | - 5i + 7j | = \sqrt{(- 5)^2 + 7^2} = \sqrt{74}$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise D, Question 3

Question:

Given that a = 4i + 3j, b = 5i − 12j, c = − 7i + 24j and d = i − 3j, find a unit vector in the direction of **a**, **b**, **c** and **d**.

Solution:

$$| a | = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

$$\text{Unit vector} = \frac{a}{| a |} = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$| b | = \sqrt{5^2 + (- 12) ^2} = \sqrt{169} = 13$$

$$\text{Unit vector} = \frac{b}{| b |} = \frac{1}{13}\begin{pmatrix} 5 \\ - 12 \end{pmatrix}$$

$$| c | = \sqrt{(- 7) ^2 + 24^2} = \sqrt{625} = 25$$

$$\text{Unit vector} = \frac{c}{| c |} = \frac{1}{25}\begin{pmatrix} - 7 \\ 24 \end{pmatrix}$$

$$| d | = \sqrt{1^2 + (- 3) ^2} = \sqrt{10}$$

$$\text{Unit vector} = \frac{d}{| d |} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ - 3 \end{pmatrix}$$

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Vectors
Exercise D, Question 4

Question:

Given that a = 5i + j and b = λ i + 3j, and that $|3a + b| = 10$, find the possible values of λ.

Solution:

$$3a + b = 3\begin{pmatrix}5\\1\end{pmatrix} + \begin{pmatrix}\lambda\\3\end{pmatrix} = \begin{pmatrix}15\\3\end{pmatrix} + \begin{pmatrix}\lambda\\3\end{pmatrix} = \begin{pmatrix}15+\lambda\\6\end{pmatrix}$$

$|3a + b| = 10$, so

$$\sqrt{(15+\lambda)^2 + 6^2} = 10$$

$$(15+\lambda)^2 + 6^2 = 100$$

$$225 + 30\lambda + \lambda^2 + 36 = 100$$

$$\lambda^2 + 30\lambda + 161 = 0$$

$$(\lambda + 7)(\lambda + 23) = 0$$

$$\lambda = -7,\ \lambda = -23$$

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Vectors
Exercise E, Question 1

Question:

Find the distance from the origin to the point $P(2, 8, -4)$.

Solution:

Distance $= \sqrt{2^2 + 8^2 + (-4)^2} = \sqrt{4 + 64 + 16} = \sqrt{84} \approx 9.17$ (3 s.f.)

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Vectors

Exercise E, Question 2

Question:

Find the distance from the origin to the point $P(7, 7, 7)$.

Solution:

Distance $= \sqrt{7^2 + 7^2 + 7^2} = \sqrt{49 + 49 + 49} = \sqrt{147} = 7\sqrt{3} \approx 12.1$ (3 s.f.)

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Vectors
Exercise E, Question 3

Question:

Find the distance between A and B when they have the following coordinates:

(a) $A(3, 0, 5)$ and $B(1, -1, 8)$

(b) $A(8, 11, 8)$ and $B(-3, 1, 6)$

(c) $A(3, 5, -2)$ and $B(3, 10, 3)$

(d) $A(-1, -2, 5)$ and $B(4, -1, 3)$

Solution:

(a) $AB = \sqrt{(3-1)^2 + [0-(-1)]^2 + (5-8)^2}$
$= \sqrt{2^2 + 1^2 + (-3)^2}$
$= \sqrt{14} \approx 3.74$

(b) $AB = \sqrt{[8-(-3)]^2 + (11-1)^2 + (8-6)^2}$
$= \sqrt{11^2 + 10^2 + 2^2}$
$= \sqrt{225} = 15$

(c) $AB = \sqrt{(3-3)^2 + (5-10)^2 + [(-2)-3]^2}$
$= \sqrt{0^2 + (-5)^2 + (-5)^2}$
$= \sqrt{50} = 5\sqrt{2} \approx 7.07$

(d) $AB = \sqrt{[(-1)-4]^2 + [(-2)-(-1)]^2 + (5-3)^2}$
$= \sqrt{(-5)^2 + (-1)^2 + 2^2}$
$= \sqrt{30} \approx 5.48$

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Vectors

Exercise E, Question 4

Question:

The coordinates of *A* and *B* are $(7, -1, 2)$ and $(k, 0, 4)$ respectively. Given that the distance from *A* to *B* is 3 units, find the possible values of *k*.

Solution:

$\text{AB} = \sqrt{(7-k)^2 + (-1-0)^2 + (2-4)^2} = 3$

$\sqrt{(49 - 14k + k^2) + 1 + 4} = 3$

$49 - 14k + k^2 + 1 + 4 = 9$

$k^2 - 14k + 45 = 0$

$(k-5)(k-9) = 0$

$k = 5$ or $k = 9$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise E, Question 5

Question:

The coordinates of *A* and *B* are $(5, 3, -8)$ and $(1, k, -3)$ respectively.
Given that the distance from *A* to *B* is $3\sqrt{10}$ units, find the possible values of *k*.

Solution:

$\text{AB} = \sqrt{(5-1)^2 + (3-k)^2 + [-8-(-3)]^2} = 3\sqrt{10}$

$\sqrt{16 + (9 - 6k + k^2) + 25} = 3\sqrt{10}$

$16 + 9 - 6k + k^2 + 25 = 9 \times 10$

$k^2 - 6k - 40 = 0$

$(k + 4)(k - 10) = 0$

$k = -4$ or $k = 10$

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Vectors
Exercise F, Question 1

Question:

Find the modulus of:

(a) 3i + 5j + k

(b) 4i − 2k

(c) i + j − k

(d) 5i − 9j − 8k

(e) i + 5j − 7k

Solution:

(a) $| 3i + 5j + k | = \sqrt{3^2 + 5^2 + 1^2}$
$= \sqrt{9 + 25 + 1} = \sqrt{35}$

(b) $| 4i - 2k | = \sqrt{4^2 + 0^2 + (- 2)^2}$
$= \sqrt{16 + 4} = \sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

(c) $| i + j - k | = \sqrt{1^2 + 1^2 + (- 1)^2}$
$= \sqrt{1 + 1 + 1} = \sqrt{3}$

(d) $| 5i - 9j - 8k | = \sqrt{5^2 + (- 9)^2 + (- 8)^2}$
$= \sqrt{25 + 81 + 64} = \sqrt{170}$

(e) $| i + 5j - 7k | = \sqrt{1^2 + 5^2 + (- 7)^2}$
$= \sqrt{1 + 25 + 49} = \sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise F, Question 2

Question:

Given that a = $\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$, b = $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and c = $\begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix}$, find in column matrix form:

(a) a + b

(b) b − c

(c) a + b + c

(d) 3a − c

(e) a − 2b + c

(f) | a − 2b + c |

Solution:

(a) a + b = $\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ -1 \end{pmatrix}$

(b) b − c = $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ -5 \end{pmatrix}$

(c) a + b + c = $\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ -3 \\ 1 \end{pmatrix}$

(d) 3a − c = 3 $\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$

(e) $\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \\ 10 \end{pmatrix}$

(f) $| \mathbf{a} - 2\mathbf{b} + \mathbf{c} | = \sqrt{8^2 + (-6)^2 + 10^2}$

$= \sqrt{64 + 36 + 100}$

$= \sqrt{200} = \sqrt{100}\sqrt{2} = 10\sqrt{2}$

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise F, Question 3

Question:

The position vector of the point *A* is 2i − 7j + 3k and AB = 5i + 4j − k. Find the position of the point *B*.

Solution:

AB = b − a, so b = AB + a

$$b = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix}$$

Position vector of *B* is 7i − 3j + 2k

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise F, Question 4

Question:

Given that a = ti + 2j + 3k, and that $|\mathbf{a}| = 7$, find the possible values of t.

Solution:

$|\mathbf{a}| = \sqrt{t^2 + 2^2 + 3^2} = 7$

$\sqrt{t^2 + 4 + 9} = 7$

$t^2 + 4 + 9 = 49$

$t^2 = 36$

$t = 6$ or $t = -6$

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise F, Question 5

Question:

Given that a = 5*t*i + 2*t*j + *t*k, and that $|\mathbf{a}| = 3\sqrt{10}$, find the possible values of *t*.

Solution:

$$|\mathbf{a}| = \sqrt{(5t)^2 + (2t)^2 + t^2} = 3\sqrt{10}$$
$$\sqrt{25t^2 + 4t^2 + t^2} = 3\sqrt{10}$$
$$\sqrt{30t^2} = 3\sqrt{10}$$
$$30t^2 = 9 \times 10$$
$$t^2 = 3$$
$$t = \sqrt{3} \text{ or } t = -\sqrt{3}$$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise F, Question 6

Question:

The points *A* and *B* have position vectors $\begin{pmatrix} 2 \\ 9 \\ t \end{pmatrix}$ and $\begin{pmatrix} 2t \\ 5 \\ 3t \end{pmatrix}$ respectively.

(a) Find AB.

(b) Find, in terms of *t*, $| \text{AB} |$.

(c) Find the value of *t* that makes $| \text{AB} |$ a minimum.

(d) Find the minimum value of $| \text{AB} |$.

Solution:

(a) $\text{AB} = \text{b} - \text{a} = \begin{pmatrix} 2t \\ 5 \\ 3t \end{pmatrix} - \begin{pmatrix} 2 \\ 9 \\ t \end{pmatrix} = \begin{pmatrix} 2t - 2 \\ -4 \\ 2t \end{pmatrix}$

(b) $| \text{AB} | = \sqrt{(2t - 2)^2 + (-4)^2 + (2t)^2}$

$= \sqrt{4t^2 - 8t + 4 + 16 + 4t^2}$

$= \sqrt{8t^2 - 8t + 20}$

(c) Let $| \text{AB} |^2 = p$, then $p = 8t^2 - 8t + 20$

$\frac{\mathrm{d}p}{\mathrm{d}t} = 16t - 8$

For a minimum, $\frac{\mathrm{d}p}{\mathrm{d}t} = 0$, so $16t - 8 = 0$, i.e. $t = \frac{1}{2}$

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = 16$, positive, $\therefore$ minimum

(d) When $t = \frac{1}{2}$,

$| \text{AB} | = \sqrt{8t^2 - 8t + 20} = \sqrt{2 - 4 + 20} = \sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise F, Question 7

Question:

The points *A* and *B* have position vectors $\begin{pmatrix} 2t+1 \\ t+1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} t+1 \\ 5 \\ 2 \end{pmatrix}$ respectively.

(a) Find AB.

(b) Find, in terms of *t*, $|\text{AB}|$.

(c) Find the value of *t* that makes $|\text{AB}|$ a minimum.

(d) Find the minimum value of $|\text{AB}|$.

Solution:

(a) $\text{AB} = \text{b} - \text{a} = \begin{pmatrix} t+1 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 2t+1 \\ t+1 \\ 3 \end{pmatrix} = \begin{pmatrix} -t \\ 4-t \\ -1 \end{pmatrix}$

(b) $|\text{AB}| = \sqrt{(-t)^2 + (4-t)^2 + (-1)^2}$
$= \sqrt{t^2 + 16 - 8t + t^2 + 1}$
$= \sqrt{2t^2 - 8t + 17}$

(c) Let $|\text{AB}|^2 = P$, then $P = 2t^2 - 8t + 17$
$\frac{\mathrm{d}P}{\mathrm{d}t} = 4t - 8$

For a minimum, $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$, so $4t - 8 = 0$, i.e. $t = 2$

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = 4$, positive, $\therefore$ minimum

(d) When $t = 2$,
$|\text{AB}| = \sqrt{2t^2 - 8t + 17} = \sqrt{8 - 16 + 17} = \sqrt{9} = 3$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise G, Question 1

Question:

The vectors **a** and **b** each have magnitude 3 units, and the angle between **a** and **b** is 60 ° . Find **a.b**.

Solution:

$$a\ .\ b = |a|\ |b|\ \cos\theta = 3 \times 3 \times \cos 60\ ° = 3 \times 3 \times \frac{1}{2} = \frac{9}{2}$$

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise G, Question 2

Question:

In each part, find **a.b**:

(a) a = 5i + 2j + 3k, b = 2i − j − 2k

(b) a = 10i − 7j + 4k, b = 3i − 5j − 12k

(c) a = i + j − k, b = − i − j + 4k

(d) a = 2i − k, b = 6i − 5j − 8k

(e) a = 3j + 9k, b = i + 12j − 4k

Solution:

(a) $a\ .\ b = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 10 - 2 - 6 = 2$

(b) $a\ .\ b = \begin{pmatrix} 10 \\ -7 \\ 4 \end{pmatrix} . \begin{pmatrix} 3 \\ -5 \\ -12 \end{pmatrix} = 30 + 35 - 48 = 17$

(c) $a\ .\ b = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} . \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} = -1 - 1 - 4 = -6$

(d) $a\ .\ b = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} . \begin{pmatrix} 6 \\ -5 \\ -8 \end{pmatrix} = 12 + 0 + 8 = 20$

(e) $a\ .\ b = \begin{pmatrix} 0 \\ 3 \\ 9 \end{pmatrix} . \begin{pmatrix} 1 \\ 12 \\ -4 \end{pmatrix} = 0 + 36 - 36 = 0$

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise G, Question 3

Question:

In each part, find the angle between **a** and **b**, giving your answer in degrees to 1 decimal place:

(a) a = 3i + 7j, b = 5i + j

(b) a = 2i − 5j, b = 6i + 3j

(c) a = i − 7j + 8k, b = 12i + 2j + k

(d) a = − i − j + 5k, b = 11i − 3j + 4k

(e) a = 6i − 7j + 12k, b = − 2i + j + k

(f) a = 4i + 5k, b = 6i − 2j

(g) a = − 5i + 2j − 3k, b = 2i − 2j + 11k

(h) a = i + j + k, b = i − j + k

Solution:

(a) $a \, . \, b = \begin{pmatrix} 3 \\ 7 \end{pmatrix} . \begin{pmatrix} 5 \\ 1 \end{pmatrix} = 15 + 7 = 22$

$| a | = \sqrt{3^2 + 7^2} = \sqrt{58}$

$| b | = \sqrt{5^2 + 1^2} = \sqrt{26}$

$\sqrt{58}\sqrt{26} \cos \theta = 22$

$\cos \theta = \frac{22}{\sqrt{58}\sqrt{26}}$

$\theta = 55.5 °$ (1 d.p.)

(b) $a \, . \, b = \begin{pmatrix} 2 \\ -5 \end{pmatrix} . \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 12 - 15 = -3$

$| a | = \sqrt{2^2 + (- 5)^2} = \sqrt{29}$

$| b | = \sqrt{6^2 + 3^2} = \sqrt{45}$

$\sqrt{29}\sqrt{45} \cos \theta = - 3$

$\cos \theta = \frac{-3}{\sqrt{29}\sqrt{45}}$

$\theta = 94.8°$ (1 d.p.)

(c) $a . b = \begin{pmatrix} 1 \\ -7 \\ 8 \end{pmatrix} . \begin{pmatrix} 12 \\ 2 \\ 1 \end{pmatrix} = 12 - 14 + 8 = 6$

$|a| = \sqrt{1^2 + (-7)^2 + 8^2} = \sqrt{114}$

$|b| = \sqrt{12^2 + 2^2 + 1^2} = \sqrt{149}$

$\sqrt{114}\sqrt{149} \cos\theta = 6$

$\cos\theta = \frac{6}{\sqrt{114}\sqrt{149}}$

$\theta = 87.4°$ (1 d.p.)

(d) $a . b = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix} . \begin{pmatrix} 11 \\ -3 \\ 4 \end{pmatrix} = -11 + 3 + 20 = 12$

$|a| = \sqrt{(-1)^2 + (-1)^2 + 5^2} = \sqrt{27}$

$|b| = \sqrt{11^2 + (-3)^2 + 4^2} = \sqrt{146}$

$\sqrt{27}\sqrt{146} \cos\theta = 12$

$\cos\theta = \frac{12}{\sqrt{27}\sqrt{146}}$

$\theta = 79.0°$ (1 d.p.)

(e) $a . b = \begin{pmatrix} 6 \\ -7 \\ 12 \end{pmatrix} . \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = -12 - 7 + 12 = -7$

$|a| = \sqrt{6^2 + (-7)^2 + 12^2} = \sqrt{229}$

$|b| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$

$\sqrt{229}\sqrt{6} \cos\theta = -7$

$\cos\theta = \frac{-7}{\sqrt{229}\sqrt{6}}$

$\theta = 100.9°$ (1 d.p.)

(f) $a . b = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} . \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} = 24 + 0 + 0 = 24$

$|a| = \sqrt{4^2 + 5^2} = \sqrt{41}$

$|b| = \sqrt{6^2 + (-2)^2} = \sqrt{40}$

$\sqrt{41}\sqrt{40} \cos\theta = 24$

$\cos\theta = \frac{24}{\sqrt{41}\sqrt{40}}$

$\theta = 53.7°$ (1 d.p.)

(g) $\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} -5 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 11 \end{pmatrix} = -10-4-33 = -47$

$|\mathbf{a}| = \sqrt{(-5)^2 + 2^2 + (-3)^2} = \sqrt{38}$

$|\mathbf{b}| = \sqrt{2^2 + (-2)^2 + 11^2} = \sqrt{129}$

$\sqrt{38}\sqrt{129}\cos\theta = -47$

$\cos\theta = \frac{-47}{\sqrt{38}\sqrt{129}}$

$\theta = 132.2°$ (1 d.p.)

(h) $\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1-1+1 = 1$

$|\mathbf{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$|\mathbf{b}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

$\sqrt{3}\sqrt{3}\cos\theta = 1$

$\cos\theta = \frac{1}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$

$\theta = 70.5°$ (1 d.p.)

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise G, Question 4

Question:

Find the value, or values, of λ for which the given vectors are perpendicular:

(a) 3i + 5j and λ i + 6j

(b) 2i + 6j − k and λ i − 4j − 14k

(c) 3i + λ j − 8k and 7i − 5j + k

(d) 9i − 3j + 5k and λ i + λ j + 3k

(e) λ i + 3j − 2k and λ i + λ j + 5k

Solution:

(a) $\begin{pmatrix} 3 \\ 5 \end{pmatrix} . \begin{pmatrix} \lambda \\ 6 \end{pmatrix} = 3\lambda + 30 = 0$

$\Rightarrow \quad 3\lambda = -30$

$\Rightarrow \quad \lambda = -10$

(b) $\begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} . \begin{pmatrix} \lambda \\ -4 \\ -14 \end{pmatrix} = 2\lambda - 24 + 14 = 0$

$\Rightarrow \quad 2\lambda = 10$

$\Rightarrow \quad \lambda = 5$

(c) $\begin{pmatrix} 3 \\ \lambda \\ -8 \end{pmatrix} . \begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix} = 21 - 5\lambda - 8 = 0$

$\Rightarrow \quad 5\lambda = 13$

$\Rightarrow \quad \lambda = 2\frac{3}{5}$

(d) $\begin{pmatrix} 9 \\ -3 \\ 5 \end{pmatrix} . \begin{pmatrix} \lambda \\ \lambda \\ 3 \end{pmatrix} = 9\lambda - 3\lambda + 15 = 0$

$$\Rightarrow \quad 6\lambda = -15$$

$$\Rightarrow \quad \lambda = -2\frac{1}{2}$$

(e) $\begin{pmatrix} \lambda \\ 3 \\ -2 \end{pmatrix} . \begin{pmatrix} \lambda \\ \lambda \\ 5 \end{pmatrix} = \lambda^2 + 3\lambda - 10 = 0$

$$\Rightarrow \quad (\lambda + 5)(\lambda - 2) = 0$$

$$\Rightarrow \quad \lambda = -5 \text{ or } \lambda = 2$$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors
Exercise G, Question 5

Question:

Find, to the nearest tenth of a degree, the angle that the vector 9i − 5j + 3k makes with:

(a) the positive *x*-axis

(b) the positive *y*-axis

Solution:

(a) Using a = 9i − 5j + 3k and b = i,

$$a \,.\, b = \begin{pmatrix} 9 \\ -5 \\ 3 \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 9$$

$$|a| = \sqrt{9^2 + (-5)^2 + 3^2} = \sqrt{115}$$

$$|b| = 1$$

$$\sqrt{115}\cos\theta = 9$$

$$\cos\theta = \frac{9}{\sqrt{115}}$$

$$\theta = 32.9\ ^\circ$$

(b) Using a = 9i − 5j + 3k and b = j,

$$a \,.\, b = \begin{pmatrix} 9 \\ -5 \\ 3 \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -5$$

$$|a| = \sqrt{115},\ |b| = 1$$

$$\sqrt{115}\cos\theta = -5$$

$$\cos\theta = \frac{-5}{\sqrt{115}}$$

$$\theta = 117.8\ ^\circ$$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise G, Question 6

Question:

Find, to the nearest tenth of a degree, the angle that the vector i + 11j − 4k makes with:

(a) the positive *y*-axis

(b) the positive *z*-axis

Solution:

(a) Using a = i + 11j − 4k and b = j,

$$a \, . \, b = \begin{pmatrix} 1 \\ 11 \\ -4 \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 11$$

$$|a| = \sqrt{1^2 + 11^2 + (-4)^2} = \sqrt{138}$$

$$|b| = 1$$

$$\sqrt{138} \cos \theta = 11$$

$$\cos \theta = \frac{11}{\sqrt{138}}$$

$$\theta = 20.5 °$$

(b) Using a = i + 11j − 4k and b = k,

$$a \, . \, b = \begin{pmatrix} 1 \\ 11 \\ -4 \end{pmatrix} . \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -4$$

$$|a| = \sqrt{138}, \ |b| = 1$$

$$\sqrt{138} \cos \theta = -4$$

$$\cos \theta = \frac{-4}{\sqrt{138}}$$

$$\theta = 109.9 °$$

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise G, Question 7

Question:

The angle between the vectors i + j + k and 2i + j + k is θ.
Calculate the exact value of $\cos\theta$.

Solution:

Using a = i + j + k and b = 2i + j + k,

$$a \, . \, b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 2 + 1 + 1 = 4$$

$$| \, a \, | = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$| \, b \, | = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

$$\sqrt{3}\sqrt{6}\cos\theta = 4$$

$$\cos\theta = \frac{4}{\sqrt{3}\sqrt{6}} = \frac{4}{\sqrt{3}\sqrt{3}\sqrt{2}} = \frac{4}{3\sqrt{2}}$$

$$= \frac{4}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$$

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise G, Question 8

Question:

The angle between the vectors i + 3j and j + λ k is 60 ° .

Show that $\lambda = \pm \sqrt{\frac{13}{5}}$.

Solution:

Using a = i + 3j and b = j + λ k,

$$a . b = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ \lambda \end{pmatrix} = 0 + 3 + 0 = 3$$

$$| a | = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$| b | = \sqrt{1^2 + \lambda^2} = \sqrt{1 + \lambda^2}$$

$$\sqrt{10}\sqrt{1 + \lambda^2}\cos 60° = 3$$

$$\sqrt{1 + \lambda^2} = \frac{3}{\sqrt{10}\cos 60°} = \frac{6}{\sqrt{10}}$$

Squaring both sides:

$$1 + \lambda^2 = \frac{36}{10}$$

$$\lambda^2 = \frac{26}{10} = \frac{13}{5}$$

$$\lambda = \pm \sqrt{\frac{13}{5}}$$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise G, Question 9

Question:

Simplify as far as possible:

(a) $a\ .\ (b+c) + b\ .\ (a-c)$, given that **b** is perpendicular to **c**.

(b) $(a+b)\ .\ (a+b)$, given that $|a| = 2$ and $|b| = 3$.

(c) $(a+b)\ .\ (2a-b)$, given that **a** is perpendicular to **b**.

Solution:

(a) $a\ .\ (b+c) + b\ .\ (a-c)$
$= a\ .\ b + a\ .\ c + b\ .\ a - b\ .\ c$
$= 2a\ .\ b + a\ .\ c$ (because $b\ .\ c = 0$)

(b) $(a+b)\ .\ (a+b)$
$= a\ .\ (a+b) + b\ .\ (a+b)$
$= a\ .\ a + a\ .\ b + b\ .\ a + b\ .\ b$
$= |a|^2 + 2a\ .\ b + |b|^2$
$= 4 + 2a\ .\ b + 9$
$= 13 + 2a\ .\ b$

(c) $(a+b)\ .\ (2a-b)$
$= a\ .\ (2a-b) + b\ .\ (2a-b)$
$= 2a\ .\ a - a\ .\ b + 2b\ .\ a - b\ .\ b$
$= 2|a|^2 - |b|^2$ (because $a\ .\ b = 0$)

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise G, Question 10

Question:

Find a vector which is perpendicular to both **a** and **b**, where:

(a) a = i + j − 3k, b = 5i − 2j − k

(b) a = 2i + 3j − 4k, b = i − 6j + 3k

(c) a = 4i − 4j − k, b = − 2i − 9j + 6k

Solution:

(a) Let the required vector be xi + yj + zk. Then

$$\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$x + y - 3z = 0$
$5x - 2y - z = 0$
Let $z = 1$:
$x + y = 3 \qquad (\times 2)$
$5x - 2y = 1$
$2x + 2y = 6$
$5x - 2y = 1$
Adding, $7x = 7 \Rightarrow x = 1$
$1 + y = 3$, so $y = 2$
So $x = 1$, $y = 2$ and $z = 1$
A possible vector is i + 2j + k.

(b) Let the required vector be xi + yj + zk. Then

$$\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$2x + 3y - 4z = 0$
$x - 6y + 3z = 0$
Let $z = 1$:
$2x + 3y = 4$
$x - 6y = -3 \qquad (\times 2)$
$2x + 3y = 4$
$2x - 12y = -6$

Subtracting, $15y = 10 \quad \Rightarrow \quad y = \frac{2}{3}$

$2x + 2 = 4$, so $x = 1$

So $x = 1$, $y = \frac{2}{3}$ and $z = 1$

A possible vector is $\mathrm{i} + \frac{2}{3}\mathrm{j} + \mathrm{k}$.

Another possible vector is $3\left(\mathrm{i} + \frac{2}{3}\mathrm{j} + \mathrm{k}\right) = 3\mathrm{i} + 2\mathrm{j} + 3\mathrm{k}$.

(c) Let the required vector be $x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$. Then

$$\begin{pmatrix} 4 \\ -4 \\ -1 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} -2 \\ -9 \\ 6 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$4x - 4y - z = 0$
$-2x - 9y + 6z = 0$
Let $z = 1$:
$4x - 4y = 1$
$-2x - 9y = -6 \qquad (\times 2)$
$4x - 4y = 1$
$-4x - 18y = -12$

Adding, $-22y = -11 \quad \Rightarrow \quad y = \frac{1}{2}$

$4x - 2 = 1$, so $x = \frac{3}{4}$

So $x = \frac{3}{4}$, $y = \frac{1}{2}$ and $z = 1$

A possible vector is $\frac{3}{4}\mathrm{i} + \frac{1}{2}\mathrm{j} + \mathrm{k}$

Another possible vector is $4\left(\frac{3}{4}\mathrm{i} + \frac{1}{2}\mathrm{j} + \mathrm{k}\right) = 3\mathrm{i} + 2\mathrm{j} + 4\mathrm{k}$.

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

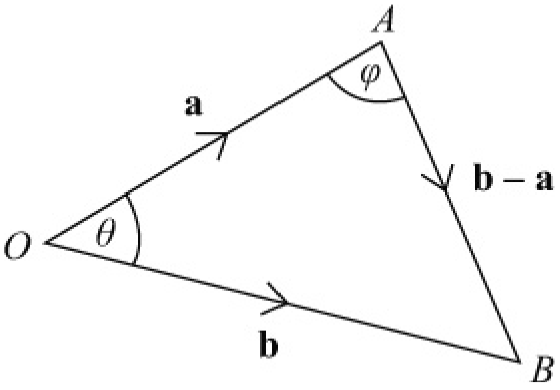
Vectors

Exercise G, Question 11

Question:

The points *A* and *B* have position vectors 2i + 5j + k and 6i + j − 2k respectively, and *O* is the origin.
Calculate each of the angles in △ OAB, giving your answers in degrees to 1 decimal place.

Solution:



Using **a** and **b** to find θ:

$$a\ .\ b = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} . \begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix} = 12 + 5 - 2 = 15$$

$$| a | = \sqrt{2^2 + 5^2 + 1^2} = \sqrt{30}$$

$$| b | = \sqrt{6^2 + 1^2 + (- 2)^2} = \sqrt{41}$$

$$\sqrt{30}\sqrt{41}\cos\theta = 15$$

$$\cos\theta = \frac{15}{\sqrt{30}\sqrt{41}}$$

$$\theta = 64.7\,°$$

Using AO and AB to find ϕ:

$$AO = -a = \begin{pmatrix} -2 \\ -5 \\ -1 \end{pmatrix}$$

$$AB = b - a = \begin{pmatrix} 6 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} \\ -\mathbf{a} \\ \\ \end{pmatrix} . \begin{pmatrix} \\ \mathbf{b}-\mathbf{a} \\ \\ \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ -1 \end{pmatrix} . \begin{pmatrix} 4 \\ -4 \\ -3 \end{pmatrix} = -8+20+3=15$$

$$|-\mathbf{a}| = \sqrt{(-2)^2+(-5)^2+(-1)^2} = \sqrt{30}$$

$$|\mathbf{b}-\mathbf{a}| = \sqrt{4^2+(-4)^2+(-3)^2} = \sqrt{41}$$

$$\sqrt{30}\sqrt{41}\cos\phi = 15$$

$$\cos\phi = \frac{15}{\sqrt{30}\sqrt{41}}$$

$\phi = 64.7°$ (1 d.p.)

(Since $|\mathbf{b}-\mathbf{a}| = |\mathbf{b}|$, AB = OB, so the triangle is isosceles).

$\angle$ OBA = 180° − 64.7° − 64.7° = 50.6° (1 d.p.)

Angles are 64.7°, 64.7° and 50.6° (all 1 d.p.)

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise G, Question 12

Question:

The points *A*, *B* and *C* have position vectors i + 3j + k, 2i + 7j − 3k and 4i − 5j + 2k respectively.

(a) Find, as surds, the lengths of *AB* and *BC*.

(b) Calculate, in degrees to 1 decimal place, the size of ∠ ABC.

Solution:

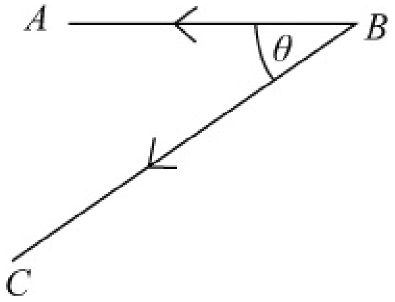
(a) $AB = b - a = \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix}$

Length of $AB = | AB | = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{33}$

$BC = c - b = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -12 \\ 5 \end{pmatrix}$

Length of $BC = | BC | = \sqrt{2^2 + (-12)^2 + 5^2} = \sqrt{173}$

(b)



θ is the angle between BA and BC.

$BA . BC = \begin{pmatrix} -1 \\ -4 \\ 4 \end{pmatrix} . \begin{pmatrix} 2 \\ -12 \\ 5 \end{pmatrix} = -2 + 48 + 20 = 66$

$\sqrt{33}\sqrt{173}\cos\theta = 66$

$\cos\theta = \frac{66}{\sqrt{33}\sqrt{173}}$

$\theta = 29.1°$ (1 d.p.)

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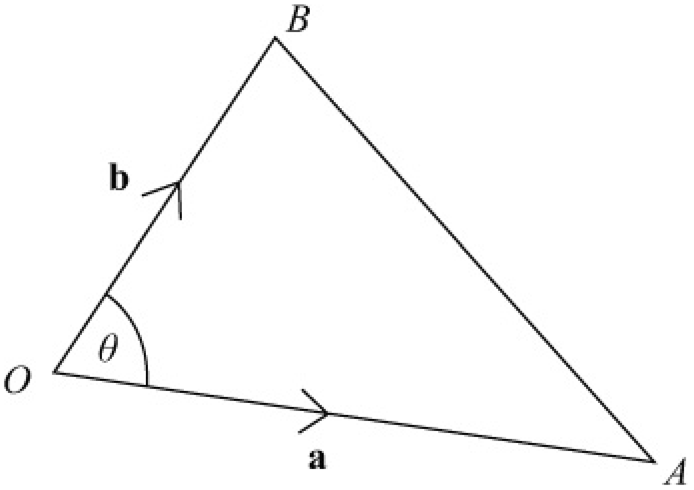
Vectors
Exercise G, Question 13

Question:

Given that the points *A* and *B* have coordinates $(7, 4, 4)$ and $(2, -2, -1)$ respectively, use a vector method to find the value of $\cos AOB$, where *O* is the origin.

Prove that the area of $\triangle AOB$ is $\frac{5\sqrt{29}}{2}$.

Solution:



The position vectors of *A* and *B* are

$$a = \begin{pmatrix} 7 \\ 4 \\ 4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

$$a . b = \begin{pmatrix} 7 \\ 4 \\ 4 \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = 14 - 8 - 4 = 2$$

$$|a| = \sqrt{7^2 + 4^2 + 4^2} = \sqrt{81} = 9$$

$$|b| = \sqrt{2^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$9 \times 3 \times \cos\theta = 2$$

$$\cos\theta = \frac{2}{27}$$

$$\cos \angle AOB = \frac{2}{27}$$

$$\text{Area of } \angle AOB = \frac{1}{2} |a| \ |b| \sin \angle AOB$$

Using $\sin^2\theta + \cos^2\theta = 1$:

$$\sin^2 \angle AOB = 1 - \left(\frac{2}{27}\right)2 = \frac{725}{27^2}$$

$$\sin \angle AOB = \sqrt{\frac{725}{27^2}} = \frac{\sqrt{25}\sqrt{29}}{27} = \frac{5\sqrt{29}}{27}$$

$$\text{Area of } \triangle AOB = \frac{1}{2} \times 9 \times 3 \times \frac{5\sqrt{29}}{27} = \frac{5\sqrt{29}}{2}$$

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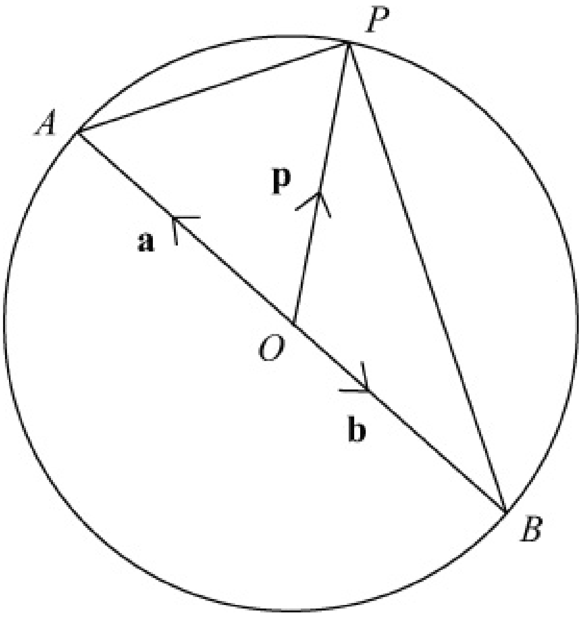
Vectors
Exercise G, Question 14

Question:

AB is a diameter of a circle centred at the origin *O*, and *P* is any point on the circumference of the circle.
Using the position vectors of *A*, *B* and *P*, prove (using a scalar product) that *AP* is perpendicular to *BP* (i.e. the angle in the semicircle is a right angle).

Solution:



Let the position vectors, referred to origin *O*, of *A*, *B* and *P* be **a**, **b** and **p** respectively.
Since $|OA| = |OB|$ and *AB* is a straight line, $b = -a$
$AP = p - a$
$BP = p - b = p - (-a) = p + a$
$AP . BP = (p - a) . (p + a) = p . (p + a) - a . (p + a)$
$= p . p + p . a - a . p - a . a$
$= p . p - a . a$
$p . p = |p|^2$ and $a . a = |a|^2$
Also $|p| = |a|$, since the magnitude of each vector equals the radius of the circle.
So $AP . BP = |p|^2 - |a|^2 = 0$
Since the scalar product is zero, *AP* is perpendicular to *BP*.

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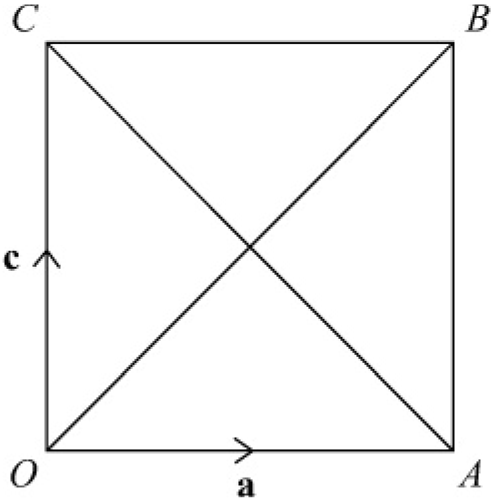
Vectors
Exercise G, Question 15

Question:

Use a vector method to prove that the diagonals of the square *OABC* cross at right angles.

Solution:



Let the position vectors, referred to origin *O*, of *A* and *C* be **a** and **c** respectively.
AB = OC = c
AC = c − a
OB = OA + AB = a + c
AC . OB = (c − a) . (a + c) = c . (a + c) − a . (a + c)
= c . a + c . c − a . a − a . c
= c . c − a . a
= $| c |^2 - | a |^2$
But | c | = | a | , since the magnitude of each vector equals the length of the side of the square.
So AC . OB = $| c |^2 - | a |^2 = 0$
Since the scalar product is zero; the diagonals cross at right angles.

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Vectors
Exercise H, Question 1

Question:

Find a vector equation of the straight line which passes through the point *A*, with position vector **a**, and is parallel to the vector **b**:

(a) a = 6i + 5j − k, b = 2i − 3j − k

(b) a = 2i + 5j, b = i + j + k

(c) a = − 7i + 6j + 2k, b = 3i + j + 2k

(d) a = $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$, b = $\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

(e) a = $\begin{pmatrix} 6 \\ -11 \\ 2 \end{pmatrix}$, b = $\begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$

Solution:

(a) r = $\begin{pmatrix} 6 \\ 5 \\ -1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$

(b) r = $\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(c) r = $\begin{pmatrix} -7 \\ 6 \\ 2 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

(d) r = $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

(e) $\mathbf{r} = \begin{pmatrix} 6 \\ -11 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$

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Vectors
Exercise H, Question 2

Question:

Calculate, to 1 decimal place, the distance between the point *P*, where *t* = 1, and the point *Q*, where *t* = 5, on the line with equation:

(a) $r = (2i - j + k) + t (3i - 8j - k)$

(b) $r = (i + 4j + k) + t (6i - 2j + 3k)$

(c) $r = (2i + 5k) + t (- 3i + 4j - k)$

Solution:

(a) $t = 1$: $$p = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \\ 0 \end{pmatrix}$$

$t = 5$: $$q = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix} = \begin{pmatrix} 17 \\ -41 \\ -4 \end{pmatrix}$$

$$PQ = q - p = \begin{pmatrix} 17 \\ -41 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -9 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ -32 \\ -4 \end{pmatrix}$$

$$\text{Distance} = | PQ | = \sqrt{12^2 + (- 32)^2 + (- 4)^2}$$
$$= \sqrt{1184} = 34.4 \text{ (1 d.p.)}$$

(b) $t = 1$: $$p = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 4 \end{pmatrix}$$

$t = 5$: $$q = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 31 \\ -6 \\ 16 \end{pmatrix}$$

$$PQ = q - p = \begin{pmatrix} 31 \\ -6 \\ 16 \end{pmatrix} - \begin{pmatrix} 7 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ -8 \\ 12 \end{pmatrix}$$

$$\text{Distance} = | PQ | = \sqrt{24^2 + (- 8)^2 + 12^2}$$
$$= \sqrt{784} = 28 \text{ (exact)}$$

(c) $t = 1$: $\quad \mathbf{p} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + 1 \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$

$t = 5$: $\quad \mathbf{q} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + 5 \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 20 \\ 0 \end{pmatrix}$

$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{pmatrix} -13 \\ 20 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} -12 \\ 16 \\ -4 \end{pmatrix}$

$$\text{Distance} = |\overrightarrow{PQ}| = \sqrt{(-12)^2 + 16^2 + (-4)^2}$$

$$= \sqrt{416} = 20.4 \text{ (1 d.p.)}$$

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Vectors
Exercise H, Question 3

Question:

Find a vector equation for the line which is parallel to the z-axis and passes through the point $(4 , \ -3 , \ 8)$.

Solution:

Vector k = $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is in the direction of the z-axis.

The point $(4 , \ -3 , \ 8)$ has position vector $\begin{pmatrix} 4 \\ -3 \\ 8 \end{pmatrix}$.

The equation of the line is

$$\mathrm{r} = \begin{pmatrix} 4 \\ -3 \\ 8 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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Vectors
Exercise H, Question 4

Question:

Find a vector equation for the line which passes through the points:

(a) (2, 1, 9) and $(4 , - 1 , 8)$

(b) $(- 3 , 5 , 0)$ and (7, 2, 2)

(c) $(1 , 11 , - 4)$ and (5, 9, 2)

(d) $(- 2 , - 3 , - 7)$ and $(12 , 4 , - 3)$

Solution:

(a) $a = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}, b = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix}$

$$b - a = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

Equation is

$$r = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

(b) $a = \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix}, b = \begin{pmatrix} 7 \\ 2 \\ 2 \end{pmatrix}$

$$b - a = \begin{pmatrix} 7 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \\ 2 \end{pmatrix}$$

Equation is

$$r = \begin{pmatrix} -3 \\ 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 10 \\ -3 \\ 2 \end{pmatrix}$$

(c) $\mathbf{a} = \begin{pmatrix} 1 \\ 11 \\ -4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix}$

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ 9 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 11 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$$

Equation is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 11 \\ -4 \end{pmatrix} + t \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$$

(d) $\mathbf{a} = \begin{pmatrix} -2 \\ -3 \\ -7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 12 \\ 4 \\ -3 \end{pmatrix}$

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 12 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \\ -7 \end{pmatrix} = \begin{pmatrix} 14 \\ 7 \\ 4 \end{pmatrix}$$

Equation is

$$\mathbf{r} = \begin{pmatrix} -2 \\ -3 \\ -7 \end{pmatrix} + t \begin{pmatrix} 14 \\ 7 \\ 4 \end{pmatrix}$$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise H, Question 5

Question:

The point (1, *p*, *q*) lies on the line *l*. Find the values of *p* and *q*, given that the equation is *l* is:

(a) $\mathrm{r} = \left(2\mathrm{i} - 3\mathrm{j} + \mathrm{k} \right) + t \left(\mathrm{i} - 4\mathrm{j} - 9\mathrm{k} \right)$

(b) $\mathrm{r} = \left(-4\mathrm{i} + 6\mathrm{j} - \mathrm{k} \right) + t \left(2\mathrm{i} - 5\mathrm{j} - 8\mathrm{k} \right)$

(c) $\mathrm{r} = \left(16\mathrm{i} - 9\mathrm{j} - 10\mathrm{k} \right) + t \left(3\mathrm{i} + 2\mathrm{j} + \mathrm{k} \right)$

Solution:

(a) $\mathrm{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -4 \\ -9 \end{pmatrix}$

$x = 1: \quad 2 + t = 1 \quad \Rightarrow \quad t = -1$

$$\mathrm{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} \\ -1 \\ \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix}$$

So $p = 1$ and $q = 10$.

(b) $\mathrm{r} = \begin{pmatrix} -4 \\ 6 \\ -1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -5 \\ -8 \end{pmatrix}$

$x = 1: \quad -4 + 2t = 1 \quad \Rightarrow \quad 2t = 5 \quad \Rightarrow \quad t = \frac{5}{2}$

$$\mathrm{r} = \begin{pmatrix} -4 \\ 6 \\ -1 \end{pmatrix} + \frac{5}{2} \begin{pmatrix} 2 \\ -5 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ -6\frac{1}{2} \\ -21 \end{pmatrix}$$

So $p = -6\frac{1}{2}$ and $q = -21$.

(c) $\mathrm{r} = \begin{pmatrix} 16 \\ -9 \\ -10 \end{pmatrix} + t \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$x = 1: \quad 16 + 3t = 1 \quad \Rightarrow \quad 3t = -15 \quad \Rightarrow \quad t = -5$

$$r = \begin{pmatrix} 16 \\ -9 \\ -10 \end{pmatrix} + \left(\vphantom{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} -5 \right) \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -19 \\ -15 \end{pmatrix}$$

So $p = -19$ and $q = -15$.

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise I, Question 1

Question:

Determine whether the lines with the given equations intersect. If they do intersect, find the coordinates of their point of intersection.

$$\mathrm{r} = \begin{pmatrix} 2 \\ 4 \\ -7 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \text{ and } \mathrm{r} = \begin{pmatrix} 1 \\ 14 \\ 16 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

Solution:

$$\mathrm{r} = \begin{pmatrix} 2+2t \\ 4+t \\ -7+3t \end{pmatrix}, \mathrm{r} = \begin{pmatrix} 1+s \\ 14-s \\ 16-2s \end{pmatrix}.$$

At an intersection point: $\begin{pmatrix} 2+2t \\ 4+t \\ -7+3t \end{pmatrix} = \begin{pmatrix} 1+s \\ 14-s \\ 16-2s \end{pmatrix}$

$2 + 2t = 1 + s$

$4 + t = 14 - s$

Adding: $6 + 3t = 15$

$\Rightarrow \quad 3t = 9$

$\Rightarrow \quad t = 3$

$2 + 6 = 1 + s$

$\Rightarrow \quad s = 7$

If the lines intersect, $-7 + 3t = 16 - 2s$ must be true.

$-7 + 3t = -7 + 9 = 2$

$16 - 2s = 16 - 14 = 2$

The z components are equal, so the lines do intersect.

Intersection point:

$$\begin{pmatrix} 2+2t \\ 4+t \\ -7+3t \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \\ 2 \end{pmatrix}$$

Coordinates (8, 7, 2)

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise I, Question 2

Question:

Determine whether the lines with the given equations intersect. If they do intersect, find the coordinates of their point of intersection.

$$r = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 9 \\ -2 \\ -1 \end{pmatrix} \text{ and } r = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Solution:

$$r = \begin{pmatrix} 2+9t \\ 2-2t \\ -3-t \end{pmatrix}, r = \begin{pmatrix} 3+2s \\ -1-s \\ 2+3s \end{pmatrix}$$

At an intersection point: $\begin{pmatrix} 2+9t \\ 2-2t \\ -3-t \end{pmatrix} = \begin{pmatrix} 3+2s \\ -1-s \\ 2+3s \end{pmatrix}$

$2+9t=3+2s$
$2-2t=-1-s \quad (\times 2)$
$2+9t=3+2s$
$4-4t=-2-2s$
Adding: $6+5t=1$
$\Rightarrow \quad 5t=-5$
$\Rightarrow \quad t=-1$
$2-9=3+2s$
$\Rightarrow \quad 2s=-10$
$\Rightarrow \quad s=-5$
If the lines intersect, $-3-t=2+3s$ must be true.
$-3-t=-3+1=-2$
$2+3s=2-15=-13$
The z components are not equal, so the lines do not intersect.

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Vectors
Exercise I, Question 3

Question:

Determine whether the lines with the given equations intersect. If they do intersect, find the coordinates of their point of intersection.

$$r = \begin{pmatrix} 12 \\ 4 \\ -6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \text{ and } r = \begin{pmatrix} 8 \\ -2 \\ 6 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$$

Solution:

$$r = \begin{pmatrix} 12-2t \\ 4+t \\ -6+4t \end{pmatrix}, r = \begin{pmatrix} 8+2s \\ -2+s \\ 6-5s \end{pmatrix}$$

At an intersection point: $\begin{pmatrix} 12-2t \\ 4+t \\ -6+4t \end{pmatrix} = \begin{pmatrix} 8+2s \\ -2+s \\ 6-5s \end{pmatrix}$

$12-2t = 8+2s$
$4+t = -2+s \quad (\times 2)$
$12-2t = 8+2s$
$8+2t = -4+2s$
Adding: $20 = 4+4s$
$\Rightarrow \quad 4s = 16$
$\Rightarrow \quad s = 4$
$12-2t = 8+8$
$\Rightarrow \quad 2t = -4$
$\Rightarrow \quad t = -2$

If the lines intersect, $-6+4t = 6-5s$ must be true.
$-6+4t = -6-8 = -14$
$6-5s = 6-20 = -14$
The z components are equal, so the lines do intersect. Intersection point:

$$\begin{pmatrix} 12-2t \\ 4+t \\ -6+4t \end{pmatrix} = \begin{pmatrix} 16 \\ 2 \\ -14 \end{pmatrix}.$$

Coordinates $(16, 2, -14)$

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise I, Question 4

Question:

Determine whether the lines with the given equations intersect. If they do intersect, find the coordinates of their point of intersection.

$$\mathrm{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} \text{ and } \mathrm{r} = \begin{pmatrix} -2 \\ -9 \\ 12 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Solution:

$$\mathrm{r} = \begin{pmatrix} 1+4t \\ 2t \\ 4+6t \end{pmatrix}, \mathrm{r} = \begin{pmatrix} -2+s \\ -9+2s \\ 12-s \end{pmatrix}$$

At an intersection point: $\begin{pmatrix} 1+4t \\ 2t \\ 4+6t \end{pmatrix} = \begin{pmatrix} -2+s \\ -9+2s \\ 12-s \end{pmatrix}$

$1 + 4t = -2 + s$
$2t = -9 + 2s \quad (\times 2)$
$1 + 4t = -2 + s$
$4t = -18 + 4s$
Subtracting: $1 = 16 - 3s$
$\Rightarrow \quad 3s = 15$
$\Rightarrow \quad s = 5$
$1 + 4t = -2 + 5$
$\Rightarrow \quad 4t = 2$
$\Rightarrow \quad t = \frac{1}{2}$

If the lines intersect, $4 + 6t = 12 - s$ must be true.
$4 + 6t = 4 + 3 = 7$
$12 - s = 12 - 5 = 7$
The z components are equal, so the lines do intersect. Intersection point:

$$\begin{pmatrix} 1+4t \\ 2t \\ 4+6t \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}.$$

Coordinates (3, 1, 7)

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise I, Question 5

Question:

Determine whether the lines with the given equations intersect. If they do intersect, find the coordinates of their point of intersection.

$$\mathrm{r} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \text{ and } \mathrm{r} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + s \begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix}$$

Solution:

$$\mathrm{r} = \begin{pmatrix} 3+2t \\ -3+t \\ 1-4t \end{pmatrix}, \mathrm{r} = \begin{pmatrix} 3+6s \\ 4-4s \\ 2+s \end{pmatrix}$$

At an intersection point: $\begin{pmatrix} 3+2t \\ -3+t \\ 1-4t \end{pmatrix} = \begin{pmatrix} 3+6s \\ 4-4s \\ 2+s \end{pmatrix}$

$3 + 2t = 3 + 6s$
$-3 + t = 4 - 4s \quad (\times 2)$
$3 + 2t = 3 + 6s$
$-6 + 2t = 8 - 8s$
Subtracting: $9 = -5 + 14s$
$\Rightarrow \quad 14s = 14$
$\Rightarrow \quad s = 1$
$3 + 2t = 3 + 6$
$\Rightarrow \quad 2t = 6$
$\Rightarrow \quad t = 3$
If the lines intersect, $1 - 4t = 2 + s$ must be true.
$1 - 4t = 1 - 12 = -11$
$2 + s = 2 + 1 = 3$
The z components are not equal, so the lines do not intersect.

Solutionbank 4
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Vectors
Exercise J, Question 1

Question:

Find, to 1 decimal place, the acute angle between the lines with the given vector equations:
r = (2i + j + k) + *t* (3i − 5j − k)
and r = (7i + 4j + k) + *s* (2i + j − 9k)

Solution:

Direction vectors are $a = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 1 \\ -9 \end{pmatrix}$

$$\cos\theta = \frac{a \cdot b}{|a| \ |b|}$$

$$a \cdot b = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -9 \end{pmatrix} = 6 - 5 + 9 = 10$$

$$|a| = \sqrt{3^2 + (-5)^2 + (-1)^2} = \sqrt{35}$$

$$|b| = \sqrt{2^2 + 1^2 + (-9)^2} = \sqrt{86}$$

$$\cos\theta = \frac{10}{\sqrt{35}\sqrt{86}}$$

$\theta = 79.5$ ° (1 d.p.)

The acute angle between the lines is 79.5 ° (1 d.p.)

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise J, Question 2

Question:

Find, to 1 decimal place, the acute angle between the lines with the given vector equations:
r = (i − j + 7k) + t (− 2i − j + 3k)
and r = (8i + 5j − k) + s (− 4i − 2j + k)

Solution:

Direction vectors are $a = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$ and $b = \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{a \,.\, b}{|a|\ |b|}$$

$$a \,.\, b = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} . \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix} = 8 + 2 + 3 = 13$$

$$|a| = \sqrt{(-2)^2 + (-1)^2 + 3^2} = \sqrt{14}$$
$$|b| = \sqrt{(-4)^2 + (-2)^2 + 1^2} = \sqrt{21}$$

$$\cos\theta = \frac{13}{\sqrt{14}\sqrt{21}}$$

$\theta = 40.7°$ (1 d.p.)
The acute angle between the lines is 40.7 ° (1 d.p.)

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise J, Question 3

Question:

Find, to 1 decimal place, the acute angle between the lines with the given vector equations:
r = (3i + 5j − k) + *t* (i + j + k)
and r = (− i + 11j + 5k) + *s* (2i − 7j + 3k)

Solution:

Direction vectors are $a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}$

$$\cos\theta = \frac{a \cdot b}{|a| \ |b|}$$

$$a \cdot b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix} = 2 - 7 + 3 = -2$$

$$|a| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$
$$|b| = \sqrt{2^2 + (-7)^2 + 3^2} = \sqrt{62}$$

$$\cos\theta = \frac{-2}{\sqrt{3}\sqrt{62}}$$

$\theta = 98.4°$ (1 d.p.)
This is the angle between the two vectors.
The acute angle between the lines is 180 ° − 98.4 ° = 81.6 ° (1 d.p.).

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise J, Question 4

Question:

Find, to 1 decimal place, the acute angle between the lines with the given vector equations:
$\mathbf{r} = (\mathbf{i} + 6\mathbf{j} - \mathbf{k}) + t (8\mathbf{i} - \mathbf{j} - 2\mathbf{k})$
and $\mathbf{r} = (6\mathbf{i} + 9\mathbf{j}) + s (\mathbf{i} + 3\mathbf{j} - 7\mathbf{k})$

Solution:

Direction vectors are $\mathbf{a} = \begin{pmatrix} 8 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}$

$\cos\theta = \frac{\mathbf{a} . \mathbf{b}}{| \mathbf{a} | \ | \mathbf{b} |}$

$\mathbf{a} . \mathbf{b} = \begin{pmatrix} 8 \\ -1 \\ -2 \end{pmatrix} . \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix} = 8 - 3 + 14 = 19$

$| \mathbf{a} | = \sqrt{8^2 + (-1)^2 + (-2)^2} = \sqrt{69}$

$| \mathbf{b} | = \sqrt{1^2 + 3^2 + (-7)^2} = \sqrt{59}$

$\cos\theta = \frac{19}{\sqrt{69}\sqrt{59}}$

$\theta = 72.7°$ (1 d.p.)

The acute angle between the lines is 72.7 ° (1 d.p.)

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise J, Question 5

Question:

Find, to 1 decimal place, the acute angle between the lines with the given vector equations:
$\mathbf{r} = (2\mathbf{i} + \mathbf{k}) + t (11\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$
and $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + s (- 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$

Solution:

Direction vectors are $\mathbf{a} = \begin{pmatrix} 11 \\ 5 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{| \mathbf{a} | \ | \mathbf{b} |}$$

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 11 \\ 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} = -33 + 25 - 12 = -20$$

$$| \mathbf{a} | = \sqrt{11^2 + 5^2 + (-3)^2} = \sqrt{155}$$
$$| \mathbf{b} | = \sqrt{(-3)^2 + 5^2 + 4^2} = \sqrt{50}$$

$$\cos\theta = \frac{-20}{\sqrt{155}\sqrt{50}}$$

$\theta = 103.1°$ (1 d.p.)
This is the angle between the two vectors.
The acute angle between the lines is $180° - 103.1° = 76.9°$ (1 d.p.).

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise J, Question 6

Question:

The straight lines l_1 and l_2 have vector equations
$\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + t(8\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + s(3\mathbf{i} + \mathbf{j})$ respectively, and P is the point with coordinates $(1, 4, 2)$.

(a) Show that the point $Q(9, 9, 3)$ lies on l_1.

(b) Find the cosine of the acute angle between l_1 and l_2.

(c) Find the possible coordinates of the point R, such that R lies on l_2 and PQ = PR.

Solution:

(a) Line l_1: $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix}$

When $t = 1$, $\mathbf{r} = \begin{pmatrix} 9 \\ 9 \\ 3 \end{pmatrix}$

So the point (9, 9, 3) lies on l_1.

(b) Direction vectors are $\mathbf{a} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = 24 + 5 + 0 = 29$$

$$|\mathbf{a}| = \sqrt{8^2 + 5^2 + 1^2} = \sqrt{90}$$
$$|\mathbf{b}| = \sqrt{3^2 + 1^2 + 0^2} = \sqrt{10}$$
$$\cos\theta = \frac{29}{\sqrt{90}\sqrt{10}} = \frac{29}{\sqrt{900}} = \frac{29}{30}$$

(c) $PQ = \sqrt{(9-1)^2 + (9-4)^2 + (3-2)^2}$
$= \sqrt{8^2 + 5^2 + 1^2} = \sqrt{90}$

Line l_2: $\quad \mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + s \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + 3s \\ 4 + s \\ 2 \end{pmatrix}$

Let the coordinates of R be $\left(1 + 3s , 4 + s , 2 \right)$

$PR = \sqrt{\left(1 + 3s - 1 \right)^2 + \left(4 + s - 4 \right)^2 + \left(2 - 2 \right)^2}$

$= \sqrt{9s^2 + s^2} = \sqrt{10s^2}$

$PQ^2 = PR^2$: $\quad 90 = 10s^2$

$\Rightarrow \quad s^2 = 9$

$\Rightarrow \quad s = \pm 3$

When $s = 3$, $\mathbf{r} = \begin{pmatrix} 10 \\ 7 \\ 2 \end{pmatrix} \qquad R : \left(10 , 7 , 2 \right)$

When $s = -3$, $\mathbf{r} = \begin{pmatrix} -8 \\ 1 \\ 2 \end{pmatrix} \qquad R : \left(-8 , 1 , 2 \right)$

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 1

Question:

With respect to an origin *O*, the position vectors of the points *L*, *M* and *N* are (4i + 7j + 7k) , (i + 3j + 2k) and (2i + 4j + 6k) respectively.

(a) Find the vectors ML and MN.

(b) Prove that $\cos \angle \text{LMN} = \frac{9}{10}$.

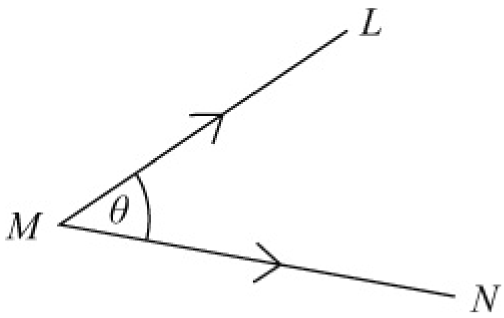
Solution:

$$l = \begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix}, m = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, n = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

(a) $\text{ML} = l - m = \begin{pmatrix} 4 \\ 7 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

$\text{MN} = n - m = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

(b)



$$\cos\theta = \frac{\text{ML} . \text{MN}}{|\text{ML}| \ |\text{MN}|}$$

$$\text{ML} . \text{MN} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = 3 + 4 + 20 = 27$$

$$| ML | = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50}$$

$$| MN | = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18}$$

$$\cos\theta = \frac{27}{\sqrt{50}\sqrt{18}} = \frac{27}{\sqrt{25}\sqrt{2}\sqrt{9}\sqrt{2}} = \frac{27}{5 \times 3 \times 2} = \frac{9}{10}.$$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

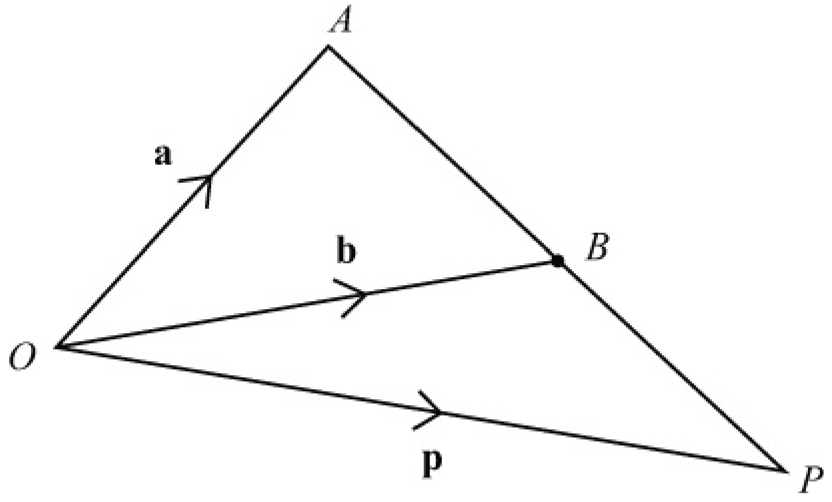
Exercise K, Question 2

Question:

The position vectors of the points *A* and *B* relative to an origin *O* are 5i + 4j + k, − i + j − 2k respectively. Find the position vector of the point *P* which lies on *AB* produced such that AP = 2BP.

E

Solution:



$$a = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

OP = OA + AP = OA + 2AB

p = a + 2 (b − a) = 2b − a

$$p = 2 \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix}$$

The position vector of *P* is − 7i − 2j − 5k.

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 3

Question:

Points *A*, *B*, *C*, *D* in a plane have position vectors a = 6i + 8j, b = $\frac{3}{2}$a, c = 6i + 3j, d = $\frac{5}{3}$c respectively. Write down vector equations of the lines *AD* and *BC* and find the position vector of their point of intersection.

E

Solution:

$$a = \begin{pmatrix} 6 \\ 8 \end{pmatrix}, b = \frac{3}{2}a = \begin{pmatrix} 9 \\ 12 \end{pmatrix}$$

$$c = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, d = \frac{5}{3}c = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

Line *AD*: $AD = d - a = \begin{pmatrix} 10 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$$r = \begin{pmatrix} 6 \\ 8 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

Line *BC*: $BC = c - b = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 9 \\ 12 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$

$$r = \begin{pmatrix} 9 \\ 12 \end{pmatrix} + s\begin{pmatrix} -3 \\ -9 \end{pmatrix}$$

or

$$r = \begin{pmatrix} 9 \\ 12 \end{pmatrix} + s\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Where *AD* and *BC* intersect, $\begin{pmatrix} 6 + 4t \\ 8 - 3t \end{pmatrix} = \begin{pmatrix} 9 + s \\ 12 + 3s \end{pmatrix}$ (Using the last version of *BC*)

$6 + 4t = 9 + s \quad (\times 3)$
$8 - 3t = 12 + 3s$
$18 + 12t = 27 + 3s$
$8 - 3t = 12 + 3s$
Subtracting: $10 + 15t = 15$
$\Rightarrow \quad 15t = 5$

$$\Rightarrow \quad t = \frac{1}{3}$$

Intersection: $\mathbf{r} = \begin{pmatrix} 6 + 4t \\ 8 - 3t \end{pmatrix} = \begin{pmatrix} \frac{22}{3} \\ 7 \end{pmatrix}$

$\mathbf{r} = \frac{22}{3}\mathbf{i} + 7\mathbf{j}$

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 4

Question:

Find the point of intersection of the line through the points (2, 0, 1) and $(-1 , 3 , 4)$ and the line through the points $(-1 , 3 , 0)$ and $(4 , -2 , 5)$.
Calculate the acute angle between the two lines.

E

Solution:

Line through (2, 0, 1) and $(-1 , 3 , 4)$.

Let a = $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and b = $\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

$$b - a = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$$

Equation: $r = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$

or $r = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Line through $(-1 , 3 , 0)$ and $(4 , -2 , 5)$.

Let c = $\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$ and d = $\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$

$$d - c = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$$

Equation: $r = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + s \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$

or $r = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

At the intersection point: $\begin{pmatrix} 2-t \\ t \\ 1+t \end{pmatrix} = \begin{pmatrix} -1+s \\ 3-s \\ s \end{pmatrix}$

$2-t = -1+s$

$t = 3-s$

$1+t = s$

Adding the second and third equations:

$1 + 2t = 3$

$2t = 2$

$t = 1$

$s = 2$

Intersection point:

$\mathrm{r} = \begin{pmatrix} 2-t \\ t \\ 1+t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ Coordinates (1, 1, 2)

Direction vectors of the lines are $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Calling these **m** and **n**:

$\cos\theta = \frac{\mathrm{m}\,.\,\mathrm{n}}{|\mathrm{m}|\ |\mathrm{n}|}$

$\mathrm{m}\,.\,\mathrm{n} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = -1 - 1 + 1 = -1$

$|\mathrm{m}| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$

$|\mathrm{n}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

$\cos\theta = \frac{-1}{\sqrt{3}\sqrt{3}} = \frac{-1}{3}$

$\theta = 109.5°$ (1 d.p.)

This is the angle between the two vectors.

The acute angle between the lines is $180° - 109.5° = 70.5°$ (1 d.p.).

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 5

Question:

Show that the lines
$\mathbf{r} = (-2\mathbf{i} + 5\mathbf{j} - 11\mathbf{k}) + \lambda (3\mathbf{i} + \mathbf{j} + 3\mathbf{k})$
$\mathbf{r} = 8\mathbf{i} + 9\mathbf{j} + \mu (4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})$
intersect. Find the position vector of their common point.

Solution:

$$\mathbf{r} = \begin{pmatrix} -2 + 3\lambda \\ 5 + \lambda \\ -11 + 3\lambda \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 8 + 4\mu \\ 9 + 2\mu \\ 5\mu \end{pmatrix}$$

At an intersection point: $\begin{pmatrix} -2 + 3\lambda \\ 5 + \lambda \\ -11 + 3\lambda \end{pmatrix} = \begin{pmatrix} 8 + 4\mu \\ 9 + 2\mu \\ 5\mu \end{pmatrix}$

$-2 + 3\lambda = 8 + 4\mu$
$5 + \lambda = 9 + 2\mu \quad (\times 2)$
$-2 + 3\lambda = 8 + 4\mu$
$10 + 2\lambda = 18 + 4\mu$
Subtracting: $-12 + \lambda = -10$
$\Rightarrow \lambda = 12 - 10$
$\Rightarrow \lambda = 2$
$-2 + 6 = 8 + 4\mu$
$\Rightarrow 4\mu = -4$
$\Rightarrow \lambda = -1$
If the lines intersect, $-11 + 3\lambda = 5\mu$:
$-11 + 3\lambda = -11 + 6 = -5$
$5\mu = -5$
The *z* components are equal, so the lines do intersect. Intersection point:

$$\mathbf{r} = \begin{pmatrix} -2 + 3\lambda \\ 5 + \lambda \\ -11 + 3\lambda \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -5 \end{pmatrix} = 4\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}.$$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise K, Question 6

Question:

Find a vector that is perpendicular to both 2i + j − k and i + j − 2k.

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Solution:

Let the required vector be xi + yj + zk.

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$2x + y - z = 0$

$x + y - 2z = 0$

Let $z = 1$:

$2x + y = 1$

$x + y = 2$

Subtracting: $x = -1, y = 3$

So $x = -1, y = 3$ and $z = 1$

A possible vector is − i + 3j + k.

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 7

Question:

State a vector equation of the line passing through the points *A* and *B* whose position vectors are i − j + 3k and i + 2j + 2k respectively. Determine the position vector of the point *C* which divides the line segment *AB* internally such that AC = 2CB.

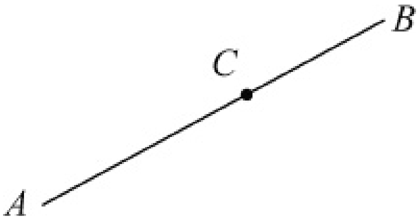
Solution:

$$a = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Equation of line:

r = a + *t* (b − a)

$$r = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$



but AC = 2CB

Position vector of *C*:

$$c = a + \frac{2}{3}\left(b - a \right)$$

$$= \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \frac{7}{3} \end{pmatrix}$$

$$= i + j + \frac{7}{3}k$$

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Vectors
Exercise K, Question 8

Question:

Vectors **r** and **s** are given by
$r = \lambda i + (2\lambda - 1) j - k$
$s = (1 - \lambda) i + 3\lambda j + (4\lambda - 1) k$
where λ is a scalar.

(a) Find the values of λ for which **r** and **s** are perpendicular.
When $\lambda = 2$, **r** and **s** are the position vectors of the points *A* and *B* respectively, referred to an origin *0*.

(b) Find AB.

(c) Use a scalar product to find the size of ∠ BAO, giving your answer to the nearest degree.

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Solution:

$$r = \begin{pmatrix} \lambda \\ 2\lambda - 1 \\ -1 \end{pmatrix}, \text{ and } \quad s = \begin{pmatrix} 1 - \lambda \\ 3\lambda \\ 4\lambda - 1 \end{pmatrix}$$

(a) If **r** and **s** are perpendicular, r . s = 0

$$r . s = \begin{pmatrix} \lambda \\ 2\lambda - 1 \\ -1 \end{pmatrix} . \begin{pmatrix} 1 - \lambda \\ 3\lambda \\ 4\lambda - 1 \end{pmatrix}$$

$$= \lambda(1 - \lambda) + 3\lambda(2\lambda - 1) - 1(4\lambda - 1)$$

$$= \lambda - \lambda^2 + 6\lambda^2 - 3\lambda - 4\lambda + 1$$

$$= 5\lambda^2 - 6\lambda + 1$$

$$\therefore 5\lambda^2 - 6\lambda + 1 = 0$$

$$(5\lambda - 1)(\lambda - 1) = 0$$

$$\lambda = \frac{1}{5} \text{or } \lambda = 1$$

(b) $\lambda = 2$: $\quad a = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, b = \begin{pmatrix} -1 \\ 6 \\ 7 \end{pmatrix}$

$$AB = b - a = \begin{pmatrix} -1 \\ 6 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 8 \end{pmatrix}$$

$$= -3i + 3j + 8k$$

(c) Using vectors AB and AO:

$$AB = \begin{pmatrix} -3 \\ 3 \\ 8 \end{pmatrix}, AO = -a = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

$$\cos \angle BAO = \frac{AB \,.\, AO}{|AB|\ |AO|}$$

$$AB \,.\, AO = \begin{pmatrix} -3 \\ 3 \\ 8 \end{pmatrix} . \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = 6 - 9 + 8 = 5$$

$$|AB| = \sqrt{(-3)^2 + 3^2 + 8^2} = \sqrt{82}$$

$$|AO| = \sqrt{(-2)^2 + (-3)^2 + 1^2} = \sqrt{14}$$

$$\cos \angle BAO = \frac{5}{\sqrt{82}\sqrt{14}}$$

$\angle BAO = 82°$ (nearest degree)

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 9

Question:

With respect to an origin *O*, the position vectors of the points *L* and *M* are $2\mathrm{i} - 3\mathrm{j} + 3\mathrm{k}$ and $5\mathrm{i} + \mathrm{j} + c\mathrm{k}$ respectively, where *c* is a constant. The point *N* is such that *OLMN* is a rectangle.

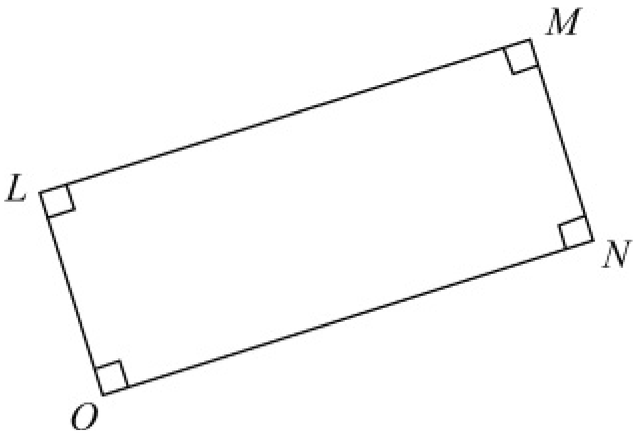
(a) Find the value of *c*.

(b) Write down the position vector of *N*.

(c) Find, in the form $\mathrm{r} = \mathrm{p} + t\mathrm{q}$, an equation of the line *MN*.

E

Solution:



(a) $\mathrm{l} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$ and $\mathrm{m} = \begin{pmatrix} 5 \\ 1 \\ c \end{pmatrix}$

$$\mathrm{LM} = \mathrm{m} - \mathrm{l} = \begin{pmatrix} 5 \\ 1 \\ c \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ c-3 \end{pmatrix}$$

Since OL and LM are perpendicular, OL . LM = *O*

$$\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} . \begin{pmatrix} 3 \\ 4 \\ c-3 \end{pmatrix} = 0$$

$6 - 12 + 3(c - 3) = 0$
$6 - 12 + 3c - 9 = 0$
$3c = 15$

$c = 5$

(b) $\mathrm{n} = \mathrm{ON} = \mathrm{LM} = \begin{pmatrix} 3 \\ 4 \\ c-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$

$\mathrm{n} = 3\mathrm{i} + 4\mathrm{j} + 2\mathrm{k}$

(c) The line *MN* is parallel to *OL*.
Using the point *M* and the direction vector **l**:

$$\mathrm{r} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise K, Question 10

Question:

The point *A* has coordinates $(7, -1, 3)$ and the point *B* has coordinates $(10, -2, 2)$. The line *l* has vector equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + \mathbf{k})$, where λ is a real parameter.

(a) Show that the point *A* lies on the line *l*.

(b) Find the length of *AB*.

(c) Find the size of the acute angle between the line *l* and the line segment *AB*, giving your answer to the nearest degree.

(d) Hence, or otherwise, calculate the perpendicular distance from *B* to the line *l*, giving your answer to two significant figures.

Solution:

(a) Line *l*: $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

Point *A* is $(7, -1, 3)$

Using $\lambda = 2$, $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}$

So *A* lies on the line *l*.

(b) $AB = \sqrt{(10-7)^2 + [-2-(-1)]^2 + (2-3)^2}$

$= \sqrt{3^2 + (-1)^2 + (-1)^2} = \sqrt{11}$

(c) $AB = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 10 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

Angle between the vectors $\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$:

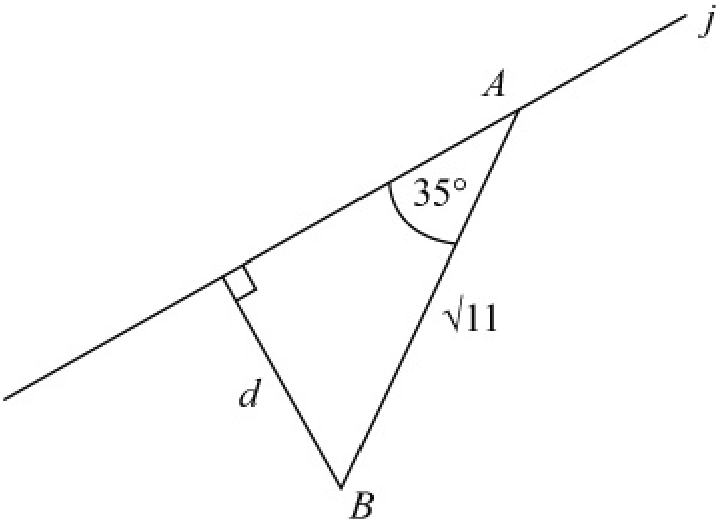
$$\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} . \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 9 + 1 - 1 = 9$$

The magnitude of each of the vectors is $\sqrt{11}$

So $\cos\theta = \frac{9}{\sqrt{11}\sqrt{11}} = \frac{9}{11}$

$\Rightarrow \quad \theta = 35°$ (nearest degree)

(d)



$\sin 35° = \frac{d}{\sqrt{11}}$

$d = \sqrt{11} \sin 35° = 1.9$ (2 s.f.)

Solutionbank 4
Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 11

Question:

Referred to a fixed origin *O*, the points *A* and *B* have position vectors (5i − j − k) and (i − 5j + 7k) respectively.

(a) Find an equation of the line *AB*.

(b) Show that the point *C* with position vector 4i − 2j + k lies on *AB*.

(c) Show that *OC* is perpendicular to *AB*.

(d) Find the position vector of the point *D*, where $D \not\equiv A$, on *AB* such that
$| OD | = | OA |$.

E

Solution:

(a) $a = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix}$

$$b - a = \begin{pmatrix} 1 \\ -5 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix}$$

Equation of *AB*:

$$r = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix}$$

or

$$r = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

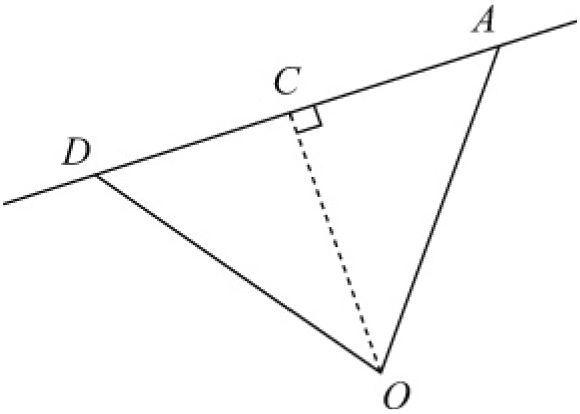
(b) Using $t = 1$: $r = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

So the point with position vector 4i − 2j + k lies on *AB*.

(c) $OC \, . \, AB = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} . \begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix} = -16 + 8 + 8 = 0$

Since the scalar product is zero, *OC* is perpendicular to *AB*.

(d)



Since OD = OA, DC = CA, so DC = CA.

$CA = a - c = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$DC = c - d = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

So $d = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$

$d = 3i - 3j + 3k$

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Edexcel AS and A Level Modular Mathematics

Vectors
Exercise K, Question 12

Question:

Referred to a fixed origin O, the points A, B and C have position vectors $(9\text{i} - 2\text{j} + \text{k})$, $(6\text{i} + 2\text{j} + 6\text{k})$ and $(3\text{i} + p\text{j} + q\text{k})$ respectively, where p and q are constants.

(a) Find, in vector form, an equation of the line l which passes through A and B.
Given that C lies on l:

(b) Find the value of p and the value of q.

(c) Calculate, in degrees, the acute angle between OC and AB,
The point D lies on AB and is such that OD is perpendicular to AB.

(d) Find the position vector of D.

Solution:

$$\text{a} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix}, \text{b} = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix}, \text{c} = \begin{pmatrix} 3 \\ p \\ q \end{pmatrix}$$

(a) $\text{b} - \text{a} = \begin{pmatrix} 6 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$

Equation of l:

$$\text{r} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

(b) Since C lies on l,

$$\begin{pmatrix} 3 \\ p \\ q \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

$3 = 9 - 3t$
$3t = 6$
$t = 2$

So $p = -2 + 4t = 6$
and $q = 1 + 5t = 11$

(c) $\cos\theta = \frac{\mathrm{OC} \,.\, \mathrm{AB}}{|\mathrm{OC}| \ |\mathrm{AB}|}$

$$\mathrm{OC} \,.\, \mathrm{AB} = \begin{pmatrix} 3 \\ 6 \\ 11 \end{pmatrix} . \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} = -9 + 24 + 55 = 70$$

$$|\mathrm{OC}| = \sqrt{3^2 + 6^2 + 11^2} = \sqrt{166}$$

$$|\mathrm{AB}| = \sqrt{(-3)^2 + 4^2 + 5^2} = \sqrt{50}$$

$$\cos\theta = \frac{70}{\sqrt{166}\sqrt{50}}$$

$\theta = 39.8°$ (1 d.p.)

(d) If *OD* and *AB* are perpendicular, $\mathrm{d} \,.\, (\mathrm{b} - \mathrm{a}) = 0$

Since **d** lies on *AB*, use $\mathrm{d} = \begin{pmatrix} 9 - 3t \\ -2 + 4t \\ 1 + 5t \end{pmatrix}$

$$\begin{pmatrix} 9 - 3t \\ -2 + 4t \\ 1 + 5t \end{pmatrix} . \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} = 0$$

$$-3(9 - 3t) + 4(-2 + 4t) + 5(1 + 5t) = 0$$

$$-27 + 9t - 8 + 16t + 5 + 25t = 0$$

$$50t = 30$$

$$t = \frac{3}{5}$$

$$\mathrm{d} = \begin{pmatrix} 9 - \frac{9}{5} \\ -2 + \frac{12}{5} \\ 1 + 3 \end{pmatrix} = \frac{36}{5}\mathrm{i} + \frac{2}{5}\mathrm{j} + 4\mathrm{k}$$

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise K, Question 13

Question:

Referred to a fixed origin O, the points A and B have position vectors $(\mathrm{i} + 2\mathrm{j} - 3\mathrm{k})$ and $(5\mathrm{i} - 3\mathrm{j})$ respectively.

(a) Find, in vector form, an equation of the line l_1 which passes through A and B.
The line l_2 has equation $\mathrm{r} = (4\mathrm{i} - 4\mathrm{j} + 3\mathrm{k}) + \lambda (\mathrm{i} - 2\mathrm{j} + 2\mathrm{k})$, where λ is a scalar parameter.

(b) Show that A lies on l_2.

(c) Find, in degrees, the acute angle between the lines l_1 and l_2.
The point C with position vector $(2\mathrm{i} - \mathrm{k})$ lies on l_2.

(d) Find the shortest distance from C to the line l_1.

Solution:

(a) $\mathrm{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, $\mathrm{b} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix}$

$$\mathrm{b} - \mathrm{a} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$$

Equation of l_1:

$$\mathrm{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$$

(b) Equation of l_2:

$$\mathrm{r} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

Using $\lambda = -3$, $r = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

So *A* lies on the line l_2.

(c) Direction vectors of l_1 and l_2 are $\begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

Calling these **m** and **n**:

$$\cos \theta = \frac{\mathrm{m} \cdot \mathrm{n}}{| \mathrm{m} | \; | \mathrm{n} |}$$

$$\mathrm{m} \cdot \mathrm{n} = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 4 + 10 + 6 = 20$$

$$| \mathrm{m} | = \sqrt{4^2 + (-5)^2 + 3^2} = \sqrt{50}$$

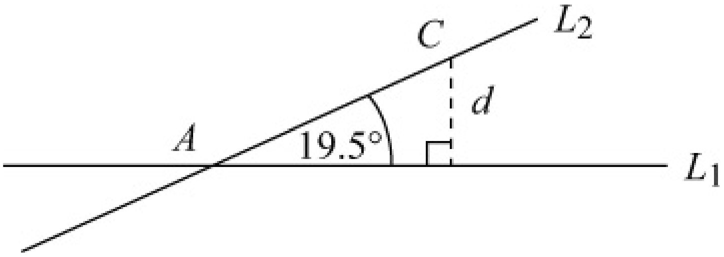
$$| \mathrm{n} | = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

$$\cos \theta = \frac{20}{3\sqrt{50}}$$

$\theta = 19.5°$ (1 d.p.)

The angle between l_1 and l_2 is 19.5° (1 d.p.).

(d) $\mathrm{c} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$



$$| \mathrm{AC} | = \sqrt{(2 - 1)^2 + (0 - 2)^2 + [-1 - (-3)]^2}$$

$$= \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

$$\sin \theta = \frac{d}{\mathrm{AC}}$$

$$d = \mathrm{AC} \sin \theta = 3 \times \frac{1}{3} = 1$$

The shortest distance from *C* to l_1 is 1 unit.

Solutionbank 4

Edexcel AS and A Level Modular Mathematics

Vectors

Exercise K, Question 14

Question:

Two submarines are travelling in straight lines through the ocean. Relative to a fixed origin, the vector equations of the two lines, l_1 and l_2, along which they travel are

$r = 3i + 4j - 5k + \lambda\ (i - 2j + 2k)$

and $r = 9i + j - 2k + \mu\ (4i + j - k)$

where λ and μ are scalars.

(a) Show that the submarines are moving in perpendicular directions.

(b) Given that l_1 and l_2 intersect at the point *A*, find the position vector of *A*.
The point *B* has position vector $10j - 11k$.

(c) Show that only one of the submarines passes through the point *B*.

(d) Given that 1 unit on each coordinate axis represents 100 m, find, in km, the distance *AB*.

Solution:

(a) Line l_1: $r = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

Line l_2: $r = \begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$

Using the direction vectors:

$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} . \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} = 4 - 2 - 2 = 0$$

Since the scalar product is zero, the directions are perpendicular.

(b) At an intersection point: $\begin{pmatrix} 3 + \lambda \\ 4 - 2\lambda \\ -5 + 2\lambda \end{pmatrix} = \begin{pmatrix} 9 + 4\mu \\ 1 + \mu \\ -2 - \mu \end{pmatrix}$

$3+\lambda=9+4\mu \quad (\times 2)$
$4-2\lambda=1+\mu$
$6+2\lambda=18+8\mu$
$4-2\lambda=1+\mu$
Adding: $10=19+9\mu$
$\Rightarrow \quad 9\mu=-9$
$\Rightarrow \quad \mu=-1$
$3+\lambda=9-4$
$\Rightarrow \quad \lambda=2$

Intersection point: $\begin{pmatrix} 3+\lambda \\ 4-2\lambda \\ -5+2\lambda \end{pmatrix}=\begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$

Position vector of A is $\mathbf{a}=5\mathbf{i}-\mathbf{k}$.

(c) Position vector of B: $\mathbf{b}=10\mathbf{j}-11\mathbf{k}=\begin{pmatrix} 0 \\ 10 \\ -11 \end{pmatrix}$

For l_1, to give zero as the x component, $\lambda=-3$.

$$\mathbf{r}=\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}-3\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}=\begin{pmatrix} 0 \\ 10 \\ -11 \end{pmatrix}$$

So B lies on l_1.

For l_2, to give -11 as the z component, $\mu=9$.

$$\mathbf{r}=\begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix}+9\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}=\begin{pmatrix} 45 \\ 10 \\ -11 \end{pmatrix}$$

So B does not lie on l_2.

So only one of the submarines passes through B.

(d) $|AB|=\sqrt{(0-5)^2+(10-0)^2+[-11-(-1)]^2}$
$=\sqrt{(-5)^2+10^2+(-10)^2}$
$=\sqrt{225}=15$

Since 1 unit represents 100 m, the distance AB is
$15\times 100=1500\,\text{m}=1.5\,\text{km}$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise A, Question 1

Question:

Integrate the following with respect to x:

(a) $3\sec^2 x + \frac{5}{x} + \frac{2}{x^2}$

(b) $5e^x - 4\sin x + 2x^3$

(c) $2(\sin x - \cos x + x)$

(d) $3\sec x\tan x - \frac{2}{x}$

(e) $5e^x + 4\cos x - \frac{2}{x^2}$

(f) $\frac{1}{2x} + 2\,\text{cosec}^2 x$

(g) $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$

(h) $e^x + \sin x + \cos x$

(i) $2\,\text{cosec}\, x\cot x - \sec^2 x$

(j) $e^x + \frac{1}{x} - \text{cosec}^2 x$

Solution:

(a) $\int \left(3\sec^2 x + \frac{5}{x} + \frac{2}{x^2} \right) dx$

$= \int \left(3\sec^2 x + \frac{5}{x} + 2x^{-2} \right) dx$

$= 3\tan x + 5\ln|x| - \frac{2}{x} + C$

(b) $\int \left(5e^{x} - 4\sin x + 2x^{3} \right) dx$

$= 5e^{x} + 4\cos x + \frac{2x^{4}}{4} + C$

$= 5e^{x} + 4\cos x + \frac{x^{4}}{2} + C$

(c) $\int 2 \left(\sin x - \cos x + x \right) dx$
$= \int \left(2\sin x - 2\cos x + 2x \right) dx$
$= - 2\cos x - 2\sin x + x^{2} + C$

(d) $\int \left(3\sec x \tan x - \frac{2}{x} \right) dx$

$= 3\sec x - 2\ln | x | + C$

(e) $\int \left(5e^{x} + 4\cos x - \frac{2}{x^{2}} \right) dx$

$= \int \left(5e^{x} + 4\cos x - 2x^{-2} \right) dx$

$= 5e^{x} + 4\sin x + \frac{2}{x} + C$

(f) $\int \left(\frac{1}{2x} + 2\operatorname{cosec}^{2} x \right) dx$

$= \int \left(\frac{1}{2} \times \frac{1}{x} + 2\operatorname{cosec}^{2} x \right) dx$

$= \frac{1}{2} \ln | x | - 2\cot x + C$

(g) $\int \left(\frac{1}{x} + \frac{1}{x^{2}} + \frac{1}{x^{3}} \right) dx$

$= \int \left(\frac{1}{x} + x^{-2} + x^{-3} \right) dx$

$= \ln | x | + \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + C$

$= \ln | x | - \frac{1}{x} - \frac{1}{2x^{2}} + C$

(h) $\int \left(e^{x} + \sin x + \cos x \right) dx$
$= e^{x} - \cos x + \sin x + C$

(i) $\int \left(2\operatorname{cosec} x \cot x - \sec^2 x \right) \mathrm{d}x$
$= -2\operatorname{cosec} x - \tan x + C$

(j) $\int \left(\mathrm{e}^x + \frac{1}{x} - \operatorname{cosec}^2 x \right) \mathrm{d}x$
$= \mathrm{e}^x + \ln |x| + \cot x + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise A, Question 2

Question:

Find the following integrals:

(a) $\int \left(\frac{1}{\cos^2 x} + \frac{1}{x^2} \right) \mathrm{d}x$

(b) $\int \left(\frac{\sin x}{\cos^2 x} + 2e^x \right) \mathrm{d}x$

(c) $\int \left(\frac{1 + \cos x}{\sin^2 x} + \frac{1 + x}{x^2} \right) \mathrm{d}x$

(d) $\int \left(\frac{1}{\sin^2 x} + \frac{1}{x} \right) \mathrm{d}x$

(e) $\int \sin x \left(1 + \sec^2 x \right) \mathrm{d}x$

(f) $\int \cos x \left(1 + \operatorname{cosec}^2 x \right) \mathrm{d}x$

(g) $\int \operatorname{cosec}^2 x \left(1 + \tan^2 x \right) \mathrm{d}x$

(h) $\int \sec^2 x \left(1 - \cot^2 x \right) \mathrm{d}x$

(i) $\int \sec^2 x \left(1 + e^x \cos^2 x \right) \mathrm{d}x$

(j) $\int \left(\frac{1 + \sin x}{\cos^2 x} + \cos^2 x \sec x \right) \mathrm{d}x$

Solution:

(a) $\int \left(\frac{1}{\cos^2 x} + \frac{1}{x^2} \right) \mathrm{d}x$

$= \int \left(\sec^2 x + x^{-2} \right) \mathrm{d}x$

$= \tan x - \frac{1}{x} + C$

(b) $\int \left(\frac{\sin x}{\cos^2 x} + 2e^x \right) dx$

$= \int (\tan x \sec x + 2e^x) \, dx$

$= \sec x + 2e^x + C$

(c) $\int \left(\frac{1 + \cos x}{\sin^2 x} + \frac{1 + x}{x^2} \right) dx$

$= \int (\text{cosec}^2 x + \cot x \, \text{cosec} \, x + x^{-2} + x^{-1}) \, dx$

$= - \cot x - \text{cosec} \, x - \frac{1}{x} + \ln | x | + C$

(d) $\int \left(\frac{1}{\sin^2 x} + \frac{1}{x} \right) dx$

$= \int \left(\text{cosec}^2 x + \frac{1}{x} \right) dx$

$= - \cot x + \ln | x | + C$

(e) $\int \sin x (1 + \sec^2 x) \, dx$

$= \int (\sin x + \sin x \sec^2 x) \, dx$

$= \int (\sin x + \tan x \sec x) \, dx$

$= - \cos x + \sec x + C$

(f) $\int \cos x (1 + \text{cosec}^2 x) \, dx$

$= \int (\cos x + \cos x \, \text{cosec}^2 x) \, dx$

$= \int (\cos x + \cot x \, \text{cosec} \, x) \, dx$

$= \sin x - \text{cosec} \, x + C$

(g) $\int \text{cosec}^2 x (1 + \tan^2 x) \, dx$

$= \int (\text{cosec}^2 x + \text{cosec}^2 x \tan^2 x) \, dx$

$= \int (\text{cosec}^2 x + \sec^2 x) \, dx$

$= - \cot x + \tan x + C$

(h) $\int \sec^2 x (1 - \cot^2 x) \, dx$

$= \int (\sec^2 x - \sec^2 x \cot^2 x) \, dx$

$= \int (\sec^2 x - \text{cosec}^2 x) \, dx$

$= \tan x + \cot x + C$

(i) $\int \sec^2 x (1 + e^x \cos^2 x) \, dx$

$= \int (\sec^2 x + e^x \cos^2 x \sec^2 x) \, dx$

$= \int (\sec^2 x + e^x) \, dx$

$$= \tan x + e^{x} + C$$

(j) $$\int \left(\frac{1 + \sin x}{\cos^2 x} + \cos^2 x \sec x \right) dx$$

$$= \int (\sec^2 x + \tan x \sec x + \cos x) \, dx$$

$$= \tan x + \sec x + \sin x + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise B, Question 1

Question:

Integrate the following:

(a) $\sin \left(2x + 1 \right)$

(b) $3e^{2x}$

(c) $4e^{x + 5}$

(d) $\cos \left(1 - 2x \right)$

(e) $\operatorname{cosec}^2 3x$

(f) $\sec 4x \tan 4x$

(g) $3 \sin \left(\frac{1}{2}x + 1 \right)$

(h) $\sec^2 \left(2 - x \right)$

(i) $\operatorname{cosec} 2x \cot 2x$

(j) $\cos 3x - \sin 3x$

Solution:

(a) $\int \sin \left(2x + 1 \right) dx = - \frac{1}{2} \cos \left(2x + 1 \right) + C$

(b) $\int 3e^{2x} dx = \frac{3}{2} e^{2x} + C$

(c) $\int 4e^{x + 5} dx = 4e^{x + 5} + C$

(d) $\int \cos \left(1 - 2x \right) dx = - \frac{1}{2} \sin \left(1 - 2x \right) + C$

OR Let $y = \sin \left(1 - 2x \right)$

then $\frac{dy}{dx} = \cos\left(1 - 2x\right) \times \left(-2\right)$ (by chain rule)

$\therefore \int \cos\left(1 - 2x\right) dx = -\frac{1}{2}\sin\left(1 - 2x\right) + C$

(e) $\int \operatorname{cosec}^2 3x \; dx = -\frac{1}{3}\cot 3x + C$

(f) $\int \sec 4x \tan 4x \; dx = \frac{1}{4}\sec 4x + C$

(g) $\int 3\sin\left(\frac{1}{2}x + 1\right) dx = -6\cos\left(\frac{1}{2}x + 1\right) + C$

(h) $\int \sec^2 (2 - x) dx = -\tan (2 - x) + C$
OR Let $y = \tan (2 - x)$
then $\frac{dy}{dx} = \sec^2\left(2 - x\right) \times \left(-1\right)$ (by chain rule)
$\therefore \int \sec^2 (2 - x) dx = -\tan (2 - x) + C$

(i) $\int \operatorname{cosec} 2x \cot 2x \; dx = -\frac{1}{2}\operatorname{cosec} 2x + C$

(j) $\int (\cos 3x - \sin 3x) dx$

$= \frac{1}{3}\sin 3x + \frac{1}{3}\cos 3x + C$

$= \frac{1}{3}\left(\sin 3x + \cos 3x\right) + C$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise B, Question 2

Question:

Find the following integrals:

(a) $\int \left(e^{2x} - \frac{1}{2} \sin \left(2x - 1 \right) \right) dx$

(b) $\int \left(e^{x} + 1 \right)^{2} dx$

(c) $\int \sec^{2} 2x \left(1 + \sin 2x \right) dx$

(d) $\int \left(\frac{3 - 2\cos \left(\frac{1}{2}x \right)}{\sin^{2} \left(\frac{1}{2}x \right)} \right) dx$

(e) $\int \left[e^{3 - x} + \sin \left(3 - x \right) + \cos \left(3 - x \right) \right] dx$

Solution:

(a) $\int \left[e^{2x} - \frac{1}{2} \sin \left(2x - 1 \right) \right] dx = \frac{1}{2} e^{2x} + \frac{1}{4} \cos \left(2x - 1 \right) + C$

(b) $\int \left(e^{x} + 1 \right)^{2} dx$
$= \int \left(e^{2x} + 2e^{x} + 1 \right) dx$
$= \frac{1}{2} e^{2x} + 2e^{x} + x + C$

(c) $\int \sec^{2} 2x \left(1 + \sin 2x \right) dx$
$= \int \left(\sec^{2} 2x + \sec^{2} 2x \sin 2x \right) dx$
$= \int \left(\sec^{2} 2x + \sec 2x \tan 2x \right) dx$
$= \frac{1}{2} \tan 2x + \frac{1}{2} \sec 2x + C$

(d) $\int \left[\frac{3 - 2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)} \right] \mathrm{d}x$

$= \int \left(3\operatorname{cosec}^2 \frac{1}{2}x - 2\operatorname{cosec} \frac{1}{2}x\cot \frac{1}{2}x \right) \mathrm{d}x$

$= -6\cot\left(\frac{1}{2}x\right) + 4\operatorname{cosec}\left(\frac{1}{2}x\right) + C$

(e) $\int \left[e^{3 - x} + \sin(3 - x) + \cos(3 - x) \right] \mathrm{d}x$

$= -e^{3 - x} + \cos(3 - x) - \sin(3 - x) + C$

Note: extra minus signs from $-x$ terms and chain rule.

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise B, Question 3

Question:

Integrate the following:

(a) $\frac{1}{2x+1}$

(b) $\frac{1}{(2x+1)^2}$

(c) $(2x+1)^2$

(d) $\frac{3}{4x-1}$

(e) $\frac{3}{1-4x}$

(f) $\frac{3}{(1-4x)^2}$

(g) $(3x+2)^5$

(h) $\frac{3}{(1-2x)^3}$

(i) $\frac{6}{(3-2x)^4}$

(j) $\frac{5}{3-2x}$

Solution:

(a) $\int \frac{1}{2x+1} dx = \frac{1}{2} \ln | 2x+1 | + C$

(b) $\int \frac{1}{(2x+1)^2}\mathrm{d}x$

$= \int (2x+1)^{-2}\mathrm{d}x$

$= \frac{(2x+1)^{-1}}{-1} \times \frac{1}{2} + C$

$= -\frac{1}{2(2x+1)} + C$

(c) $\int (2x+1)^2\mathrm{d}x$

$= \frac{(2x+1)^3}{3} \times \frac{1}{2} + C$

$= \frac{(2x+1)^3}{6} + C$

(d) $\int \frac{3}{4x-1}\mathrm{d}x = \frac{3}{4}\ln|4x-1| + C$

(e) $\int \frac{3}{1-4x}\mathrm{d}x$

$= -\int \frac{3}{4x-1}\mathrm{d}x$

$= -\frac{3}{4}\ln|4x-1| + C$

OR Let $y = \ln|1-4x|$

then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-4x} \times \left(-4\right)$ (by chain rule)

$\therefore \int \frac{3}{1-4x}\mathrm{d}x = -\frac{3}{4}\ln|1-4x| + C$

Note: $\ln|1-4x| = \ln|4x-1|$ because of $|\quad|$ sign.

(f) $\int \frac{3}{(1-4x)^2}\mathrm{d}x$

$= \int 3(1-4x)^{-2}\mathrm{d}x$

$= \frac{3}{-4} \times \frac{(1-4x)^{-1}}{-1}$

$= \frac{3}{4(1-4x)} + C$

(g) $\int (3x + 2)^{5} dx = \frac{(3x + 2)^{6}}{18} + C$

(h) $\int \frac{3}{(1 - 2x)^{3}} dx = \frac{3}{-2} \times \frac{(1 - 2x)^{-2}}{-2} + C = \frac{3}{4 (1 - 2x)^{2}} + C$

OR Let $y = (1 - 2x)^{-2}$

then $\frac{dy}{dx} = -2 (1 - 2x)^{-3} \times \left(-2 \right)$ (by chain rule)

$\therefore \int \frac{3}{(1 - 2x)^{3}} dx = \frac{3}{4} (1 - 2x)^{-2} + C$

(i) $\int \frac{6}{(3 - 2x)^{4}} dx = \frac{6}{-2} \times \frac{(3 - 2x)^{-3}}{-3} + C = \frac{1}{(3 - 2x)^{3}} + C$

OR Let $y = (3 - 2x)^{-3}$

then $\frac{dy}{dx} = -3 (3 - 2x)^{-4} \times \left(-2 \right)$

$\therefore \int \frac{6}{(3 - 2x)^{4}} dx = \frac{1}{(3 - 2x)^{3}} + C$

(j) $\int \frac{5}{(3 - 2x)} dx = -\frac{5}{2} \ln | 3 - 2x | + C$

OR Let $y = \ln | 3 - 2x |$

then $\frac{dy}{dx} = \frac{1}{3 - 2x} \times \left(-2 \right)$ (by chain rule)

$\therefore \int \frac{5}{3 - 2x} dx = -\frac{5}{2} \ln | 3 - 2x | + C$

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Integration
Exercise B, Question 4

Question:

Find the following integrals

(a) $\int \left(3\sin \left(2x + 1 \right) + \frac{4}{2x+1} \right) \mathrm{d}x$

(b) $\int \left[e^{5x} + \left(1 - x \right)^{5} \right] \mathrm{d}x$

(c) $\int \left(\frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{\left(1 + 2x \right)^{2}} \right) \mathrm{d}x$

(d) $\int \left[\left(3x + 2 \right)^{2} + \frac{1}{\left(3x + 2 \right)^{2}} \right] \mathrm{d}x$

Solution:

(a) $\int \left[3\sin \left(2x + 1 \right) + \frac{4}{2x+1} \right] \mathrm{d}x$

$$= - \frac{3}{2}\cos \left(2x + 1 \right) + \frac{4}{2}\ln \left| 2x + 1 \right| + C$$

$$= - \frac{3}{2}\cos \left(2x + 1 \right) + 2\ln \left| 2x + 1 \right| + C$$

(b) $\int \left[e^{5x} + \left(1 - x \right)^{5} \right] \mathrm{d}x$

$$= \int e^{5x}\mathrm{d}x + \int \left(1 - x \right)^{5}\mathrm{d}x$$

$$= \frac{1}{5}e^{5x} - \frac{1}{6}\left(1 - x \right)^{6} + C \qquad \text{(from ⑪ and ⑩)}$$

OR Let $y = \left(1 - x \right)^{6}$

then $\frac{\mathrm{d}y}{\mathrm{d}x} = 6\left(1 - x \right)^{5} \times \left(- 1 \right)$ (by chain rule)

$$\therefore \int \left(1 - x \right)^{5}\mathrm{d}x = - \frac{1}{6}\left(1 - x \right)^{6} + C$$

(c) $\int \left[\frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{\left(1 + 2x \right)^{2}} \right] \mathrm{d}x$

$$= \int \left[\text{cosec}^2 2x + \frac{1}{1+2x} + (1+2x)^{-2} \right] dx$$

$$= -\frac{1}{2}\cot 2x + \frac{1}{2}\ln|1+2x| + \frac{(1+2x)^{-1}}{-1} \times \frac{1}{2} + C$$

$$= -\frac{1}{2}\cot 2x + \frac{1}{2}\ln|1+2x| - \frac{1}{2(1+2x)} + C$$

(d) $\int \left[(3x+2)^2 + \frac{1}{(3x+2)^2} \right] dx$

$$= \int \left[(3x+2)^2 + (3x+2)^{-2} \right] dx$$

$$= \frac{(3x+2)^3}{9} - \frac{(3x+2)^{-1}}{3} + C$$

$$= \frac{(3x+2)^3}{9} - \frac{1}{3(3x+2)} + C$$

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Integration
Exercise C, Question 1

Question:

Integrate the following:

(a) $\cot^2 x$

(b) $\cos^2 x$

(c) $\sin 2x \cos 2x$

(d) $(1 + \sin x)^2$

(e) $\tan^2 3x$

(f) $(\cot x - \operatorname{cosec} x)^2$

(g) $(\sin x + \cos x)^2$

(h) $\sin^2 x \cos^2 x$

(i) $\dfrac{1}{\sin^2 x \quad \cos^2 x}$

(j) $(\cos 2x - 1)^2$

Solution:

(a) $\int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1) \, dx$
$= - \cot x - x + C$

(b) $\int \cos^2 x \, dx = \int \frac{1}{2} \left(1 + \cos 2x \right) dx$

$= \frac{1}{2} x + \frac{1}{4} \sin 2x + C$

(c) $\int \sin 2x \cos 2x \, dx = \int \frac{1}{2} \sin 4x \, dx$

$= -\frac{1}{8}\cos 4x + C$

(d) $\int (1+\sin x)^2 \mathrm{d}x = \int (1 + 2\sin x + \sin^2 x)\,\mathrm{d}x$

But $\cos 2x = 1 - 2\sin^2 x$

$\therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\therefore \int (1+\sin x)^2 \mathrm{d}x = \int \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x \right) \mathrm{d}x$

$= \frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x + C$

(e) $\int \tan^2 3x\,\mathrm{d}x = \int (\sec^2 3x - 1)\,\mathrm{d}x$

$= \frac{1}{3}\tan 3x - x + C$

(f) $\int (\cot x - \operatorname{cosec} x)^2 \mathrm{d}x = \int (\cot^2 x - 2\cot x \operatorname{cosec} x + \operatorname{cosec}^2 x)\,\mathrm{d}x$

$= \int (2\operatorname{cosec}^2 x - 1 - 2\cot x \operatorname{cosec} x)\,\mathrm{d}x$

$= -2\cot x - x + 2\operatorname{cosec} x + C$

(g) $\int (\sin x + \cos x)^2 \mathrm{d}x = \int (\sin^2 x + 2\sin x\cos x + \cos^2 x)\,\mathrm{d}x$

$= \int (1 + \sin 2x)\,\mathrm{d}x$

$= x - \frac{1}{2}\cos 2x + C$

(h) $\int \sin^2 x \cos^2 x\,\mathrm{d}x = \int \left(\frac{1}{2}\sin 2x \right)^2 \mathrm{d}x$

$= \int \frac{1}{4}\sin^2 2x\,\mathrm{d}x$

$= \int \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\cos 4x \right) \mathrm{d}x$

$= \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x \right) \mathrm{d}x$

$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$

(i) $\frac{1}{\sin^2 x \cos^2 x} = \frac{1}{\left(\frac{1}{2}\sin 2x \right)^2} = 4\operatorname{cosec}^2 2x$

$$\therefore \int \frac{1}{\sin^2 x \cos^2 x} \, dx = \int 4 \operatorname{cosec}^2 2x \, dx$$

$$= -2 \cot 2x + C$$

(j) $\int (\cos 2x - 1)^2 dx = \int (\cos^2 2x - 2 \cos 2x + 1) \, dx$

$$= \int \left(\frac{1}{2} \cos 4x + \frac{1}{2} - 2 \cos 2x + 1 \right) dx$$

$$= \int \left(\frac{1}{2} \cos 4x + \frac{3}{2} - 2 \cos 2x \right) dx$$

$$= \frac{1}{8} \sin 4x + \frac{3}{2} x - \sin 2x + C$$

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Integration
Exercise C, Question 2

Question:

Find the following integrals:

(a) $\int \left(\frac{1 - \sin x}{\cos^2 x} \right) \mathrm{d}x$

(b) $\int \left(\frac{1 + \cos x}{\sin^2 x} \right) \mathrm{d}x$

(c) $\int \frac{\cos 2x}{\cos^2 x} \mathrm{d}x$

(d) $\int \frac{\cos^2 x}{\sin^2 x} \mathrm{d}x$

(e) $\int \frac{(1 + \cos x)^2}{\sin^2 x} \mathrm{d}x$

(f) $\int \frac{(1 + \sin x)^2}{\cos^2 x} \mathrm{d}x$

(g) $\int (\cot x - \tan x)^2 \mathrm{d}x$

(h) $\int (\cos x - \sin x)^2 \mathrm{d}x$

(i) $\int (\cos x - \sec x)^2 \mathrm{d}x$

(j) $\int \frac{\cos 2x}{1 - \cos^2 2x} \mathrm{d}x$

Solution:

(a) $\int \left(\frac{1 - \sin x}{\cos^2 x} \right) \mathrm{d}x = \int \left(\sec^2 x - \tan x \sec x \right) \mathrm{d}x$

$= \tan x - \sec x + C$

(b) $\int \left(\frac{1 + \cos x}{\sin^2 x} \right) dx = \int \left(\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x \right) dx$

$= - \cot x - \operatorname{cosec} x + C$

(c) $\int \frac{\cos 2x}{\cos^2 x} dx = \int \frac{2\cos^2 x - 1}{\cos^2 x} dx$

$= \int (2 - \sec^2 x) dx$

$= 2x - \tan x + C$

(d) $\int \frac{\cos^2 x}{\sin^2 x} dx = \int \cot^2 x \, dx$

$= \int (\operatorname{cosec}^2 x - 1) dx$

$= - \cot x - x + C$

(e) $I = \int \frac{(1 + \cos x)^2}{\sin^2 x} dx = \int \frac{1 + 2\cos x + \cos^2 x}{\sin^2 x} dx$

$= \int (\operatorname{cosec}^2 x + 2 \cot x \operatorname{cosec} x + \cot^2 x) dx$

But $\operatorname{cosec}^2 x = 1 + \cot^2 x \Rightarrow \cot^2 x = \operatorname{cosec}^2 x - 1$

$\therefore I = \int (2 \operatorname{cosec}^2 x - 1 + 2 \cot x \operatorname{cosec} x) dx$

$= - 2 \cot x - x - 2 \operatorname{cosec} x + C$

(f) $I = \int \frac{(1 + \sin x)^2}{\cos^2 x} dx = \int \frac{1 + 2\sin x + \sin^2 x}{\cos^2 x} dx$

$= \int (\sec^2 x + 2 \tan x \sec x + \tan^2 x) dx$

But $\sec^2 x = 1 + \tan^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$\therefore I = \int (2 \sec^2 x - 1 + 2 \tan x \sec x) dx$

$= 2 \tan x - x + 2 \sec x + C$

(g) $\int (\cot x - \tan x)^2 dx = \int (\cot^2 x - 2 \cot x \tan x + \tan^2 x) dx$

$= \int (\operatorname{cosec}^2 x - 1 - 2 + \sec^2 x - 1) dx$

$= \int (\operatorname{cosec}^2 x - 4 + \sec^2 x) dx$

$= - \cot x - 4x + \tan x + C$

(h) $\int (\cos x - \sin x)^2 dx = \int (\cos^2 x - 2 \cos x \sin x + \sin^2 x) dx$

$= \int (1 - \sin 2x) dx$

$= x + \frac{1}{2} \cos 2x + C$

(i) $\int (\cos x - \sec x)^2 \mathrm{d}x = \int (\cos^2 x - 2\cos x \sec x + \sec^2 x)\, \mathrm{d}x$

$= \int \left(\frac{1}{2}\cos 2x + \frac{1}{2} - 2 + \sec^2 x \right) \mathrm{d}x$

$= \int \left(\frac{1}{2}\cos 2x - \frac{3}{2} + \sec^2 x \right) \mathrm{d}x$

$= \frac{1}{4}\sin 2x - \frac{3}{2}x + \tan x + C$

(j) $\int \frac{\cos 2x}{1 - \cos^2 2x} \mathrm{d}x = \int \frac{\cos 2x}{\sin^2 2x} \mathrm{d}x$

$= \int \cot 2x \operatorname{cosec} 2x \, \mathrm{d}x$

$= -\frac{1}{2}\operatorname{cosec} 2x + C$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise C, Question 3

Question:

Find the following integrals:

(a) $\int \cos 2x \cos x \, dx$

(b) $\int 2 \sin 5x \cos 3x \, dx$

(c) $\int 2 \sin 3x \cos 5x \, dx$

(d) $\int 2 \sin 2x \sin 5x \, dx$

(e) $4 \int \cos 3x \cos 7x \, dx$

(f) $\int 2 \cos 4x \cos 4x \, dx$

(g) $\int 2 \cos 4x \sin 4x \, dx$

(h) $\int 2 \sin 4x \sin 4x \, dx$

Solution:

(a) $\cos 3x + \cos x = 2 \cos \frac{3x + x}{2} \cos \frac{3x - x}{2} = 2 \cos 2x \cos x$

$\therefore \int \cos 2x \cos x \, dx = \frac{1}{2} \int \left(\cos 3x + \cos x \right) dx$

$= \frac{1}{2} \left(\frac{1}{3} \sin 3x + \sin x \right) + C$

$= \frac{1}{6} \sin 3x + \frac{1}{2} \sin x + C$

(b) $\sin 8x + \sin 2x = 2 \sin 5x \cos 3x$

$\therefore \int 2 \sin 5x \cos 3x \, dx = \int \left(\sin 8x + \sin 2x \right) dx$

$= - \frac{1}{8} \cos 8x - \frac{1}{2} \cos 2x + C$

(c) $\sin 8x - \sin 2x = 2 \sin 3x \cos 5x$

$\therefore \int 2 \sin 3x \cos 5x \, dx = \int \left(\sin 8x - \sin 2x \right) dx$

$$= -\frac{1}{8}\cos 8x + \frac{1}{2}\cos 2x + C$$

(d) $\cos 7x - \cos 3x = -2\sin 5x\sin 2x$

$$\therefore \int 2\sin 2x\sin 5x\,dx = \int (\cos 3x - \cos 7x)\,dx$$

$$= \frac{1}{3}\sin 3x - \frac{1}{7}\sin 7x + C$$

(e) $\cos 10x + \cos 4x = 2\cos 7x\cos 3x$

$$\therefore \int 4\cos 3x\cos 7x\,dx = 2\int (\cos 10x + \cos 4x)\,dx$$

$$= 2\left(\frac{1}{10}\sin 10x + \frac{1}{4}\sin 4x\right) + C$$

$$= \frac{1}{5}\sin 10x + \frac{1}{2}\sin 4x + C$$

(f) $\cos 8x + \cos 0x = 2\cos 4x\cos 4x$

i.e. $\cos 8x + 1 = 2\cos 4x\cos 4x$

$$\therefore \int 2\cos 4x\cos 4x\,dx = \int (1 + \cos 8x)\,dx$$

$$= x + \frac{1}{8}\sin 8x + C$$

(g) $\sin 8x + \sin 0x = 2\sin 4x\cos 4x$

$$\therefore \int 2\cos 4x\sin 4x\,dx = \int \sin 8x\,dx$$

$$= -\frac{1}{8}\cos 8x + C$$

(h) $\cos 8x - \cos 0x = -2\sin 4x\sin 4x$

i.e. $\cos 8x - 1 = -2\sin 4x\sin 4x$

$$\therefore \int 2\sin 4x\sin 4x\,dx = \int (1 - \cos 8x)\,dx$$

$$= x - \frac{1}{8}\sin 8x + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise D, Question 1

Question:

Use partial fractions to integrate the following:

(a) $\frac{3x+5}{(x+1)(x+2)}$

(b) $\frac{3x-1}{(2x+1)(x-2)}$

(c) $\frac{2x-6}{(x+3)(x-1)}$

(d) $\frac{3}{(2+x)(1-x)}$

(e) $\frac{4}{(2x+1)(1-2x)}$

(f) $\frac{3(x+1)}{9x^2-1}$

(g) $\frac{3-5x}{(1-x)(2-3x)}$

(h) $\frac{x^2-3}{(2+x)(1+x)^2}$

(i) $\frac{5+3x}{(x+2)(x+1)^2}$

(j) $\frac{17-5x}{(3+2x)(2-x)^2}$

Solution:

(a) $\frac{3x+5}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$

$$\Rightarrow \quad 3x + 5 \equiv A (x + 2) + B (x + 1)$$

$$x = -1 \quad \Rightarrow \quad 2 = A$$

$$x = -2 \quad \Rightarrow \quad -1 = -B \quad \Rightarrow \quad B = 1$$

$$\therefore \int \frac{3x + 5}{(x + 1) (x + 2)} dx = \int \left(\frac{2}{x + 1} + \frac{1}{x + 2} \right) dx$$

$$= 2 \ln | x + 1 | + \ln | x + 2 | + C$$

$$= \ln [| x + 1 | ^{2}] + \ln | x + 2 | + C$$

$$= \ln | (x + 1) ^{2} (x + 2) | + C$$

(b) $\frac{3x - 1}{(2x + 1) (x - 2)} \equiv \frac{A}{2x + 1} + \frac{B}{x - 2}$

$$\Rightarrow \quad 3x - 1 \equiv A (x - 2) + B (2x + 1)$$

$$x = 2 \quad \Rightarrow \quad 5 = 5B \quad \Rightarrow \quad B = 1$$

$$x = -\frac{1}{2} \quad \Rightarrow \quad -\frac{5}{2} = -\frac{5}{2}A \quad \Rightarrow \quad A = 1$$

$$\therefore \int \frac{3x - 1}{(2x + 1) (x - 2)} dx = \int \left(\frac{1}{2x + 1} + \frac{1}{x - 2} \right) dx$$

$$= \frac{1}{2} \ln | 2x + 1 | + \ln | x - 2 | + C$$

$$= \ln | (x - 2) \sqrt{2x + 1} | + C$$

(c) $\frac{2x - 6}{(x + 3) (x - 1)} \equiv \frac{A}{x + 3} + \frac{B}{x - 1}$

$$\Rightarrow \quad 2x - 6 \equiv A (x - 1) + B (x + 3)$$

$$x = 1 \quad \Rightarrow \quad -4 = 4B \quad \Rightarrow \quad B = -1$$

$$x = -3 \quad \Rightarrow \quad -12 = -4A \quad \Rightarrow \quad A = 3$$

$$\therefore \int \frac{2x - 6}{(x + 3) (x - 1)} dx = \int \left(\frac{3}{x + 3} - \frac{1}{x - 1} \right) dx$$

$$= 3 \ln | x + 3 | - \ln | x - 1 | + C$$

$$= \ln \left| \frac{(x + 3) ^{3}}{x - 1} \right| + C$$

(d) $\frac{3}{(2 + x) (1 - x)} \equiv \frac{A}{(2 + x)} + \frac{B}{1 - x}$

$$\Rightarrow \quad 3 \equiv A (1 - x) + B (2 + x)$$

$$x = 1 \quad \Rightarrow \quad 3 = 3B \quad \Rightarrow \quad B = 1$$

$$x = -2 \quad \Rightarrow \quad 3 = 3A \quad \Rightarrow \quad A = 1$$

$$\therefore \int \frac{3}{(2+x)(1-x)}\mathrm{d}x = \int \left(\frac{1}{2+x} + \frac{1}{1-x}\right)\mathrm{d}x.$$

$$= \ln|2+x| - \ln|1-x| + C$$

$$= \ln\left|\frac{2+x}{1-x}\right| + C$$

(e) $\frac{4}{(2x+1)(1-2x)} \equiv \frac{A}{2x+1} + \frac{B}{1-2x}$

$$\Rightarrow \quad 4 \equiv A(1-2x) + B(2x+1)$$

$$x = \frac{1}{2} \quad \Rightarrow \quad 4 = 2B \quad \Rightarrow \quad B = 2$$

$$x = -\frac{1}{2} \quad \Rightarrow \quad 4 = 2A \quad \Rightarrow \quad A = 2$$

$$\therefore \int \frac{4}{(2x+1)(1-2x)}\mathrm{d}x = \int \left(\frac{2}{2x+1} + \frac{2}{1-2x}\right)\mathrm{d}x$$

$$= \ln|2x+1| - \ln|1-2x| + C$$

$$= \ln\left|\frac{2x+1}{1-2x}\right| + C$$

(f) $\frac{3(x+1)}{9x^2-1} \equiv \frac{3(x+1)}{(3x-1)(3x+1)} \equiv \frac{A}{3x-1} + \frac{B}{3x+1}$

$$\Rightarrow \quad 3x+3 \equiv A(3x+1) + B(3x-1)$$

$$x = -\frac{1}{3} \quad \Rightarrow \quad 2 = -2B \quad \Rightarrow \quad B = -1$$

$$x = \frac{1}{3} \quad \Rightarrow \quad 4 = 2A \quad \Rightarrow \quad A = 2$$

$$\therefore \int \frac{3(x+1)}{9x^2-1}\mathrm{d}x = \int \left(\frac{2}{3x-1} - \frac{1}{3x+1}\right)\mathrm{d}x$$

$$= \frac{2}{3}\ln|3x-1| - \frac{1}{3}\ln|3x+1| + C$$

$$= \frac{1}{3}\ln\left|\frac{(3x-1)^2}{3x+1}\right| + C$$

(g) $\frac{3-5x}{(1-x)(2-3x)} \equiv \frac{A}{1-x} + \frac{B}{2-3x}$

$$\Rightarrow \quad 3-5x \equiv A(2-3x) + B(1-x)$$

$$x = \frac{2}{3} \quad \Rightarrow \quad -\frac{1}{3} = \frac{1}{3}B \quad \Rightarrow \quad B = -1$$

$x = 1 \quad \Rightarrow \quad -2 = -A \quad \Rightarrow \quad A = 2$

$$\therefore \int \frac{3-5x}{(1-x)(2-3x)}\,dx = \int \left(\frac{2}{1-x} - \frac{1}{2-3x} \right) dx$$

$$= -2\ln|1-x| + \frac{1}{3}\ln|2-3x| + C$$

$$= \ln\left| \frac{(2-3x)^{\frac{1}{3}}}{(1-x)^2} \right| + C$$

(h) $\dfrac{x^2-3}{(2+x)(1+x)^2} \equiv \dfrac{A}{2+x} + \dfrac{B}{1+x} + \dfrac{C}{(1+x)^2}$

$\Rightarrow \quad x^2 - 3 \equiv A(1+x)^2 + B(2+x)(1+x) + C(2+x)$

$x = -1 \quad \Rightarrow \quad -2 = C \quad \Rightarrow \quad C = -2$

$x = -2 \quad \Rightarrow \quad 1 = 1A \quad \Rightarrow \quad A = 1$

Coefficient of $x^2 \quad \Rightarrow \quad 1 = A + B \quad \Rightarrow \quad B = 0$

$$\therefore \int \frac{x^2-3}{(2+x)(1+x)^2}\,dx = \int \left(\frac{1}{2+x} - \frac{2}{(1+x)^2} \right) dx$$

$$= \ln|2+x| - 2\frac{(1+x)^{-1}}{-1} + C$$

$$= \ln|2+x| + \frac{2}{1+x} + C$$

(i) $\dfrac{5+3x}{(x+2)(x+1)^2} \equiv \dfrac{A}{x+2} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$\Rightarrow \quad 5 + 3x \equiv A(x+1)^2 + B(x+2)(x+1) + C(x+2)$

$x = -1 \quad \Rightarrow \quad 2 = C \quad \Rightarrow \quad C = 2$

$x = -2 \quad \Rightarrow \quad -1 = A \quad \Rightarrow \quad A = -1$

Coefficient of $x^2 \quad \Rightarrow \quad 0 = A + B \quad \Rightarrow \quad B = 1$

$$\therefore \int \frac{5+3x}{(x+2)(x+1)^2}\,dx = \int \left(-\frac{1}{x+2} + \frac{1}{x+1} + \frac{2}{(x+1)^2} \right) dx$$

$$= -\ln|x+2| + \ln|x+1| - \frac{2}{x+1} + C$$

$$= \ln\left| \frac{x+1}{x+2} \right| - \frac{2}{x+1} + C$$

(j) $\frac{17-5x}{(3+2x)(2-x)^2} \equiv \frac{A}{3+2x} + \frac{B}{2-x} + \frac{C}{(2-x)^2}$

$\Rightarrow \quad 17-5x \equiv A(2-x)^2 + B(3+2x)(2-x) + C(3+2x)$

$x = 2 \quad \Rightarrow \quad 7 = 7C \quad \Rightarrow \quad C = 1$

$x = -\frac{3}{2} \quad \Rightarrow \quad \frac{49}{2} = \frac{49}{4}A \quad \Rightarrow \quad A = 2$

Coefficient of $x^2 \quad \Rightarrow \quad 0 = A - 2B \quad \Rightarrow \quad B = 1$

$\therefore \int \frac{17-5x}{(3+2x)(2-x)^2}dx = \int \left(\frac{2}{3+2x} + \frac{1}{2-x} + \frac{1}{(2-x)^2} \right) dx$

$= \frac{2}{2}\ln|3+2x| - \ln|2-x| + \frac{1}{2-x} + C$

$= \ln\left|\frac{3+2x}{2-x}\right| + \frac{1}{2-x} + C$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise D, Question 2

Question:

Find the following integrals:

(a) $\int \frac{2(x^2+3x-1)}{(x+1)(2x-1)}dx$

(b) $\int \frac{x^3+2x^2+2}{x(x+1)}dx$

(c) $\int \frac{x^2}{x^2-4}dx$

(d) $\int \frac{x^2+x+2}{3-2x-x^2}dx$

(e) $\int \frac{6+3x-x^2}{x^3+2x^2}dx$

Solution:

(a) $\frac{2(x^2+3x-1)}{(x+1)(2x-1)} \equiv 1 + \frac{A}{x+1} + \frac{B}{2x-1}$

$\Rightarrow \quad 2x^2+6x-2 \equiv (x+1)(2x-1) + A(2x-1) + B(x+1)$

$x = -1 \quad \Rightarrow \quad -6 = -3A \quad \Rightarrow \quad A = 2$

$x = \frac{1}{2} \quad \Rightarrow \quad \frac{3}{2} = \frac{3}{2}B \quad \Rightarrow \quad B = 1$

$\therefore \int \frac{2(x^2+3x-1)}{(x+1)(2x-1)}dx = \int \left(1 + \frac{2}{x+1} + \frac{1}{2x-1}\right)dx$

$= x + 2\ln|x+1| + \frac{1}{2}\ln|2x-1| + C$

$= x + \ln|(x+1)^2\sqrt{2x-1}| + C$

(b) $\frac{x^3+2x^2+2}{x(x+1)} \quad \Rightarrow$

$$\begin{array}{rl}
 & x+1 \\ \cline{2-2}
x^2+x \big) & x^3+2x^2+2 \\
 & \underline{x^3+x^2} \\
 & \qquad x^2+2 \\
 & \qquad \underline{x^2+x} \\
 & \qquad\ \ 2-x
\end{array}$$

$$\frac{x^3+2x^2+2}{x(x+1)} \equiv x+1+\frac{2-x}{x(x+1)}$$

$$\equiv x+1+\frac{A}{x}+\frac{B}{x+1}$$

$$\Rightarrow \quad x^3+2x^2+2 \equiv (x+1)x(x+1)+A(x+1)+Bx$$

$$x=0 \quad \Rightarrow \quad 2=A \quad \Rightarrow \quad A=2$$

$$x=-1 \quad \Rightarrow \quad 3=-B \quad \Rightarrow \quad B=-3$$

$$\therefore \int \frac{x^3+2x^2+2}{x(x+1)}\mathrm{d}x = \int \left(x+1+\frac{2}{x}-\frac{3}{x+1} \right) \mathrm{d}x$$

$$= \frac{x^2}{2}+x+2\ln|x| - 3\ln|x+1| + C$$

$$= \frac{x^2}{2}+x+\ln\left|\frac{x^2}{(x+1)^3}\right| + C$$

(c) $\frac{x^2}{x^2-4} \equiv 1+\frac{A}{x-2}+\frac{B}{x+2}$

$$\Rightarrow \quad x^2 \equiv (x-2)(x+2)+A(x+2)+B(x-2)$$

$$x=2 \quad \Rightarrow \quad 4=4A \quad \Rightarrow \quad A=1$$

$$x=-2 \quad \Rightarrow \quad 4=-4B \quad \Rightarrow \quad B=-1$$

$$\therefore \int \frac{x^2}{x^2-4}\mathrm{d}x = \int \left(1+\frac{1}{x-2}-\frac{1}{x+2} \right) \mathrm{d}x$$

$$= x+\ln|x-2| - \ln|x+2| + C$$

$$= x+\ln\left|\frac{x-2}{x+2}\right| + C$$

(d) $\frac{x^2+x+2}{3-2x-x^2} \equiv \frac{x^2+x+2}{(3+x)(1-x)} \equiv -1+\frac{A}{3+x}+\frac{B}{1-x}$

$$\Rightarrow \quad x^2+x+2 \equiv -1(3+x)(1-x)+A(1-x)+B(3+x)$$

$$x=1 \quad \Rightarrow \quad 4=4B \quad \Rightarrow \quad B=1$$

$$x=-3 \quad \Rightarrow \quad 8=4A \quad \Rightarrow \quad A=2$$

$$\therefore \int \frac{x^2+x+2}{3-2x-x^2}\mathrm{d}x = \int \left(-1+\frac{2}{3+x}+\frac{1}{1-x} \right) \mathrm{d}x$$

$$= -x+2\ln|3+x| - \ln|1-x| + C$$

$$= -x+\ln\left|\frac{(3+x)^2}{1-x}\right| + C$$

(e) $\frac{6+3x-x^2}{x^3+2x^2} \equiv \frac{6+3x-x^2}{x^2(x+2)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$

$\Rightarrow \quad 6+3x-x^2 \equiv Ax(x+2) + B(x+2) + Cx^2$

$x = 0 \quad \Rightarrow \quad 6 = 2B \quad \Rightarrow \quad B = 3$

$x = -2 \quad \Rightarrow \quad -4 = 4C \quad \Rightarrow \quad C = -1$

Coefficient of $x^2 \quad \Rightarrow \quad -1 = A + C \quad \Rightarrow \quad A = 0$

$\therefore \int \frac{6+3x-x^2}{x^3+2x^2} \mathrm{d}x = \int \left(\frac{3}{x^2} - \frac{1}{x+2} \right) \mathrm{d}x$

$= -\frac{3}{x} - \ln|x+2| + C$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 1

Question:

Integrate the following functions:

(a) $\frac{x}{x^2+4}$

(b) $\frac{e^{2x}}{e^{2x}+1}$

(c) $\frac{x}{(x^2+4)^3}$

(d) $\frac{e^{2x}}{(e^{2x}+1)^3}$

(e) $\frac{\cos 2x}{3+\sin 2x}$

(f) $\frac{\sin 2x}{(3+\cos 2x)^3}$

(g) xe^{x^2}

(h) $\cos 2x(1+\sin 2x)^4$

(i) $\sec^2 x \tan^2 x$

(j) $\sec^2 x(1+\tan^2 x)$

Solution:

(a) $y=\ln|x^2+4|$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x^2+4}\times 2x \qquad \text{(chain rule)}$$

$$\therefore \int \frac{x}{x^2+4} dx = \frac{1}{2} \ln | x^2 + 4 | + C$$

(b) $y = \ln | e^{2x} + 1 |$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^{2x}+1} \times e^{2x} \times 2 \quad \text{(chain rule)}$$

$$\therefore \int \frac{e^{2x}}{e^{2x}+1} dx = \frac{1}{2} \ln | e^{2x} + 1 | + C$$

(c) $y = (x^2 + 4)^{-2}$

$$\Rightarrow \frac{dy}{dx} = -2 (x^2 + 4)^{-3} \times 2x \quad \text{(chain rule)}$$

$$\therefore \int \frac{x}{(x^2 + 4)^3} dx = -\frac{1}{4} (x^2 + 4)^{-2} + C$$

or $-\frac{1}{4 (x^2 + 4)^2} + C$

(d) $y = (e^{2x} + 1)^{-2}$

$$\Rightarrow \frac{dy}{dx} = -2 (e^{2x} + 1)^{-3} \times e^{2x} \times 2 \quad \text{(chain rule)}$$

$$\therefore \int \frac{e^{2x}}{(e^{2x} + 1)^3} dx = -\frac{1}{4} (e^{2x} + 1)^{-2} + C$$

or $-\frac{1}{4 (e^{2x} + 1)^2} + C$

(e) $y = \ln | 3 + \sin 2x |$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3 + \sin 2x} \times \cos 2x \times 2 \quad \text{(chain rule)}$$

$$\therefore \int \frac{\cos 2x}{3 + \sin 2x} dx = \frac{1}{2} \ln | 3 + \sin 2x | + C$$

(f) $y = (3 + \cos 2x)^{-2}$

$$\Rightarrow \frac{dy}{dx} = -2 (3 + \cos 2x)^{-3} \times \left(-\sin 2x \right) \times 2 \quad \text{(chain rule)}$$

$$\therefore \int \frac{\sin 2x}{(3 + \cos 2x)^3} dx = \frac{1}{4} (3 + \cos 2x)^{-2} + C$$

or $\frac{1}{4 (3 + \cos 2x)^2} + C$

(g) $y = e^{x^2}$

$\Rightarrow \quad \frac{dy}{dx} = e^{x^2} \times 2x \qquad$ (chain rule)

$\therefore \int xe^{x^2} dx = \frac{1}{2} e^{x^2} + C$

(h) $y = (1 + \sin 2x)^5$

$\Rightarrow \quad \frac{dy}{dx} = 5 (1 + \sin 2x)^4 \times \cos 2x \times 2 \qquad$ (chain rule)

$\therefore \int \cos 2x (1 + \sin 2x)^4 dx = \frac{1}{10} (1 + \sin 2x)^5 + C$

(i) $y = \tan^3 x$

$\Rightarrow \quad \frac{dy}{dx} = 3 \tan^2 x \times \sec^2 x \qquad$ (chain rule)

$\therefore \int \sec^2 x \tan^2 x \, dx = \frac{1}{3} \tan^3 x + C$

(j) $\sec^2 x (1 + \tan^2 x) = \sec^2 x + \sec^2 x \tan^2 x$

$\therefore \int \sec^2 x (1 + \tan^2 x) dx = \int (\sec^2 x + \sec^2 x \tan^2 x) dx$

$= \tan x + \frac{1}{3} \tan^3 x + C$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise E, Question 2

Question:

Find the following integrals:

(a) $\int \left(x + 1 \right) \left(x^2 + 2x + 3 \right)^4 dx$

(b) $\int \operatorname{cosec}^2 2x \cot 2x \, dx$

(c) $\int \sin^5 3x \cos 3x \, dx$

(d) $\int \cos x \, e^{\sin x} dx$

(e) $\int \frac{e^{2x}}{e^{2x} + 3} dx$

(f) $\int x \left(x^2 + 1 \right)^{\frac{3}{2}} dx$

(g) $\int \left(2x + 1 \right) \sqrt{x^2 + x + 5} \, dx$

(h) $\int \frac{2x + 1}{\sqrt{x^2 + x + 5}} dx$

(i) $\int \frac{\sin x \cos x}{\sqrt{\cos 2x + 3}} dx$

(j) $\int \frac{\sin x \cos x}{\cos 2x + 3} dx$

Solution:

(a) $y = \left(x^2 + 2x + 3 \right)^5$

$$\Rightarrow \quad \frac{dy}{dx} = 5 \left(x^2 + 2x + 3 \right)^4 \times \left(2x + 2 \right)$$

$$= 5 \left(x^2 + 2x + 3 \right)^4 \times 2 \left(x + 1 \right)$$

$$\therefore \int \left(x + 1 \right) \left(x^2 + 2x + 3 \right)^4 dx = \frac{1}{10} \left(x^2 + 2x + 3 \right)^5 + C$$

(b) $y = \cot^2 2x$

$$\Rightarrow \quad \frac{dy}{dx} = 2\cot 2x \times \left(-\operatorname{cosec}^2 2x \right) \times 2$$

$$= -4\operatorname{cosec}^2 2x \cot 2x$$

$$\therefore \int \operatorname{cosec}^2 2x \cot 2x \, dx = -\frac{1}{4}\cot^2 2x + C$$

(c) $y = \sin^6 3x$

$$\Rightarrow \quad \frac{dy}{dx} = 6\sin^5 3x \times \cos 3x \times 3$$

$$\therefore \int \sin^5 3x \cos 3x \, dx = \frac{1}{18}\sin^6 3x + C$$

(d) $y = e^{\sin x}$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\sin x} \times \cos x$$

$$\therefore \int \cos x e^{\sin x} dx = e^{\sin x} + C$$

(e) $y = \ln | e^{2x} + 3 |$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{e^{2x} + 3} \times e^{2x} \times 2$$

$$\therefore \int \frac{e^{2x}}{e^{2x} + 3} dx = \frac{1}{2}\ln | e^{2x} + 3 | + C$$

(f) $y = \left(x^2 + 1 \right)^{\frac{5}{2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{5}{2}\left(x^2 + 1 \right)^{\frac{3}{2}} \times 2x = 5x\left(x^2 + 1 \right)^{\frac{3}{2}}$$

$$\therefore \int x\left(x^2 + 1 \right)^{\frac{3}{2}} dx = \frac{1}{5}\left(x^2 + 1 \right)^{\frac{5}{2}} + C$$

(g) $y = \left(x^2 + x + 5 \right)^{\frac{3}{2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{2}\left(x^2 + x + 5 \right)^{\frac{1}{2}} \times \left(2x + 1 \right)$$

$$\therefore \int \left(2x + 1 \right)\sqrt{x^2 + x + 5} \, dx = \frac{2}{3}\left(x^2 + x + 5 \right)^{\frac{3}{2}} + C$$

(h) $y = \left(x^2 + x + 5 \right)^{\frac{1}{2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left(x^2 + x + 5 \right)^{-\frac{1}{2}} \times \left(2x + 1 \right)$$

$$= \frac{1}{2}\frac{\left(2x + 1 \right)}{\sqrt{x^2 + x + 5}}$$

$$\therefore \int \frac{2x + 1}{\sqrt{x^2 + x + 5}} dx = 2\left(x^2 + x + 5 \right)^{\frac{1}{2}} + C$$

(i) $y = \left(\cos 2x + 3 \right)^{\frac{1}{2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left(\cos 2x + 3 \right)^{-\frac{1}{2}} \times \left(-\sin 2x \right) \times 2$$

$$= -\frac{\sin 2x}{\sqrt{\cos 2x + 3}}$$

$$= -\frac{2\sin x\cos x}{\sqrt{\cos 2x + 3}}$$

$$\therefore \int \frac{\sin x\cos x}{\sqrt{\cos 2x + 3}} dx = -\frac{1}{2}\left(\cos 2x + 3 \right)^{\frac{1}{2}} + C$$

(j) $y = \ln \left| \cos 2x + 3 \right|$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos 2x + 3} \times \left(-\sin 2x \right) \times 2$$

$$= -\frac{2\sin 2x}{\cos 2x + 3}$$

$$= -\frac{4\sin x\cos x}{\cos 2x + 3}$$

$$\therefore \int \frac{\sin x\cos x}{\cos 2x + 3} dx = -\frac{1}{4}\ln \left| \cos 2x + 3 \right| + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise F, Question 1

Question:

Use the given substitution to find the following integrals:

(a) $\int x\sqrt{1+x}\,\mathrm{d}x;\ u = 1 + x$

(b) $\int \frac{x}{\sqrt{1+x}}\mathrm{d}x;\ u = 1 + x$

(c) $\int \frac{1+\sin x}{\cos x}\mathrm{d}x;\ u = \sin x$

(d) $\int x(3+2x)^5\mathrm{d}x;\ u = 3 + 2x$

(e) $\int \sin^3 x\,\mathrm{d}x;\ u = \cos x$

Solution:

(a) $u = 1 + x \quad \Rightarrow \quad \mathrm{d}u = \mathrm{d}x$ and $x = u - 1$

$$\therefore \int x(1+x)^{\frac{1}{2}}\mathrm{d}x = \int (u-1)u^{\frac{1}{2}}\mathrm{d}u$$

$$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\mathrm{d}u$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + C$$

$$\text{OR } = \frac{2}{15}(1+x)^{\frac{3}{2}}\left[3\left(1+x\right) - 5\right] + C$$

$$= \frac{2}{15}(1+x)^{\frac{3}{2}}\left(3x-2\right) + C$$

(b) $u = 1 + x \quad \Rightarrow \quad \mathrm{d}u = \mathrm{d}x$ and $x = u - 1$

$$\therefore \int \frac{x}{\sqrt{1+x}}\mathrm{d}x = \int \frac{u-1}{u^{\frac{1}{2}}}\mathrm{d}u$$

$$= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}} \right) \mathrm{d}u$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(1+x)^{\frac{3}{2}} - 2(1+x)^{\frac{1}{2}} + C$$

$$\text{OR} = \frac{2}{3}(1+x)^{\frac{1}{2}} \left[1 + x - 3 \right] + C$$

$$= \frac{2}{3}(1+x)^{\frac{1}{2}} \left(x - 2 \right) + C$$

(c) $u = \sin x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \quad \Rightarrow \quad \mathrm{d}x = \frac{\mathrm{d}u}{\cos x}$

$$\therefore \int \frac{1+\sin x}{\cos x}\mathrm{d}x = \int \frac{1+u}{\cos x}\frac{\mathrm{d}u}{\cos x}$$

$$= \int \frac{1+u}{1-\sin^2 x}\mathrm{d}u$$

$$= \int \frac{1+u}{1-u^2}\mathrm{d}u$$

$$= \int \frac{(1+u)}{(1-u)(1+u)}\mathrm{d}u$$

$$= \int \frac{1}{1-u}\mathrm{d}u$$

$$= -\ln|1-u| + C$$
$$= -\ln|1-\sin x| + C$$

(d) $u = 3 + 2x \quad \Rightarrow \quad \mathrm{d}u = 2\,\mathrm{d}x \text{ and } x = \frac{u-3}{2}$

$$\therefore \int x(3+2x)^5 \mathrm{d}x = \int \frac{u-3}{2}u^5 \frac{\mathrm{d}u}{2}$$

$$= \int \left(\frac{u^6}{4} - \frac{3u^5}{4} \right) \mathrm{d}u$$

$$= \frac{u^7}{28} - \frac{3u^6}{24} + C$$

$$= \frac{u^7}{28} - \frac{u^6}{8} + C$$

$$= \frac{(3+2x)^7}{28} - \frac{(3+2x)^6}{8} + C$$

(e) $u = \cos x \quad \Rightarrow \quad \mathrm{d}u = -\sin x\,\mathrm{d}x$

$$\therefore \int \sin^3 x\,\mathrm{d}x = \int -(1-u^2)\,\mathrm{d}u$$

$$= \int (u^2-1)\,\mathrm{d}u$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C$$

$$\text{OR} = \frac{\cos x}{3}\left(\cos^2 x - 3\right) + C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise F, Question 2

Question:

Use the given substitution to find the following integrals:

(a) $\int x\sqrt{2+x}\,\mathrm{d}x;\ u^2=2+x$

(b) $\int \frac{2}{\sqrt{x}(x-4)}\mathrm{d}x;\ u=\sqrt{x}$

(c) $\int \sec^2 x\tan x\sqrt{1+\tan x}\,\mathrm{d}x;\ u^2=1+\tan x$

(d) $\int \frac{\sqrt{x^2+4}}{x}\mathrm{d}x;\ u^2=x^2+4$

(e) $\int \sec^4 x\,\mathrm{d}x;\ u=\tan x$

Solution:

(a) $u^2=2+x \quad\Rightarrow\quad 2u\,\mathrm{d}u=\mathrm{d}x$ and $x=u^2-2$

$$\therefore \int x\sqrt{2+x}\,\mathrm{d}x=\int(u^2-2)\times u\times 2u\,\mathrm{d}u$$

$$=\int(2u^4-4u^2)\,\mathrm{d}u$$

$$=\frac{2u^5}{5}-\frac{4u^3}{3}+C$$

$$=\frac{2}{5}(2+x)^{\frac{5}{2}}-\frac{4}{3}(2+x)^{\frac{3}{2}}+C$$

(b) $u=x^{\frac{1}{2}} \quad\Rightarrow\quad \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}x^{-\frac{1}{2}} \quad\Rightarrow\quad \frac{\mathrm{d}x}{\sqrt{x}}=2\,\mathrm{d}u$

and $x-4=u^2-4$

$$\therefore I=\int\frac{2}{\sqrt{x}(x-4)}\mathrm{d}x=\int\frac{2}{u^2-4}\times 2\,\mathrm{d}u=\int\frac{4}{u^2-4}\mathrm{d}u$$

$$\frac{4}{u^2-4}=\frac{A}{u-2}+\frac{B}{u+2}$$

$$\Rightarrow\quad 4=A(u+2)+B(u-2)$$

$$u=2\quad\Rightarrow\quad 4=4A\quad\Rightarrow\quad A=1$$

$u = -2 \quad \Rightarrow \quad 4 = -4B \quad \Rightarrow \quad B = -1$

$$\therefore I = \int \left(\frac{1}{u-2} - \frac{1}{u+2} \right) \mathrm{d}u$$

$$= \ln|u-2| - \ln|u+2| + C$$

$$= \ln\left| \frac{\sqrt{x}-2}{\sqrt{x}+2} \right| + C$$

(c) $u^2 = 1 + \tan x \quad \Rightarrow \quad 2u\,\mathrm{d}u = \sec^2 x\,\mathrm{d}x$

$$\therefore \int \sec^2 x \tan x \sqrt{1+\tan x}\,\mathrm{d}x$$

$$= \int (u^2 - 1) \times u \times 2u\,\mathrm{d}u$$

$$= \int (2u^4 - 2u^2)\,\mathrm{d}u$$

$$= \frac{2u^5}{5} - \frac{2u^3}{3} + C$$

$$= \frac{2}{5}(1+\tan x)^{\frac{5}{2}} - \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + C$$

(d) $u^2 = x^2 + 4 \quad \Rightarrow \quad 2u\,\mathrm{d}u = 2x\,\mathrm{d}x \quad \Rightarrow \quad \frac{u\mathrm{d}u}{x} = \mathrm{d}x$

$$\therefore \int \frac{\sqrt{x^2+4}}{x}\mathrm{d}x = \int \frac{u}{x} \times \frac{u\mathrm{d}u}{x}$$

$$= \int \frac{u^2}{x^2}\mathrm{d}u$$

$$= \int \frac{u^2}{u^2-4}\mathrm{d}u$$

$= \int \left(1 + \frac{4}{u^2-4}\right)\mathrm{d}u$ but $\frac{4}{u^2-4} \equiv \frac{A}{u+2} + \frac{B}{u+2}$

$4 \equiv A(u+2) + B(u-2)$

$u = 2 : 4 = 4A,\ A = 1$

$u = -2 : 4 = -4B,\ B = -1$

$$= \int \left(1 + \frac{1}{u-2} - \frac{1}{u+2} \right) \mathrm{d}u$$

$$= u + \ln|u-2| - \ln|u+2| + C$$

$$= \sqrt{x^2+4} + \ln\left| \frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2} \right| + C$$

(e) $u = \tan x \quad \Rightarrow \quad \mathrm{d}u = \sec^2 x\,\mathrm{d}x$

$\therefore \int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx$

$= \int (1 + u^2) \, du$

$= u + \frac{u^3}{3} + C$

$= \tan x + \frac{\tan^3 x}{3} + C$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise F, Question 3

Question:

Evaluate the following:

(a) $\int_0^5 x\sqrt{x+4}\,dx$

(b) $\int_0^{\frac{\pi}{3}} \sec x \tan x \sqrt{\sec x + 2}\,dx$

(c) $\int_2^5 \frac{1}{1+\sqrt{x-1}}\,dx$; use $u^2 = x - 1$

(d) $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,d\theta$; let $u = 1 + \cos\theta$

(e) $\int_0^1 x(2+x)^3\,dx$

(f) $\int_1^4 \frac{1}{\sqrt{x}(4x-1)}\,dx$; let $u = \sqrt{x}$

Solution:

(a) $u^2 = x + 4 \quad \Rightarrow \quad 2u\,du = dx$ and $x = u^2 - 4$

Also $u = 3$ when $x = 5$

and $u = 2$ when $x = 0$.

$$\therefore \int_0^5 x\sqrt{x+4}\,dx = \int_2^3 (u^2 - 4) \times u \times 2u\,du$$

$$= \int_2^3 (2u^4 - 8u^2)\,du$$

$$= \left[\frac{2}{5}u^5 - \frac{8}{3}u^3\right]_2^3$$

$$= \left(\frac{2}{5} \times 243 - \frac{8}{3} \times 27\right) - \left(\frac{64}{5} - \frac{64}{3}\right)$$

$$= 25.2 - -8.53$$

$$= 33.73$$

$$= 33.7 \text{ (3 s.f.)}$$

(b) $u^2 = \sec x + 2 \quad \Rightarrow \quad 2u\,du = \sec x \tan x\,dx$

Also $u = 2$ when $x = \frac{\pi}{3}$

and $u = \sqrt{3}$ when $x = 0$.

$$\therefore \int_0^{\frac{\pi}{3}} \sec x \tan x \sqrt{\sec x + 2}\, dx = \int_{\sqrt{3}}^{2} u \times 2u\, du$$

$$= \int_{\sqrt{3}}^{2} 2u^2\, du$$

$$= \left[\frac{2}{3} u^3 \right]_{\sqrt{3}}^{2}$$

$$= \left(\frac{16}{3} \right) - \left(\frac{2}{3} \times 3\sqrt{3} \right)$$

$$= \frac{16}{3} - 2\sqrt{3}$$

(c) $u^2 = x - 1 \quad \Rightarrow \quad 2u\, du = dx$

Also $u = 2$ when $x = 5$

and $u = 1$ when $x = 2$.

$$\therefore \int_2^5 \frac{1}{1 + \sqrt{x - 1}} dx = \int_1^2 \frac{1}{1 + u} \times 2u\, du$$

$$= \int_1^2 \frac{2u}{u + 1} du$$

$$= \int_1^2 \left(2 - \frac{2}{u + 1} \right) du$$

$$= \left[2u - 2\ln | u + 1 | \right]_1^2$$

$$= (4 - 2\ln 3) - (2 - 2\ln 2)$$

$$= 2 + 2\ln \frac{2}{3}$$

(d) $u = 1 + \cos\theta \quad \Rightarrow \quad du = -\sin\theta\, d\theta$ or $-du = \sin\theta\, d\theta$

Also $u = 1$ when $\theta = \frac{\pi}{2}$

and $u = 2$ when $\theta = 0$.

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1 + \cos\theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{2\sin\theta\cos\theta}{1 + \cos\theta} d\theta = \int_2^1 - \frac{2(u - 1)}{u} du$$

Use ‘ − ’ to reverse limits:

$$I = \int_1^2 \frac{2u - 2}{u} du$$

$$= \int_1^2 \left(2 - \frac{2}{u} \right) du$$

$$= \left[2u - 2\ln | u | \right]_1^2$$

$= (4 - 2\ln 2) - (2 - 2\ln 1)$
$= 2 - 2\ln 2$

(e) $u = 2 + x \quad \Rightarrow \quad \mathrm{d}u = \mathrm{d}x$ and $x = u - 2$

Also $u = 3$ when $x = 1$
and $u = 2$ when $x = 0$.

$\therefore \int_0^1 x(2 + x)^3 \mathrm{d}x = \int_2^3 (u - 2) u^3 \mathrm{d}u$

$= \int_2^3 (u^4 - 2u^3)\,\mathrm{d}u$

$$= \left[\frac{u^5}{5} - \frac{2}{4}u^4 \right]_2^3$$

$$= \left(\frac{243}{5} - \frac{81}{2} \right) - \left(\frac{32}{5} - \frac{16}{2} \right)$$

$$= \frac{211}{5} - 32.5$$

$= 42.2 - 32.5$
$= 9.7$

(f) $u = x^{\frac{1}{2}} \quad \Rightarrow \quad \mathrm{d}u = \frac{1}{2}x^{-\frac{1}{2}}\mathrm{d}x \quad \Rightarrow \quad \frac{\mathrm{d}x}{\sqrt{x}} = 2\,\mathrm{d}u$

and $4x - 1 = 4u^2 - 1$
Also $u = 2$ when $x = 4$
and $u = 1$ when $x = 1$.

$$\therefore I = \int_1^4 \frac{1}{\sqrt{x}(4x - 1)}\mathrm{d}x = \int_1^2 \frac{1}{4u^2 - 1} \times 2\,\mathrm{d}u$$

$$\frac{2}{4u^2 - 1} = \frac{A}{2u - 1} + \frac{B}{2u + 1}$$

$\Rightarrow \quad 2 = A(2u + 1) + B(2u - 1)$

$u = \frac{1}{2} \quad \Rightarrow \quad 2 = 2A \quad \Rightarrow \quad A = 1$

$u = -\frac{1}{2} \quad \Rightarrow \quad 2 = -2B \quad \Rightarrow \quad B = -1$

$$\therefore I = \int_1^2 \left(\frac{1}{2u - 1} - \frac{1}{2u + 1} \right) \mathrm{d}u$$

$$= \left[\frac{1}{2}\ln|2u - 1| - \frac{1}{2}\ln|2u + 1| \right]_1^2$$

$$= \left[\frac{1}{2}\ln\left| \frac{2u - 1}{2u + 1} \right| \right]_1^2$$

$$= \left(\frac{1}{2} \ln \left| \frac{3}{5} \right| \right) - \left(\frac{1}{2} \ln \left| \frac{1}{3} \right| \right)$$

$$= \frac{1}{2} \ln \frac{9}{5}$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise G, Question 1

Question:

Find the following integrals:

(a) $\int x \sin x \, \mathrm{d}x$

(b) $\int x\mathrm{e}^{x} \, \mathrm{d}x$

(c) $\int x \sec^2 x \, \mathrm{d}x$

(d) $\int x \sec x \tan x \, \mathrm{d}x$

(e) $\int \frac{x}{\sin^2 x} \mathrm{d}x$

Solution:

(a) $u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \quad \Rightarrow \quad v = -\cos x$$

$$\therefore \int x \sin x \, \mathrm{d}x = -x \cos x - \int -\cos x \times 1 \, \mathrm{d}x$$
$$= -x \cos x + \int \cos x \, \mathrm{d}x$$
$$= -x \cos x + \sin x + C$$

(b) $u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{x} \quad \Rightarrow \quad v = \mathrm{e}^{x}$$

$$\therefore \int x\mathrm{e}^{x} \, \mathrm{d}x = x\mathrm{e}^{x} - \int \mathrm{e}^{x} \times 1 \, \mathrm{d}x$$

$$= x\mathrm{e}^{x} - \mathrm{e}^{x} + C$$

(c) $u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x \quad \Rightarrow \quad v = \tan x$$

$$\therefore \int x \sec^2 x \, \mathrm{d}x = x \tan x - \int \tan x \times 1 \, \mathrm{d}x$$

$= x \tan x - \ln | \sec x | + C$

(d) $u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec x \tan x \quad \Rightarrow \quad v = \sec x$

$\therefore \int x \sec x \tan x \, dx = x \sec x - \int \sec x \times 1 \, dx$

$= x \sec x - \ln | \sec x + \tan x | + C$

(e) $u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \operatorname{cosec}^2 x \quad \Rightarrow \quad v = - \cot x$

$\therefore \int \frac{x}{\sin^2 x} dx = \int x \operatorname{cosec}^2 x \, dx$

$= - x \cot x - \int - \cot x \times 1 \, dx$

$= - x \cot x + \int \cot x \, dx$

$= - x \cot x + \ln | \sin x | + C$

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Integration

Exercise G, Question 2

Question:

Find the following integrals:

(a) $\int x^2 \ln x \, dx$

(b) $\int 3 \ln x \, dx$

(c) $\int \frac{\ln x}{x^3} dx$

(d) $\int (\ln x) ^2 dx$

(e) $\int (x^2 + 1) \ln x \, dx$

Solution:

(a) $u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \quad \Rightarrow \quad v = \frac{x^3}{3}$

$\therefore \int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \times \frac{1}{x} dx$

$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx$

$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$

(b) $u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 3 \quad \Rightarrow \quad v = 3x$

$\therefore \int 3 \ln x \, dx = 3x \ln x - \int 3x \times \frac{1}{x} dx$

$= 3x \ln x - \int 3 \, dx$

$= 3x \ln x - 3x + C$

(c) $u = \ln x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^{-3} \quad \Rightarrow \quad v = \frac{x^{-2}}{-2}$$

$$\therefore \int \frac{\ln x}{x^3}\mathrm{d}x = -\frac{1}{2x^2}\ln x - \int -\frac{1}{2x^2} \times \frac{1}{x}\mathrm{d}x$$

$$= -\frac{\ln x}{2x^2} + \int \frac{1}{2}x^{-3}\mathrm{d}x$$

$$= -\frac{\ln x}{2x^2} + \frac{x^{-2}}{2 \times (-2)} + C$$

$$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$$

(d) $u = (\ln x)^2 \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2\ln x \times \frac{1}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \quad \Rightarrow \quad v = x$$

$$\therefore I = \int (\ln x)^2\mathrm{d}x = x(\ln x)^2 - \int x \times 2\ln x \times \frac{1}{x}\mathrm{d}x$$

$$= x(\ln x)^2 - \int 2\ln x\,\mathrm{d}x$$

Let $J = \int 2\ln x\,\mathrm{d}x$

$$u = \ln x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 2 \quad \Rightarrow \quad v = 2x$$

$$\therefore J = 2x\ln x - \int 2x \times \frac{1}{x}\mathrm{d}x = 2x\ln x - 2x + C$$

$$\therefore I = x(\ln x)^2 - 2x\ln x + 2x + C$$

(e) $u = \ln x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^2 + 1 \quad \Rightarrow \quad v = \frac{x^3}{3} + x$$

$$\therefore \int \left(x^2 + 1\right)\ln x\,\mathrm{d}x = \ln x\left(\frac{x^3}{3} + x\right) - \int \left(\frac{x^3}{3} + x\right) \times \frac{1}{x}\mathrm{d}x$$

$$= \left(\frac{x^3}{3} + x\right)\ln x - \int \left(\frac{x^2}{3} + 1\right)\mathrm{d}x$$

$$= \left(\frac{x^3}{3} + x \right) \ln x - \frac{x^3}{9} - x + C$$

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Integration
Exercise G, Question 3

Question:

Find the following integrals:

(a) $\int x^2 \mathrm{e}^{-x} \mathrm{d}x$

(b) $\int x^2 \cos x \, \mathrm{d}x$

(c) $\int 12x^2 (3 + 2x)^5 \mathrm{d}x$

(d) $\int 2x^2 \sin 2x \, \mathrm{d}x$

(e) $\int x^2 2 \sec^2 x \tan x \, \mathrm{d}x$

Solution:

(a) $u = x^2 \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-x} \quad \Rightarrow \quad v = -\mathrm{e}^{-x}$

$\therefore I = \int x^2 \mathrm{e}^{-x} \mathrm{d}x = -x^2 \mathrm{e}^{-x} - \int -\mathrm{e}^{-x} \times 2x \, \mathrm{d}x$

$= -x^2 \mathrm{e}^{-x} + \int 2x\mathrm{e}^{-x} \mathrm{d}x$

Let $J = \int 2x\mathrm{e}^{-x} \mathrm{d}x$

$u = 2x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-x} \quad \Rightarrow \quad v = -\mathrm{e}^{-x}$

$\therefore J = -\mathrm{e}^{-x} \; 2x - \int (-\mathrm{e}^{-x}) \times 2 \, \mathrm{d}x$

$= -2x\mathrm{e}^{-x} + \int 2\mathrm{e}^{-x} \mathrm{d}x$

$= -2x\mathrm{e}^{-x} - 2\mathrm{e}^{-x} + C$

$\therefore I = -x^2 \mathrm{e}^{-x} - 2x\mathrm{e}^{-x} - 2\mathrm{e}^{-x} + C'$

(b) $u = x^2 \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x \quad \Rightarrow \quad v = \sin x$

$\therefore I = \int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$

Let $J = \int 2x \sin x \, dx$

$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$

$\frac{dv}{dx} = \sin x \quad \Rightarrow \quad v = -\cos x$

$\therefore J = -2x \cos x - \int (-\cos x) \times 2 \, dx$

$= -2x \cos x + \int 2 \cos x \, dx$

$= -2x \cos x + 2 \sin x + C$

$\therefore I = x^2 \sin x + 2x \cos x - 2 \sin x + C'$

(c) $u = 12x^2 \quad \Rightarrow \quad \frac{du}{dx} = 24x$

$\frac{dv}{dx} = (3 + 2x)^5 \quad \Rightarrow \quad v = \frac{(3 + 2x)^6}{12}$

$\therefore I = \int 12x^2 (3 + 2x)^5 dx = 12x^2 \frac{(3 + 2x)^6}{12} - \int 24x \frac{(3 + 2x)^6}{12} dx$

$= x^2 (3 + 2x)^6 - \int 2x (3 + 2x)^6 dx$

Let $J = \int 2x (3 + 2x)^6 dx$

$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$

$v = \frac{(3 + 2x)^7}{14} \quad \Rightarrow \quad \frac{dv}{dx} = (3 + 2x)^6$

$\therefore J = 2x \frac{(3 + 2x)^7}{14} - \int \frac{(3 + 2x)^7}{14} \times 2 \, dx$

$= x \frac{(3 + 2x)^7}{7} - \int \frac{(3 + 2x)^7}{7} dx$

$= x \frac{(3 + 2x)^7}{7} - \frac{(3 + 2x)^8}{7 \times 16} + C$

$\therefore I = x^2 (3 + 2x)^6 - x \frac{(3 + 2x)^7}{7} + \frac{(3 + 2x)^8}{112} + C'$

(d) $u = 2x^2 \quad \Rightarrow \quad \frac{du}{dx} = 4x$

$\frac{dv}{dx} = \sin 2x \quad \Rightarrow \quad v = -\frac{1}{2} \cos 2x$

$$\therefore I = \int 2x^2 \sin 2x \, dx = -\frac{2x^2}{2}\cos 2x - \int \left(-\frac{1}{2}\cos 2x \right) \times 4x \, dx$$

$$= -x^2 \cos 2x + \int 2x \cos 2x \, dx$$

Let $J = \int 2x \cos 2x \, dx$

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = 2\cos 2x \quad \Rightarrow \quad v = \sin 2x$$

$$\therefore J = x \sin 2x - \int \sin 2x \, dx$$

$$= x \sin 2x + \frac{1}{2}\cos 2x + C$$

$$\therefore I = -x^2 \cos 2x + x \sin 2x + \frac{1}{2}\cos 2x + C'$$

(e) $u = x^2 \quad \Rightarrow \quad \frac{du}{dx} = 2x$

$$\frac{dv}{dx} = 2\sec x \sec x \tan x \quad \Rightarrow \quad v = \sec^2 x$$

$$\therefore I = \int x^2 \times 2\sec^2 x \tan x \, dx = x^2 \sec^2 x - \int 2x \sec^2 x \, dx$$

Let $J = \int 2x \sec^2 x \, dx$

$$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$$

$$\frac{dv}{dx} = \sec^2 x \quad \Rightarrow \quad v = \tan x$$

$$\therefore J = 2x \tan x - \int 2 \tan x \, dx$$

$$= 2x \tan x - 2 \ln | \sec x | + C$$

$$\therefore I = x^2 \sec^2 x - 2x \tan x + 2 \ln | \sec x | + C'$$

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Integration
Exercise G, Question 4

Question:

Evaluate the following:

(a) $\int_0^{\ln 2} x e^{2x} \, dx$

(b) $\int_0^{\frac{\pi}{2}} x \sin x \, dx$

(c) $\int_0^{\frac{\pi}{2}} x \cos x \, dx$

(d) $\int_1^2 \frac{\ln x}{x^2} dx$

(e) $\int_0^1 4x \left(1 + x \right)^{3} dx$

(f) $\int_0^{\pi} x \cos \left(\frac{1}{4} x \right) dx$

(g) $\int_0^{\frac{\pi}{3}} \sin x \ln \left| \sec x \right| dx$

Solution:

(a) $u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$$\frac{dv}{dx} = e^{2x} \quad \Rightarrow \quad v = \frac{1}{2} e^{2x}$$

$$\therefore \int_0^{\ln 2} x e^{2x} dx = \left[\frac{1}{2} e^{2x} \times x \right]_0^{\ln 2} - \int_0^{\ln 2} \frac{1}{2} e^{2x} dx$$

$$= \left(\frac{1}{2} e^{2 \ln 2} \ln 2 \right) - \left(0 \right) - \left[\frac{1}{4} e^{2x} \right]_0^{\ln 2}$$

$$= \frac{4}{2} \ln 2 - \left[\left(\frac{1}{4} e^{2 \ln 2} \right) - \left(\frac{1}{4} e^{0} \right) \right]$$

$$= 2 \ln 2 - \frac{4}{4} + \frac{1}{4}$$

$= 2\ln 2 - \frac{3}{4}$

(b) $u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \quad \Rightarrow \quad v = -\cos x$

$\therefore \int_0^{\frac{\pi}{2}} x \sin x \, dx = [-x \cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(-\cos x \right) dx$

$= \left(-\frac{\pi}{2} \cos \frac{\pi}{2} \right) - \left(0 \right) + \int_0^{\frac{\pi}{2}} \cos x \, dx$

$= 0 + [\sin x]_0^{\frac{\pi}{2}}$

$= \left(\sin \frac{\pi}{2} \right) - \left(\sin 0 \right)$

$= 1$

(c) $u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos x \quad \Rightarrow \quad v = \sin x$

$\therefore \int_0^{\frac{\pi}{2}} x \cos x \, dx = [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx$

$= \left(\frac{\pi}{2} \sin \frac{\pi}{2} \right) - \left(0 \right) - [-\cos x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} + \left(\cos \frac{\pi}{2} \right) - \left(\cos 0 \right)$

$= \frac{\pi}{2} - 1$

(d) $u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^{-2} \quad \Rightarrow \quad v = -x^{-1}$

$\therefore \int_1^2 \frac{\ln x}{x^2} dx = \left[-\frac{\ln x}{x} \right]_1^2 - \int_1^2 \frac{1}{x} \times \left(-x^{-1} \right) dx$

$= \left(-\frac{\ln 2}{2} \right) - \left(-\frac{\ln 1}{1} \right) + \int_1^2 \frac{1}{x^2} dx$

$= -\frac{1}{2} \ln 2 + [-x^{-1}]_1^2$

$$= -\frac{1}{2}\ln 2 + \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)$$

$$= \frac{1}{2}\left(1 - \ln 2\right)$$

(e) $u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 4(1+x)^3 \quad \Rightarrow \quad v = (1+x)^4$$

$$\therefore \int_0^1 4x(1+x)^3\mathrm{d}x = \left[x(1+x)^4\right]_0^1 - \int_0^1 (1+x)^4\mathrm{d}x$$

$$= \left(1 \times 2^4\right) - \left(0\right) - \left[\frac{(1+x)^5}{5}\right]_0^1$$

$$= 16 - \left[\left(\frac{2^5}{5}\right) - \left(\frac{1}{5}\right)\right]$$

$$= 16 - \frac{31}{5}$$

$$= 16 - 6.2$$
$$= 9.8$$

(f) $u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos\left(\frac{1}{4}x\right) \quad \Rightarrow \quad v = 4\sin\left(\frac{1}{4}x\right)$$

$$\therefore \int_0^\pi x\cos\left(\frac{1}{4}x\right)\mathrm{d}x = \left[4x\sin\frac{x}{4}\right]_0^\pi - \int_0^\pi 4\sin\left(\frac{1}{4}x\right)\mathrm{d}x$$

$$= \left(4\pi\sin\frac{\pi}{4}\right) - \left(0\right) + \left[16\cos\frac{1}{4}x\right]_0^\pi$$

$$= \frac{4\pi}{\sqrt{2}} + \left(16\cos\frac{\pi}{4}\right) - \left(16\cos 0\right)$$

$$= \frac{4\pi}{\sqrt{2}} + \frac{16}{\sqrt{2}} - 16$$

OR $= \frac{4\pi\sqrt{2}}{2} + \frac{16\sqrt{2}}{2} - 16$

$$= 2\pi\sqrt{2} + 8\sqrt{2} - 16$$

(g) $u = \ln|\sec x| \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \tan x$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \quad \Rightarrow \quad v = -\cos x$$

$$\therefore \int_0^{\frac{\pi}{3}} \sin x \ln|\sec x|\ dx = \left[-\cos x \ln|\sec x| \right]_0^{\frac{\pi}{3}} + \int_0^{\frac{\pi}{3}} \cos x \tan x\, dx$$

$$= \left(-\cos\frac{\pi}{3} \ln \left| \sec\frac{\pi}{3} \right| \right) - \left(-\cos 0 \ln \left| \sec 0 \right| \right) + \int_0^{\frac{\pi}{3}} \sin x\, dx$$

$$= -\frac{1}{2}\ln 2 + 0 + \left[-\cos x \right]_0^{\frac{\pi}{3}}$$

$$= -\frac{1}{2}\ln 2 + \left(-\frac{1}{2} \right) - \left(-1 \right)$$

$$= \frac{1}{2}\left(1 - \ln 2 \right)$$

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Integration
Exercise H, Question 1

Question:

Use the trapezium rule with *n* strips to estimate the following:

(a) $\int_0^3 \ln (1 + x^2) \, dx \, ; n = 6$

(b) $\int_0^{\frac{\pi}{3}} \sqrt{(1 + \tan x)} \, dx ; n = 4$

(c) $\int_0^2 \frac{1}{\sqrt{(e^x + 1)}} dx ; n = 4$

(d) $\int_{-1}^1 \operatorname{cosec}^2 (x^2 + 1) \, dx \, ; n = 4$

(e) $\int_{0.1}^{1.1} \sqrt{\cot x} \, dx ; n = 5$

Solution:

(a)

x	0	0.5	1	1.5	2	2.5	3
$\ln (1 + x^2)$	0	0.223	0.693	1.179	1.609	1.981	2.303

$$I = \int_0^3 \ln (1 + x^2) \, dx$$

$$\therefore I \approx \frac{1}{2} \times 0.5 \left[0 + 2.303 + 2 \left(0.223 + 0.693 + 1.179 + 1.609 + 1.981 \right) \right]$$

$$= \frac{1}{4} \left(13.673 \right)$$

$= 3.41825$

$= 3.42$ (3 s.f.)

(b)

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{\pi}{3}$
$\sqrt{1 + \tan x}$	1	1.126	1.256	1.414	1.653

$$I = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan x} \, dx$$

$$\therefore I - \frac{1}{2} \times \frac{\pi}{12} \left[1 + 1.653 + 2 \left(1.126 + 1.256 + 1.414 \right) \right]$$

$$= \frac{\pi}{24} \left(10.245 \right)$$

$= 1.3410...$

$= 1.34$ (3 s.f.)

(c)

x	0	0.5	1	1.5	2
$\frac{1}{\sqrt{e^x + 1}}$	0.707	0.614	0.519	0.427	0.345

$$I = \int_0^2 \frac{1}{\sqrt{e^x + 1}} dx$$

$$\therefore I \simeq \frac{1}{2} \times 0.5 \left[0.707 + 0.345 + 2 \left(0.614 + 0.519 + 0.427 \right) \right]$$

$$= \frac{1}{4} \left(4.172 \right)$$

$= 1.043$

$= 1.04$ (3 s.f.)

(d)

x	− 1	− 0.5	0	0.5	1
$\operatorname{cosec}^2 (x^2 + 1)$	1.209	1.110	1.412	1.110	1.209

$$I = \int_{-1}^{1} \operatorname{cosec}^2 (x^2 + 1) dx$$

$$\therefore I \simeq \frac{1}{2} \times 0.5 \left[1.209 \times 2 + 2 \left(1.110 + 1.412 + 1.110 \right) \right]$$

$$= \frac{1}{4} \left(9.682 \right)$$

$= 2.42$ (3 s.f.)

(e)

x	0.1	0.3	0.5	0.7	0.9	1.1
$\sqrt{\cot x}$	3.157	1.798	1.353	1.090	0.891	0.713

$$I = \int_{0.1}^{1.1} \sqrt{\cot x} dx$$

$$\therefore I \simeq \frac{1}{2} \times 0.2 \left[3.157 + 0.713 + 2 \left(1.798 + 1.353 + 1.090 + 0.891 \right) \right]$$

$$= \frac{1}{10} \left(14.134 \right)$$

$= 1.41$ (3 s.f.)

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise H, Question 2

Question:

(a) Find the exact value of $I = \int_1^4 x \ln x \, dx$.

(b) Find approximate values for I using the trapezium rule with
(i) 3 strips
(ii) 6 strips

(c) Compare the percentage error for these two approximations.

Solution:

(a) $I = \int_1^4 x \ln x \, dx$

$$u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x \quad \Rightarrow \quad v = \frac{1}{2}x^2$$

$$\therefore I = \left[\frac{1}{2}x^2 \ln x \right]_1^4 - \int_1^4 \frac{1}{2}x^2 \times \frac{1}{x} dx$$

$$= 8 \ln 4 - \left[\frac{x^2}{4} \right]_1^4$$

$$= 8 \ln 4 - \left(4 - \frac{1}{4} \right)$$

$$= 8 \ln 4 - \frac{15}{4}$$

(b) (i)

x	1	2	3	4
$x \ln x$	0	1.386	3.296	5.545

$$I \approx \frac{1}{2} \times 1 \left[5.545 + 2 \left(1.386 + 3.296 \right) \right]$$

$$= \frac{1}{2} \left(14.909 \right) = 7.4545 = 7.45 \text{ (3 s.f.)}$$

(ii)

x	1	1.5	2	2.5	3	3.5	4
$x\ln x$	0	0.608	1.386	2.291	3.296	4.385	5.545

$$I \approx \frac{1}{2} \times 0.5 \left[5.545 + 2 \left(0.608 + 1.386 + 2.291 + 3.296 + 4.385 \right) \right]$$

$$= \frac{1}{4} \left[29.477 \right] = 7.36925 = 7.37 \text{ (3 s.f.)}$$

(c) % error using 3 strips: $\frac{\left[7.4545 - \left(8\ln 4 - 3.75 \right) \right] \times 100}{8\ln 4 - 3.75} = 1.6\ \%\ \ 1 \text{ d.p.}$

% error using 6 strips: $\frac{\left[7.376925 - \left(8\ln 4 - 3.75 \right) \right] \times 100}{8\ln 4 - 3.75} = 0.4\ \%\ \ 1 \text{ d.p.}$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise H, Question 3

Question:

(a) Find an approximate value for $I = \int_0^1 e^x \tan x \, dx$ using

(i) 2 strips
(ii) 4 strips
(ii) 8 strips.

(b) Suggest a possible value for I.

Solution:

(a) (i)

x	0	0.5	1
$e^x \tan x$	0	0.901	4.233

$$I \simeq \frac{1}{2} \times 0.5 \left(0 + 4.233 + 2 \times 0.901 \right) = \frac{1}{4} \left(6.035 \right) = 1.509$$

(ii)

x	0	0.25	0.5	0.75	1
$e^x \tan x$	0	0.328	0.901	1.972	4.233

$$I \simeq \frac{1}{2} \times 0.25 \left[4.233 + 2 \left(0.328 + 0.901 + 1.972 \right) \right]$$

$$= \frac{1}{8} \left(10.635 \right) = 1.329$$

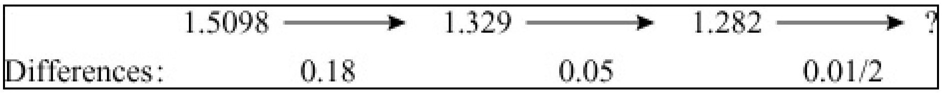
(iii)

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$e^x \tan x$	0	0.142	0.328	0.573	0.901	1.348	1.972	2.872	4.233

$$I \simeq \frac{1}{2} \times \frac{1}{8} \left[4.233 + 2 \left(0.142 + 0.328 + 0.573 + 0.901 + 1.348 + 1.972 + 2.872 \right) \right]$$

$$= \frac{1}{16}\left(20.505 \right) = 1.282$$

(b) Halving *h* reduces differences by about $\frac{1}{3}$:



	1.5098	→	1.329	→	1.282	→ ?
Differences:		0.18		0.05		0.01/2

So an answer in the range 1.25 – 1.27 seems likely.
(**Note:** Calculator gives 1.265)

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Integration
Exercise H, Question 4

Question:

(a) Find the exact value of $I = \int_0^2 x\sqrt{(2-x)}\,\mathrm{d}x$.

(b) Find an approximate value for I using the trapezium rule with
(i) 4 and
(ii) 6 strips.

(c) Compare the percentage error for these two approximations.

Solution:

(a) $u^2 = 2 - x \quad \Rightarrow \quad 2u\,\mathrm{d}u = -\mathrm{d}x$ and $x = 2 - u^2$

Also $u = 0$ when $x = 2$
and $u = \sqrt{2}$ when $x = 0$.

$$\therefore I = \int_{\sqrt{2}}^{0} (2 - u^2)\,u \times (-2u)\,\mathrm{d}u$$

$$= \int_0^{\sqrt{2}} (2 - u^2)\,2u^2\,\mathrm{d}u$$

$$= \int_0^{\sqrt{2}} (4u^2 - 2u^4)\,\mathrm{d}u$$

$$= \left[\frac{4u^3}{3} - \frac{2u^5}{5} \right]_0^{\sqrt{2}}$$

$$= \left(\frac{4 \times 2\sqrt{2}}{3} - \frac{2 \times 4\sqrt{2}}{5} \right) - \left(0 \right)$$

$$= \frac{16\sqrt{2}}{15}$$

(b) (i)

x	0	0.5	1	1.5	2
$x\sqrt{2-x}$	0	0.612	1	1.061	0

$$I \simeq \frac{1}{2} \times 0.5 \left[0 + 2 \left(0.612 + 1 + 1.061 \right) \right]$$

$$= \frac{1}{4} \left(5.346 \right) = 1.3365 = 1.34 \text{ (2 d.p.)}$$

(ii)

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
$x\sqrt{2-x}$	0	0.430	0.770	1	1.089	0.962	0

$$I \simeq \frac{1}{2} \times \frac{1}{3} \left[0 + 2 \left(0.430 + 0.770 + 1 + 1.089 + 0.962 \right) \right]$$

$$= \frac{1}{6} \left(8.502 \right) = 1.417 = 1.42 \text{ (2 d.p.)}$$

(c) (i) % error with 4 strips $= \dfrac{\frac{16}{15}\sqrt{2} - 1.3365}{\frac{16}{15}\sqrt{2}} \times 100 = 11.4\ \%$

(ii) % error with 6 strips $= \dfrac{\frac{16\sqrt{2}}{15} - 1.417}{\frac{16}{15}\sqrt{2}} \times 100 = 6.1\ \%$

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Integration

Exercise I, Question 1

Question:

The region R is bounded by the curve with equation $y = \text{f}(x)$, the x-axis and the lines $x = a$ and $x = b$. In each of the following cases find the exact value of:
(i) the area of R,
(ii) the volume of the solid of revolution formed by rotating R through 2π radians about the x-axis.

(a) $\text{f}\left(x\right) = \frac{2}{1+x}$; $a = 0$, $b = 1$

(b) $\text{f}(x) = \sec x$; $a = 0$, $b = \frac{\pi}{3}$

(c) $\text{f}(x) = \ln x$; $a = 1$, $b = 2$

(d) $\text{f}(x) = \sec x \tan x$; $a = 0$, $b = \frac{\pi}{4}$

(e) $\text{f}(x) = x\sqrt{4-x^2}$; $a = 0$, $b = 2$

Solution:

(a) (i) Area $= \int_0^1 \frac{2}{1+x}\text{d}x = \left[2\ln|1+x| \right]_0^1 = \left(2\ln 2\right) - \left(2\ln 1\right)$

$\therefore$ Area $= 2\ln 2$

(ii) Volume $= \pi\int_0^1 \left(\frac{2}{1+x}\right)^2 \text{d}x$

$= \pi\int_0^1 \frac{4}{(1+x)^2}\text{d}x$

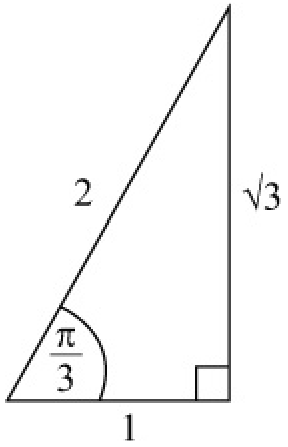
$= \pi\left[4\frac{(1+x)^{-1}}{-1} \right]_0^1$

$= \pi\left[-\frac{4}{1+x} \right]_0^1$

$$= \pi \left[\left(- \frac{4}{2} \right) - \left(- \frac{4}{1} \right) \right]$$
$$= 2\pi$$

(b) (i) Area $= \int_0^{\frac{\pi}{3}} \sec x \, dx$

$$= \left[\ln | \sec x + \tan x | \right]_0^{\frac{\pi}{3}}$$



$$= [\ln (2 + \sqrt{3})] - [\ln (1)]$$
$$\therefore \text{ Area } = \ln (2 + \sqrt{3})$$

(ii) Volume $= \pi \int_0^{\frac{\pi}{3}} \sec^2 x \, dx$

$$= \pi \left[\tan x \right]_0^{\frac{\pi}{3}}$$
$$= \pi [(\sqrt{3}) - (0)]$$
$$= \sqrt{3}\pi$$

(c) (i) Area $= \int_1^2 \ln x \, dx$

$$u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = 1 \quad \Rightarrow \quad v = x$$

$$\therefore \text{ Area } = \left[x \ln x \right]_1^2 - \int_1^2 x \times \frac{1}{x} dx$$

$$= (2 \ln 2) - (0) - [x]_1^2$$
$$= 2 \ln 2 - 1$$

(ii) Volume $= \pi \int_1^2 (\ln x)^2 dx$

$u = (\ln x)^{2} \quad \Rightarrow \quad \frac{du}{dx} = 2 \ln x \times \frac{1}{x}$

$\frac{dv}{dx} = 1 \quad \Rightarrow \quad v = x$

$\therefore V = \pi \left\{ \left[x (\ln x)^{2} \right]_{1}^{2} - 2 \int_{1}^{2} x \times \ln x \times \frac{1}{x} dx \right\}$

$= \pi \{ [2 (\ln 2)^{2}] - (0) \{ - 2\pi \int_{1}^{2} \ln x \, dx$

But $\int_{1}^{2} \ln x \, dx = 2 \ln 2 - 1$ from (i)

$\therefore V = 2\pi (\ln 2)^{2} - 2\pi (2 \ln 2 - 1)$

(d) (i) Area $= \int_{0}^{\frac{\pi}{4}} \sec x \tan x \, dx$

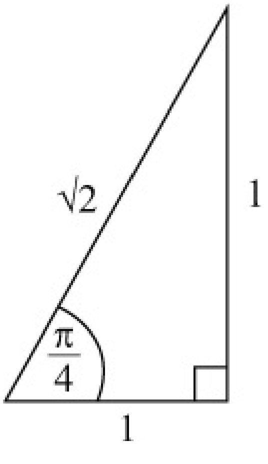
$= \left[\sec x \right]_{0}^{\frac{\pi}{4}}$

$= (\sqrt{2}) - (1)$

$\therefore$ Area $= \sqrt{2} - 1$

(ii) Volume $= \pi \int_{0}^{\frac{\pi}{4}} \sec^{2} x \tan^{2} x \, dx$

$= \pi \left[\frac{\tan^{3} x}{3} \right]_{0}^{\frac{\pi}{4}}$



$= \pi \left[\left(\frac{1^{3}}{3} \right) - \left(0 \right) \right]$

$= \frac{\pi}{3}$

(e) (i) Area $= \int_{0}^{2} x \sqrt{4 - x^{2}} \, dx$

Let $y = \left(4 - x^2 \right)^{\frac{3}{2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{2} \left(4 - x^2 \right)^{\frac{1}{2}} \times \left(-2x \right) = -3x \left(4 - x^2 \right)^{\frac{1}{2}}$$

$$\therefore \text{ Area } = \left[-\frac{1}{3} \left(4 - x^2 \right)^{\frac{3}{2}} \right]_0^2 = \left(0 \right) - \left(-\frac{1}{3} \times 2^3 \right) = \frac{8}{3}$$

(ii) Volume $= \pi \int_0^2 x^2 \left(4 - x^2 \right) dx$

$$= \pi \int_0^2 \left(4x^2 - x^4 \right) dx$$

$$= \pi \left[\frac{4}{3} x^3 - \frac{x^5}{5} \right]_0^2$$

$$= \pi \left[\left(\frac{32}{3} - \frac{32}{5} \right) - \left(0 \right) \right]$$

$$= \frac{64\pi}{15}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise I, Question 2

Question:

Find the exact area between the curve $y = \text{f}\left(x \right)$, the x-axis and the lines $x = a$ and $x = b$ where:

(a) $\text{f}\left(x \right) = \frac{4x + 3}{\left(x + 2 \right) \left(2x - 1 \right)}$; $a = 1$, $b = 2$

(b) $\text{f}\left(x \right) = \frac{x}{\left(x + 1 \right)^2}$; $a = 0$, $b = 2$

(c) $\text{f}\left(x \right) = x \sin x$; $a = 0$, $b = \frac{\pi}{2}$

(d) $\text{f}\left(x \right) = \cos x \sqrt{2 \sin x + 1}$; $a = 0$, $b = \frac{\pi}{6}$

(e) $\text{f}\left(x \right) = x\text{e}^{-x}$; $a = 0$, $b = \ln 2$

Solution:

(a) $\frac{4x + 3}{\left(x + 2 \right) \left(2x - 1 \right)} \equiv \frac{A}{x + 2} + \frac{B}{2x - 1}$

$\Rightarrow \quad 4x + 3 \equiv A \left(2x - 1 \right) + B \left(x + 2 \right)$

$x = \frac{1}{2} \quad \Rightarrow \quad 5 = \frac{5}{2}B \quad \Rightarrow \quad B = 2$

$x = -2 \quad \Rightarrow \quad -5 = -5A \quad \Rightarrow \quad A = 1$

$\therefore \text{ area } = \int_1^2 \frac{4x + 3}{\left(x + 2 \right) \left(2x - 1 \right)} \text{d}x$

$= \int_1^2 \left(\frac{1}{x + 2} + \frac{2}{2x - 1} \right) \text{d}x$

$= \left[\ln \left| x + 2 \right| + \ln \left| 2x - 1 \right| \right]_1^2$

$= \left(\ln 4 + \ln 3 \right) - \left(\ln 3 + \ln 1 \right)$

$= \ln 4 \quad \text{or} \quad 2 \ln 2$

(b) $\frac{x}{\left(x + 1 \right)^2} \equiv \frac{A}{\left(x + 1 \right)^2} + \frac{B}{x + 1}$

$\Rightarrow \quad x \equiv A + B (x + 1)$

Compare coefficient of x: $1 = B \quad \Rightarrow \quad B = 1$

Compare constants: $0 = A + B \quad \Rightarrow \quad A = - 1$

$$\therefore \text{ area } = \int_0^2 \frac{x}{(x + 1)^2} dx$$

$$= \int_0^2 \left(\frac{1}{x + 1} - \frac{1}{(x + 1)^2} \right) dx$$

$$= \left[\ln | x + 1 | + \frac{1}{x + 1} \right]_0^2$$

$$= \left(\ln 3 + \frac{1}{3} \right) - \left(\ln 1 + 1 \right)$$

$$= \ln 3 - \frac{2}{3}$$

(c) Area $= \int_0^{\frac{\pi}{2}} x \sin x dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \quad \Rightarrow \quad v = - \cos x$

$$\therefore \text{ area } = \left[- x \cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(- \cos x \right) dx$$

$$= \left(- \frac{\pi}{2} \cos \frac{\pi}{2} \right) - \left(0 \right) + \int_0^{\frac{\pi}{2}} \cos x dx$$

$$= 0 + \left[\sin x \right]_0^{\frac{\pi}{2}}$$

$$= \left(\sin \frac{\pi}{2} - 0 \right)$$

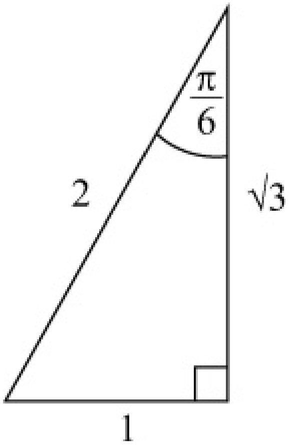
$$= 1$$

(d) Area $= \int_0^{\frac{\pi}{6}} \cos x \sqrt{2 \sin x + 1} dx$

Let $y = (2 \sin x + 1)^{\frac{3}{2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{2} (2 \sin x + 1)^{\frac{1}{2}} \times 2 \cos x = 3 \cos x (2 \sin x + 1)^{\frac{1}{2}}$$

$$\therefore \text{ area } = \left[\frac{1}{3} (2 \sin x + 1)^{\frac{3}{2}} \right]_0^{\frac{\pi}{6}}$$



$$= \left(\frac{1}{3}2^{\frac{3}{2}} \right) - \left(\frac{1}{3}1^{\frac{3}{2}} \right)$$

$$= \frac{2\sqrt{2}}{3} - \frac{1}{3}$$

$$= \frac{2\sqrt{2} - 1}{3}$$

(e) Area $= \int_0^{\ln 2} x e^{-x} dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$

$\therefore$ area $= [-xe^{-x}]_0^{\ln 2} - \int_0^{\ln 2} (-e^{-x})\ dx$

$= (-\ln 2 \times e^{-\ln 2}) - (0) + \int_0^{\ln 2} e^{-x} dx$

$= -\ln 2 \times \frac{1}{2} + \left[-e^{-x} \right]_0^{\ln 2}$

$= -\frac{1}{2}\ln 2 + \left(-e^{-\ln 2} \right) - \left(-e^{-0} \right)$

$= -\frac{1}{2}\ln 2 - \frac{1}{2} + 1$

$= \frac{1}{2}\left(1 - \ln 2 \right)$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 3

Question:

The region R is bounded by the curve C, the x-axis and the lines $x = -8$ and $x = +8$. The parametric equations for C are $x = t^3$ and $y = t^2$. Find:

(a) the area of R,

(b) the volume of the solid of revolution formed when R is rotated through 2π radians about the x-axis.

Solution:

(a) Area $= \int_{x=-8}^{x=8} y \, dx$

$x = t^3 \quad \Rightarrow \quad dx = 3t^2 \, dt$

Also $t = 2$ when $x = 8$
and $t = -2$ when $x = -8$.

$$\therefore \text{ area } = \int_{-2}^{2} t^2 \times 3t^2 \, dt$$

$$= \int_{-2}^{2} 3t^4 \, dt$$

$$= \left[\frac{3t^5}{5} \right]_{-2}^{2}$$

$$= \left(\frac{96}{5} \right) - \left(-\frac{96}{5} \right)$$

$$= \frac{192}{5}$$

(b) $V = \pi \int_{x=-8}^{x=8} y^2 \, dx$

$$= \pi \int_{-2}^{2} t^4 \times 3t^2 \, dt$$

$$= \pi \int_{-2}^{2} 3t^6 \, dt$$

$$= \pi \left[\frac{3t^7}{7} \right]_{-2}^{2}$$

$$= \pi \left[\left(\frac{3 \times 128}{7} \right) - \left(\frac{-3 \times 128}{7} \right) \right]$$

$$= \frac{768}{7} \pi$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 4

Question:

The curve C has parametric equations $x = \sin t$, $y = \sin 2t$, $0 \leq t \leq \frac{\pi}{2}$.

(a) Find the area of the region bounded by C and the x-axis.
If this region is revolved through 2π radians about the x-axis,

(b) find the volume of the solid formed.

Solution:

(a) Area $= \int_{t=0}^{t=\frac{\pi}{2}} y\,\mathrm{d}x$

$x = \sin t \Rightarrow \mathrm{d}x = \cos t\,\mathrm{d}t$

$\therefore$ area $= \int_0^{\frac{\pi}{2}} \sin 2t \times \cos t\,\mathrm{d}t$

$= \int_0^{\frac{\pi}{2}} 2\cos^2 t \sin t\,\mathrm{d}t$

$= \left[-\frac{2}{3}\cos^3 t \right]_0^{\frac{\pi}{2}}$

$= \left(0 \right) - \left(-\frac{2}{3} \right)$

$= \frac{2}{3}$

(b) $V = \pi \int_{t=0}^{t=\frac{\pi}{2}} y^2\,\mathrm{d}x$

$= \int_0^{\frac{\pi}{2}} \sin^2 2t \cos t\,\mathrm{d}t$

$= \pi \int_0^{\frac{\pi}{2}} 4\cos^3 t \sin t \times \sin t\,\mathrm{d}t$

$u = \sin t \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = \cos t$

$\frac{\mathrm{d}v}{\mathrm{d}t} = 4\cos^3 t \sin t \Rightarrow v = -\cos^4 t$

$$\therefore V = \pi \left\{ \left[-\sin t \cos^4 t \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos^5 t \, \mathrm{d}t \right\}$$

$$= \pi \int_0^{\frac{\pi}{2}} \cos^5 t \, \mathrm{d}t$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\cos^2 t \right)^2 \times \cos t \, \mathrm{d}t \qquad \text{Let } y = \sin t \quad \Rightarrow \quad \mathrm{d}y = \cos t \, \mathrm{d}t$$

$$= \pi \int_0^1 \left(1 - y^2 \right)^2 \mathrm{d}y$$

$$= \pi \int_0^1 \left(1 - 2y^2 + y^4 \right) \mathrm{d}y$$

$$= \pi \left[y - \frac{2}{3}y^3 + \frac{y^5}{5} \right]_0^1$$

$$= \pi \left(1 - \frac{2}{3} + \frac{1}{5} \right) - \left(0 \right)$$

$$= \frac{8\pi}{15}$$

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Integration
Exercise J, Question 1

Question:

Find general solutions of the following differential equations. Leave your answer in the form $y = \mathrm{f}(x)$.

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(1 + y\right)\left(1 - 2x\right)$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = y \tan x$

(c) $\cos^2 x \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \sin^2 x$

(d) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{x - y}$

(e) $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = y + xy$

Solution:

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(1 + y\right)\left(1 - 2x\right)$

$\Rightarrow \quad \int \frac{1}{1 + y}\mathrm{d}y = \int \left(1 - 2x\right)\mathrm{d}x$

$\Rightarrow \quad \ln|1 + y| = x - x^2 + C$

$\Rightarrow \quad 1 + y = \mathrm{e}^{(x - x^2 + C)}$

$\Rightarrow \quad 1 + y = A\mathrm{e}^{x - x^2}, \qquad (A = \mathrm{e}^C)$

$\Rightarrow \quad y = A\mathrm{e}^{x - x^2} - 1$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = y \tan x$

$\Rightarrow \quad \int \frac{1}{y}\mathrm{d}y = \int \tan x \, \mathrm{d}x$

$\Rightarrow \quad \ln|y| = \ln|\sec x| + C$

$$\Rightarrow \quad \ln|y| = \ln|k\sec x| \quad , \qquad (C = \ln k)$$

$$\Rightarrow \quad y = k\sec x$$

(c) $\cos^2 x \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \sin^2 x$

$$\Rightarrow \quad \int \frac{1}{y^2}\mathrm{d}y = \int \frac{\sin^2 x}{\cos^2 x}\mathrm{d}x$$

$$\Rightarrow \quad \int \frac{1}{y^2}\mathrm{d}y = \int \tan^2 x\,\mathrm{d}x = \int \left(\sec^2 x - 1 \right) \mathrm{d}x$$

$$\Rightarrow \quad -\frac{1}{y} = \tan x - x + C$$

$$\Rightarrow \quad y = \frac{-1}{\tan x - x + C}$$

(d) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{x-y} = 2\mathrm{e}^{x}\mathrm{e}^{-y}$

$$\Rightarrow \quad \int \frac{1}{\mathrm{e}^{-y}}\mathrm{d}y = \int 2\mathrm{e}^{x}\mathrm{d}x$$

i.e. $\Rightarrow \quad \int \mathrm{e}^{y}\mathrm{d}y = \int 2\mathrm{e}^{x}\mathrm{d}x$

$$\Rightarrow \quad \mathrm{e}^{y} = 2\mathrm{e}^{x} + C$$

$$\Rightarrow \quad y = \ln\left(2\mathrm{e}^{x} + C\right)$$

(e) $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = y + xy = y\left(1 + x\right)$

$$\Rightarrow \quad \int \frac{1}{y}\mathrm{d}y = \int x^{-2} + \frac{1}{x}\mathrm{d}x$$

$$\Rightarrow \quad \ln|y| = -x^{-1} + \ln|x| + C$$

$$\Rightarrow \quad \ln|y| - \ln|x| = C - \frac{1}{x}$$

$$\Rightarrow \quad \ln\left|\frac{y}{x}\right| = C - \frac{1}{x}$$

$$\Rightarrow \quad \frac{y}{x} = \mathrm{e}^{C - \frac{1}{x}}$$

$$\Rightarrow \quad \frac{y}{x} = A\mathrm{e}^{-\frac{1}{x}} , \qquad \left(\mathrm{e}^{C} = A \right)$$

$$\Rightarrow \quad y = Ax\mathrm{e}^{-\frac{1}{x}}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise J, Question 2

Question:

Find a general solution of the following differential equations. (You do not need to write the answers in the form $y = \mathrm{f}(x)$.)

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan y \tan x$

(b) $\sin y \cos x \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x \cos y}{\cos x}$

(c) $\left(1 + x^2\right) \frac{\mathrm{d}y}{\mathrm{d}x} = x \left(1 - y^2\right)$

(d) $\cos y \sin 2x \frac{\mathrm{d}y}{\mathrm{d}x} = \cot x \operatorname{cosec} y$

(e) $\mathrm{e}^{x+y} \frac{\mathrm{d}y}{\mathrm{d}x} = x \left(2 + \mathrm{e}^y\right)$

Solution:

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan y \tan x$

$\Rightarrow \quad \int \frac{1}{\tan y} \mathrm{d}y = \int \tan x \, \mathrm{d}x$

$\Rightarrow \quad \int \cot y \, \mathrm{d}y = \int \tan x \, \mathrm{d}x$

$\Rightarrow \quad \ln | \sin y | = \ln | \sec x | + C = \ln | k \sec x | \qquad (\ln k = C)$

$\Rightarrow \quad \sin y = k \sec x$

(b) $\sin y \cos x \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x \cos y}{\cos x}$

$\Rightarrow \quad \int \frac{\sin y}{\cos y} \mathrm{d}y = \int \frac{x}{\cos^2 x} \mathrm{d}x$

$\Rightarrow \quad \int \tan y \, \mathrm{d}y = \int x \sec^2 x \, \mathrm{d}x$

$\Rightarrow \quad \ln | \sec y | = \int x \sec^2 x \, \mathrm{d}x$

$$u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x \quad \Rightarrow \quad v = \tan x$$

$$\Rightarrow \quad \ln | \sec y | \; = x \tan x - \int \tan x \, \mathrm{d}x$$

$$\Rightarrow \quad \ln | \sec y | \; = x \tan x - \ln | \sec x | \; + \; C$$

(c) $\left(1 + x^2 \right) \frac{\mathrm{d}y}{\mathrm{d}x} = x \left(1 - y^2 \right)$

$$\Rightarrow \quad \int \frac{1}{1 - y^2} \mathrm{d}y = \int \frac{x}{1 + x^2} \mathrm{d}x$$

$$\frac{1}{1 - y^2} \equiv \frac{A}{1 - y} + \frac{B}{1 + y}$$

$$\Rightarrow \quad 1 \equiv A (1 + y) + B (1 - y)$$

$$y = 1 \quad \Rightarrow \quad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$$

$$y = -1 \quad \Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$$

$$\therefore \int \left(\frac{\left(\frac{1}{2} \right)}{1 - y} + \frac{\left(\frac{1}{2} \right)}{1 + y} \right) \mathrm{d}y = \int \frac{x}{1 + x^2} \mathrm{d}x$$

$$\Rightarrow \quad \frac{1}{2} \ln | 1 + y | \; - \; \frac{1}{2} \ln | 1 - y | \; = \; \frac{1}{2} \ln | 1 + x^2 | \; + \; C$$

(using $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \mathrm{d}x = \ln | \mathrm{f}(x) | \; + \; C$)

$$\Rightarrow \quad \ln \left| \frac{1 + y}{1 - y} \right| = \ln | 1 + x^2 | \; + \; 2C$$

$$\Rightarrow \quad \left| \frac{1 + y}{1 - y} \right| = k \left(1 + x^2 \right) \qquad \left(\ln k = 2C \right)$$

(d) $\cos y \sin 2x \frac{\mathrm{d}y}{\mathrm{d}x} = \cot x \operatorname{cosec} y$

$$\Rightarrow \quad \int \frac{\cos y}{\operatorname{cosec} y} \mathrm{d}y = \int \frac{\cot x}{\sin 2x} \mathrm{d}x$$

$$\Rightarrow \quad \int \sin y \cos y \, \mathrm{d}y = \int \frac{\cos x}{\sin x \quad 2 \sin x \cos x} \mathrm{d}x$$

$$\Rightarrow \quad \int \frac{1}{2}\sin 2y \, \mathrm{d}y = \int \frac{1}{2}\operatorname{cosec}^2 x \, \mathrm{d}x$$

$$\Rightarrow \quad -\frac{1}{4}\cos 2y = -\frac{1}{2}\cot x + C$$

or $\cos 2y = 2\cot x + k$

(e) $e^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(2 + e^y\right)$

$$\Rightarrow \quad e^x e^y \frac{\mathrm{d}y}{\mathrm{d}x} = x\left(2 + e^y\right)$$

$$\Rightarrow \quad \int \frac{e^y}{2 + e^y}\mathrm{d}y = \int x e^{-x}\mathrm{d}x$$

$$u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$$

$$\therefore \ln|2 + e^y| = -xe^{-x} + \int e^{-x}\mathrm{d}x$$

$$\Rightarrow \quad \ln|2 + e^y| = -xe^{-x} - e^{-x} + C$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise J, Question 3

Question:

Find general solutions of the following differential equations:

(a) $\frac{dy}{dx} = ye^x$

(b) $\frac{dy}{dx} = xe^y$

(c) $\frac{dy}{dx} = y\cos x$

(d) $\frac{dy}{dx} = x\cos y$

(e) $\frac{dy}{dx} = \left(1 + \cos 2x \right) \cos y$

(f) $\frac{dy}{dx} = \left(1 + \cos 2y \right) \cos x$

Solution:

(a) $\frac{dy}{dx} = ye^x$

$\Rightarrow \quad \int \frac{1}{y} dy = \int e^x dx$

$\Rightarrow \quad \ln | y | = e^x + C$

(b) $\frac{dy}{dx} = xe^y$

$\Rightarrow \quad \int \frac{1}{e^y} dy = \int x dx$

$\Rightarrow \quad \int e^{-y} dy = \int x dx$

$\Rightarrow \quad - e^{-y} = \frac{1}{2}x^2 + C$

(c) $\frac{dy}{dx} = y\cos x$

$$\Rightarrow \quad \int \frac{1}{y} dy = \int \cos x \, dx$$

$$\Rightarrow \quad \ln | y | = \sin x + C$$

or $y = Ae^{\sin x}$

(d) $\frac{dy}{dx} = x\cos y$

$$\Rightarrow \quad \int \frac{1}{\cos y} dy = \int x \, dx$$

$$\Rightarrow \quad \int \sec y \, dy = \int x \, dx$$

$$\Rightarrow \quad \ln | \sec y + \tan y | = \frac{x^2}{2} + C$$

(e) $\frac{dy}{dx} = \left(1 + \cos 2x \right) \cos y$

$$\Rightarrow \quad \int \frac{1}{\cos y} dy = \int \left(1 + \cos 2x \right) dx$$

$$\Rightarrow \quad \int \sec y \, dy = \int (1 + \cos 2x) \, dx$$

$$\Rightarrow \quad \ln | \sec y + \tan y | = x + \frac{1}{2} \sin 2x + C$$

(f) $\frac{dy}{dx} = \left(1 + \cos 2y \right) \cos x$

$$\Rightarrow \quad \int \frac{1}{1 + \cos 2y} dy = \int \cos x \, dx$$

$$\Rightarrow \quad \int \frac{1}{2\cos^2 y} dy = \int \cos x \, dx$$

$$\Rightarrow \quad \int \frac{1}{2} \sec^2 y \, dy = \int \cos x \, dx$$

$$\Rightarrow \quad \frac{1}{2} \tan y = \sin x + C$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise J, Question 4

Question:

Find particular solutions of the following differential equations using the given boundary conditions.

(a) $\frac{dy}{dx} = \sin x \cos^2 x; y = 0, x = \frac{\pi}{3}$

(b) $\frac{dy}{dx} = \sec^2 x \sec^2 y; y = 0, x = \frac{\pi}{4}$

(c) $\frac{dy}{dx} = 2\cos^2 y \cos^2 x; y = \frac{\pi}{4}, x = 0$

(d) $\left(1 - x^2 \right) \frac{dy}{dx} = xy + y ; x = 0.5, y = 6$

(e) $2\left(1 + x \right) \frac{dy}{dx} = 1 - y^2 ; x = 5, y = \frac{1}{2}$

Solution:

(a) $\frac{dy}{dx} = \sin x \cos^2 x$

$\Rightarrow \quad \int dy = \int \sin x \cos^2 x \, dx$

$\Rightarrow \quad y = - \frac{\cos^3 x}{3} + C$

$y = 0, x = \frac{\pi}{3} \quad \Rightarrow \quad 0 = - \frac{\left(\frac{1}{8} \right)}{3} + C \quad \Rightarrow \quad C = \frac{1}{24}$

$\therefore y = \frac{1}{24} - \frac{1}{3}\cos^3 x$

(b) $\frac{dy}{dx} = \sec^2 x \sec^2 y$

$$\Rightarrow \quad \int \frac{1}{\sec^2 y}\, dy = \int \sec^2 x\, dx$$

$$\Rightarrow \quad \int \cos^2 y\, dy = \int \sec^2 x\, dx$$

$$\Rightarrow \quad \int \left(\frac{1}{2} + \frac{1}{2}\cos 2y \right) dy = \int \sec^2 x\, dx$$

$$\Rightarrow \quad \frac{1}{2}y + \frac{1}{4}\sin 2y = \tan x + C$$

or $\sin 2y + 2y = 4\tan x + k$

$y = 0, x = \frac{\pi}{4} \Rightarrow 0 = 4 + k \Rightarrow k = -4$

$\therefore \sin 2y + 2y = 4\tan x - 4$

(c) $\frac{dy}{dx} = 2\cos^2 y \cos^2 x$

$$\Rightarrow \quad \int \frac{1}{\cos^2 y}\, dy = \int 2\cos^2 x\, dx$$

$$\Rightarrow \quad \int \sec^2 y\, dy = \int (1 + \cos 2x)\, dx$$

$$\Rightarrow \quad \tan y = x + \frac{1}{2}\sin 2x + C$$

$x = 0, \quad y = \frac{\pi}{4} \Rightarrow 1 = 0 + C$

$\therefore \tan y = x + \frac{1}{2}\sin 2x + 1$

(d) $\left(1 - x^2\right)\frac{dy}{dx} = xy + y$

$$\Rightarrow \quad \left(1 - x^2\right)\frac{dy}{dx} = \left(x + 1\right) y$$

$$\Rightarrow \quad \int \frac{1}{y}\, dy = \int \frac{1 + x}{1 - x^2}\, dx$$

$$\Rightarrow \quad \int \frac{1}{y}\, dy = \int \frac{1 + x}{(1 - x)(1 + x)}\, dx$$

$$\Rightarrow \quad \int \frac{1}{y}\, dy = \int \frac{1}{1 - x}\, dx$$

$$\Rightarrow \quad \ln |y| = -\ln|1 - x| + C$$

$x = 0.5, \quad y = 6 \Rightarrow \ln 6 = -\ln\frac{1}{2} + C \Rightarrow C = \ln 3$

$\therefore \ln|y| = \ln 3 - \ln|1 - x|$

or $y = \frac{3}{1 - x}$

(e) $2 \left(1 + x \right) \frac{dy}{dx} = 1 - y^2$

$$\Rightarrow \quad \int \frac{2}{1 - y^2} dy = \int \frac{1}{1 + x} dx$$

$$\frac{2}{1 - y^2} \equiv \frac{A}{1 - y} + \frac{B}{1 + y}$$

$$\Rightarrow \quad 2 \equiv A (1 + y) + B (1 - y)$$

$$y = 1 \quad \Rightarrow \quad 2 = 2A \quad \Rightarrow \quad A = 1$$

$$y = - 1 \quad \Rightarrow \quad 2 = 2B \quad \Rightarrow \quad B = 1$$

$$\therefore \int \left(\frac{1}{1 + y} + \frac{1}{1 - y} \right) dy = \int \frac{1}{1 + x} dx$$

$$\Rightarrow \quad \ln | 1 + y | - \ln | 1 - y | = \ln | 1 + x | + C$$

$$\Rightarrow \quad \ln \left| \frac{1 + y}{1 - y} \right| = \ln | k (1 + x) | \qquad \left(C = \ln k \right)$$

$$\Rightarrow \quad \frac{1 + y}{1 - y} = k \left(1 + x \right)$$

$$x = 5, y = \frac{1}{2} \quad \Rightarrow \quad \frac{\frac{3}{2}}{\frac{1}{2}} = 6k \quad \Rightarrow \quad k = \frac{1}{2}$$

$$\therefore \frac{1 + y}{1 - y} = \frac{1 + x}{2}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration
Exercise K, Question 1

Question:

The size of a certain population at time t is given by P. The rate of increase of P is given by $\frac{\mathrm{d}P}{\mathrm{d}t} = 2P$. Given that at time $t = 0$, the population was 3, find the population at time $t = 2$.

Solution:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 2P$$

$$\Rightarrow \quad \int \frac{1}{P}\mathrm{d}P = \int 2\,\mathrm{d}t$$

$$\Rightarrow \quad \ln|P| = 2t + C$$

$$\Rightarrow \quad P = A\mathrm{e}^{2t}$$

$$t = 0, P = 3 \quad \Rightarrow \quad 3 = A\mathrm{e}^{0} \quad \Rightarrow \quad A = 3$$

$$\therefore P = 3\mathrm{e}^{2t}$$

When $t = 2$, $P = 3\mathrm{e}^{4} = 164$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration
Exercise K, Question 2

Question:

The number of particles at time t of a certain radioactive substance is N. The substance is decaying in such a way that $\frac{\mathrm{d}N}{\mathrm{d}t} = - \frac{N}{3}$.
Given that at time $t = 0$ the number of particles is N_0, find the time when the number of particles remaining is $\frac{1}{2}N_0$.

Solution:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = - \frac{N}{3}$$

$$\Rightarrow \quad \int \frac{1}{N}\mathrm{d}N = \int - \frac{1}{3}\mathrm{d}t$$

$$\Rightarrow \quad \ln | N | = - \frac{1}{3}t + C$$

$$\Rightarrow \quad N = Ae^{- \frac{1}{3}t}$$

$$t = 0, N = N_0 \quad \Rightarrow \quad N_0 = Ae^0 \quad \Rightarrow \quad A = N_0$$

$$\therefore N = N_0 e^{- \frac{1}{3}t}$$

$$N = \frac{1}{2} N_0 \quad \Rightarrow \quad \frac{1}{2} = e^{- \frac{1}{3}t}$$

$$\Rightarrow \quad - \ln 2 = - \frac{1}{3}t$$

$$\Rightarrow \quad t = 3\ln 2 \text{ or } 2.08$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise K, Question 3

Question:

The mass M at time t of the leaves of a certain plant varies according to the differential equation $\frac{dM}{dt} = M - M^2$.

(a) Given that at time $t = 0$, $M = 0.5$, find an expression for M in terms of t.

(b) Find a value for M when $t = \ln 2$.

(c) Explain what happens to the value of M as t increases.

Solution:

$$\frac{dM}{dt} = M - M^2$$

$$\Rightarrow \quad \int \frac{1}{M(1-M)} dM = \int 1 \, dt \quad \text{but} \quad \frac{1}{M(1-M)} \equiv \frac{A}{M} + \frac{B}{1-M}$$

$$\begin{aligned} \therefore 1 \quad & \equiv A(1-M) + BM \\ M = 0 : 1 \quad & = 1A , \; A = 1 \\ M = 1 : 1 \quad & = 1B , \; B = 1 \end{aligned}$$

$$\Rightarrow \quad \int \left(\frac{1}{M} + \frac{1}{1-M} \right) dM = \int 1 \, dt$$

$$\Rightarrow \quad \ln |M| - \ln |1 - M| = t + C$$

$$\Rightarrow \quad \ln \left| \frac{M}{1-M} \right| = t + C$$

$$\Rightarrow \quad \frac{M}{1-M} = Ae^t$$

(a) $t = 0, M = 0.5 \quad \Rightarrow \quad \frac{0.5}{0.5} = Ae^0 \quad \Rightarrow \quad A = 1$

$$\therefore M = e^t - e^t M \quad \Rightarrow \quad M = \frac{e^t}{1 + e^t}$$

(b) $t = \ln 2 \quad \Rightarrow \quad M = \frac{e^{\ln 2}}{1 + e^{\ln 2}} = \frac{2}{1 + 2} = \frac{2}{3}$

(c) $t \to \infty \quad \Rightarrow \quad M = \frac{1}{e^{-t} + 1} \to \frac{1}{1} = 1$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise K, Question 4

Question:

The volume of liquid $V\text{cm}^3$ at time t seconds satisfies

$$-15\frac{dV}{dt} = 2V - 450.$$

Given that initially the volume is $300\,\text{cm}^3$, find to the nearest cm^3 the volume after 15 seconds.

Solution:

$$-15\frac{dV}{dt} = 2V - 450$$

$$\Rightarrow \quad \int \frac{1}{2V-450}dV = \int -\frac{1}{15}dt$$

$$\Rightarrow \quad \frac{1}{2}\ln|2V-450| = -\frac{1}{15}t + C$$

$$\Rightarrow \quad 2V - 450 = Ae^{-\frac{2}{15}t}$$

$$t = 0, V = 300 \quad \Rightarrow \quad 150 = Ae^0 \quad \Rightarrow \quad A = 150$$

$$\therefore 2V = 150e^{-\frac{2}{15}t} + 450$$

$$t = 15 \quad \Rightarrow \quad 2V = 150e^{-2} + 450$$

$$\Rightarrow \quad V = \frac{150}{2}\left(e^{-2} + 3\right)$$

$$\Rightarrow \quad V = 75(3 + e^{-2}) = 235$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise K, Question 5

Question:

The thickness of ice x mm on a pond is increasing and $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{20x^2}$, where t is measured in hours. Find how long it takes the thickness of ice to increase from 1 mm to 2 mm.

Solution:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{20x^2}$$

$$\Rightarrow \quad \int x^2 \mathrm{d}x = \int \frac{1}{20}\mathrm{d}t$$

$$\Rightarrow \quad \frac{1}{3}x^3 = \frac{t}{20} + C$$

$$t = 0, x = 1 \quad \Rightarrow \quad \frac{1}{3} = C$$

$$\therefore \quad \frac{20 (x^3 - 1)}{3} = t$$

$$x = 2 \quad \Rightarrow \quad t = \frac{20}{3}\left(8 - 1 \right)$$

$$\Rightarrow \quad t = \frac{140}{3} \text{or } 46 \frac{2}{3}$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise K, Question 6

Question:

The depth *h* metres of fluid in a tank at time *t* minutes satisfies $\frac{dh}{dt} = -k\sqrt{h}$, where *k* is a positive constant. Find, in terms of *k*, how long it takes the depth to decrease from 9 m to 4 m.

Solution:

$$\frac{dh}{dt} = -k\sqrt{h}$$

$$\Rightarrow \quad \int \frac{1}{h^{\frac{1}{2}}} dh = \int -k\, dt$$

$$\Rightarrow \quad \int h^{-\frac{1}{2}} dh = \int -k\, dt$$

$$\Rightarrow \quad 2h^{\frac{1}{2}} = -kt + C$$

$$t = 0, h = 9 \quad \Rightarrow \quad 2 \times 3 = 0 + C \quad \Rightarrow \quad C = 6$$

$$\therefore 2h^{\frac{1}{2}} - 6 = -kt$$

$$\text{or } t = \frac{6 - 2\sqrt{h}}{k}$$

$$h = 4 \quad \Rightarrow \quad t = \frac{6 - 2 \times 2}{k} = \frac{2}{k}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise K, Question 7

Question:

The rate of increase of the radius r kilometres of an oil slick is given by $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r^2}$, where k is a positive constant. When the slick was first observed the radius was 3 km. Two days later it was 5 km. Find, to the nearest day when the radius will be 6.

Solution:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r^2}$$

$$\Rightarrow \quad \int r^2 \, \mathrm{d}r = \int k \, \mathrm{d}t$$

$$\Rightarrow \quad \frac{1}{3} r^3 = kt + \quad C$$

$$t = 0, r = 3 \quad \Rightarrow \quad \frac{27}{3} = C \quad \Rightarrow \quad C = 9$$

$$\therefore kt = \frac{1}{3} r^3 - 9$$

$$t = 2, r = 5 \quad \Rightarrow \quad 2k = \frac{125}{3} - 9 \quad \Rightarrow \quad k = 16 \frac{1}{3}$$

$$\therefore \frac{49}{3} t = \frac{1}{3} r^3 - 9$$

or $t = \frac{r^3 - 27}{49}$

$$r = 6 \quad \Rightarrow \quad t = \frac{6^3 - 27}{49} = 3.85... = 4 \text{ days}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 1

Question:

It is given that $y = x^{\frac{3}{2}} + \frac{48}{x}, x > 0$.

(a) Find the value of x and the value of y when $\frac{dy}{dx} = 0$.

(b) Show that the value of y which you found is a minimum.

The finite region R is bounded by the curve with equation $y = x^{\frac{3}{2}} + \frac{48}{x}$, the lines $x = 1$, $x = 4$ and the x-axis.

(c) Find, by integration, the area of R giving your answer in the form $p + q \ln r$, where the numbers p, q and r are to be found.

Solution:

(a) $y = x^{\frac{3}{2}} + 48x^{-1} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 48x^{-2}$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{3}{2}x^{\frac{1}{2}} = \frac{48}{x^2}$$

$$\Rightarrow \quad x^{\frac{5}{2}} = \frac{2}{3} \times 48 = 32$$

$$\Rightarrow \quad x = 4, y = 2^3 + 12 = 20$$

$$\Rightarrow \quad x = 4\ ,\ y = 20$$

(b) $\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} + 96x^{-3} > 0$ for all $x > 0$

$\therefore$ 20 is a minimum value of y

(c) Area $= \int_1^4 \left(x^{\frac{3}{2}} + \frac{48}{x} \right) dx$

$$= \left[\frac{2}{5}x^{\frac{5}{2}} + 48\ln|x| \right]_1^4$$

$$= \left(\frac{2}{5} \times 32 + 48\ln 4 \right) - \left(\frac{2}{5} + 0 \right)$$

$$= \frac{62}{5} + 48\ln 4$$

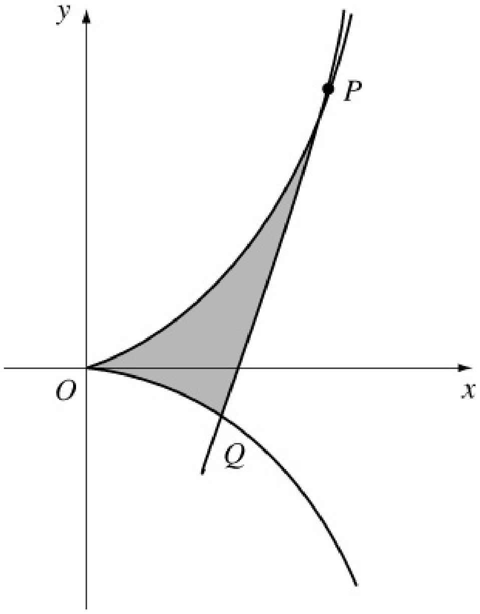
Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 2

Question:

The curve C has two arcs, as shown, and the equations
$x = 3t^2, y = 2t^3$,
where t is a parameter.



(a) Find an equation of the tangent to C at the point P where $t = 2$.
The tangent meets the curve again at the point Q.

(b) Show that the coordinates of Q are $(3, -2)$.
The shaded region R is bounded by the arcs OP and OQ of the curve C, and the line PQ, as shown.

(c) Find the area of R.

E

Solution:

(a) $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{6t^2}{6t} = t$

P is (12, 16)
$\therefore$ tangent is $y - 16 = 2(x - 12)$ or $y = 2x - 8$

(b) Substitute $x = 3t^2, y = 2t^3$ into the equation for the tangent

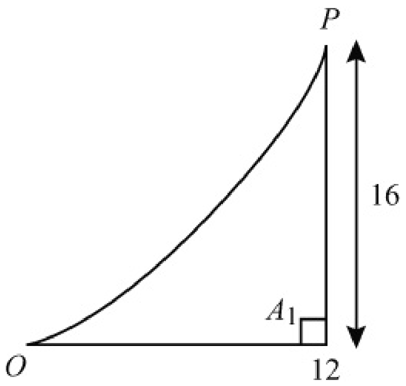
$\Rightarrow \quad 2t^3 = 6t^2 - 8$

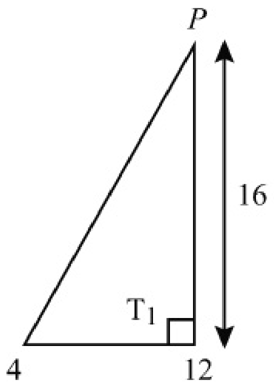
$\Rightarrow \quad t^3 - 3t^2 + 4 = 0$

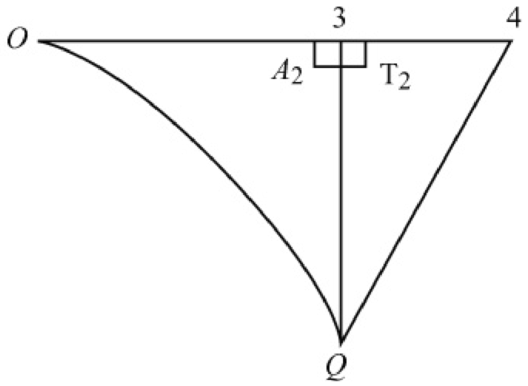
$\Rightarrow \quad (t-2)^2(t+1) = 0$

$\Rightarrow \quad t = -1$ at $Q(3, -2)$

(c)







Area of $R = A_1 - T_1 + A_2 + T_2$

$$A_1 + A_2 = \int y\,dx = \int_{t=-1}^{t=2} 2t^3 \times 6t\,dt = \int_{-1}^{2} 12t^4\,dt$$

$$= \left[\frac{12}{5}t^5\right]_{-1}^{2} = \left(\frac{12 \times 32}{5}\right) - \left(-\frac{12}{5}\right) = 79.2$$

$$T_1 = \frac{1}{2} \times 16 \times 8 = 64$$

$$T_2 = \frac{1}{2} \times 1 \times 2 = 1$$

$\therefore$ area of $R = 79.2 - 64 + 1 = 16.2$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 3

Question:

(a) Show that $(1 + \sin 2x)^2 \equiv \frac{1}{2}\left(3 + 4\sin 2x - \cos 4x \right)$.

(b) The finite region bounded by the curve with equation $y = 1 + \sin 2x$, the x-axis, the y-axis and the line with equation $x = \frac{\pi}{2}$ is rotated through 2π about the x-axis.

Using calculus, calculate the volume of the solid generated, giving your answer in terms of π.

Solution:

(a) $(1 + \sin 2x)^2 = 1 + 2\sin 2x + \sin^2 2x$

$= 1 + 2\sin 2x + \frac{1}{2}\left(1 - \cos 4x \right)$

$= \frac{3}{2} + 2\sin 2x - \frac{1}{2}\cos 4x$

$= \frac{1}{2}\left(3 + 4\sin 2x - \cos 4x \right)$

(b) $V = \pi \int y^2 \mathrm{d}x = \pi \int_0^{\frac{\pi}{2}} (1 + \sin 2x)^2 \mathrm{d}x$

$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \left(3 + 4\sin 2x - \cos 4x \right) \mathrm{d}x$

$= \frac{\pi}{2}\left[3x - 2\cos 2x - \frac{1}{4}\sin 4x \right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}\left[\left(\frac{3\pi}{2} - 2\cos \pi - \frac{1}{4}\sin 2\pi \right) - \left(0 - 2 - 0 \right) \right]$

$= \frac{\pi}{2}\left(\frac{3\pi}{2} + 2 + 2 \right)$

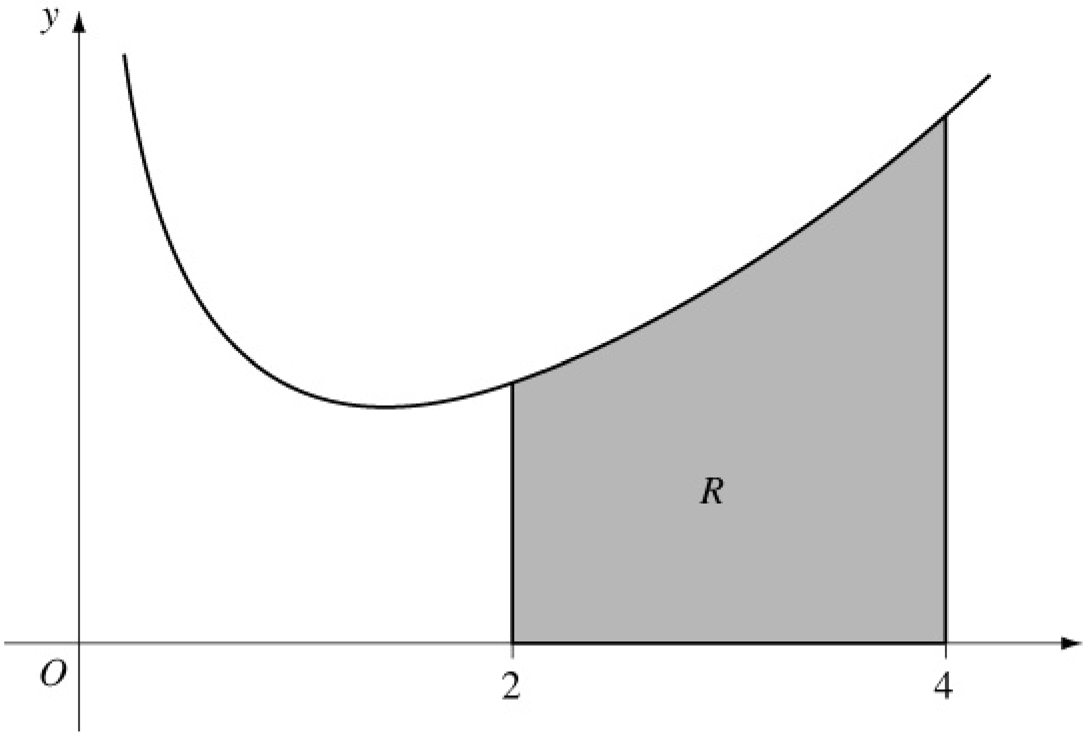
$= \frac{\pi}{4}\left(3\pi + 8 \right)$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 4

Question:

This graph shows part of the curve with equation $y = \mathrm{f}(x)$ where $\mathrm{f}\left(x\right) \equiv \mathrm{e}^{0.5x} + \frac{1}{x}, x > 0$.



The curve has a stationary point at $x = \alpha$.

(a) Find $\mathrm{f}'(x)$.

(b) Hence calculate $\mathrm{f}'(1.05)$ and $\mathrm{f}'(1.10)$ and deduce that $1.05 < \alpha < 1.10$.

(c) Find $\int \mathrm{f}(x)\,\mathrm{d}x$.
The shaded region R is bounded by the curve, the x-axis and the lines $x = 2$ and $x = 4$.

(d) Find, to 2 decimal places, the area of R.

E

Solution:

(a) $f'\left(x\right) = \frac{1}{2}e^{\frac{1}{2}x} - \frac{1}{x^2}$

(b) $f'(1.05) = -0.061... < 0$
$f'(1.10) = +0.040... > 0$
Change of sign $\therefore$ root α in interval (1.05, 1.10)

(c) $\int \left(e^{0.5x} + \frac{1}{x} \right) dx = 2e^{0.5x} + \ln|x| + C$

(d) Area $= \int_2^4 y\,dx$
$= [2e^{0.5x} + \ln|x|]_2^4$
$= (2e^2 + \ln 4) - (2e^1 + \ln 2)$
$= 2e^2 - 2e^1 + \ln 2$
$= 10.03$ (2 d.p.)

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 5

Question:

(a) Find $\int xe^{-x}\mathrm{d}x$.

(b) Given that $y = \frac{\pi}{4}$ at $x = 0$, solve the differential equation

$$e^{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\sin 2y}$$

Solution:

(a) $I = \int xe^{-x}\mathrm{d}x$

$u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\frac{\mathrm{d}v}{\mathrm{d}x} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$

$\therefore I = -xe^{-x} - \int(-e^{-x})\,\mathrm{d}x$

i.e. $I = -xe^{-x} - e^{-x} + C$

(b) $e^{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\sin 2y}$

$\Rightarrow \quad \int \sin 2y\,\mathrm{d}y = \int xe^{-x}\mathrm{d}x$

$\Rightarrow \quad -\frac{1}{2}\cos 2y = -xe^{-x} - e^{-x} + C$

$x = 0, y = \frac{\pi}{4} \quad \Rightarrow \quad 0 = 0 - 1 + C \quad \Rightarrow \quad C = 1$

$\therefore \frac{1}{2}\cos 2y = xe^{-x} + e^{-x} - 1$

or $\cos 2y = 2(xe^{-x} + e^{-x} - 1)$

Solutionbank

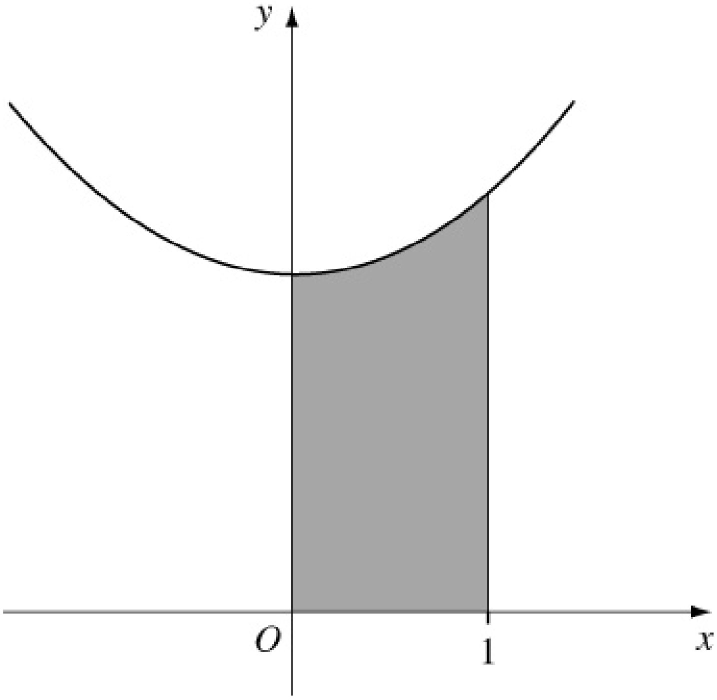
Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 6

Question:

The diagram shows the finite shaded region bounded by the curve with equation $y = x^2 + 3$, the lines $x = 1$, $x = 0$ and the x-axis. This region is rotated through 360° about the x-axis.
Find the volume generated.



Solution:

$$V = \pi \int_0^1 y^2 \mathrm{d}x = \pi \int_0^1 \left(x^2 + 3 \right)^2 \mathrm{d}x$$

$$= \pi \int_0^1 \left(x^4 + 6x^2 + 9 \right) \mathrm{d}x$$

$$= \pi \left[\frac{1}{5}x^5 + 2x^3 + 9x \right]_0^1$$

$$= \pi \left[\left(\frac{1}{5} + 2 + 9 \right) - \left(0 \right) \right]$$

$$= \frac{56\pi}{5}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 7

Question:

(a) Find $\int \frac{1}{x(x+1)} \mathrm{d}x$

(b) Using the substitution $u = \mathrm{e}^x$ and the answer to **a**, or otherwise, find $\int \frac{1}{1+\mathrm{e}^x} \mathrm{d}x$.

(c) Use integration by parts to find $\int x^2 \sin x \, \mathrm{d}x$.

Solution:

(a) $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$

$$\therefore \int \frac{1}{x(x+1)} \mathrm{d}x = \int \left(\frac{1}{x} - \frac{1}{x+1} \right) \mathrm{d}x$$

$$= \ln |x| - \ln |x+1| + C$$

$$= \ln \left| \frac{x}{x+1} \right| + C$$

(b) $I = \int \frac{1}{1+\mathrm{e}^x} \mathrm{d}x \qquad u = \mathrm{e}^x \quad \Rightarrow \quad \mathrm{d}u = \mathrm{e}^x \mathrm{d}x$

$$\therefore I = \int \frac{1}{(1+u)} \times \frac{1}{u} \mathrm{d}u = \ln \left| \frac{u}{1+u} \right| + C \quad \text{or} \quad \ln \left| \frac{\mathrm{e}^x}{1+\mathrm{e}^x} \right| + C$$

(c) $I = \int x^2 \sin x \, \mathrm{d}x$

$$u = x^2 \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \quad \Rightarrow \quad v = -\cos x$$

$$\therefore I = -x^2 \cos x - \int (-\cos x) \times 2x \, \mathrm{d}x$$

$$= -x^2 \cos x + \int 2x \cos x \, \mathrm{d}x$$

Let $J = \int 2x \cos x \, \mathrm{d}x$

$u = 2x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x \quad \Rightarrow \quad v = \sin x$

$\therefore J = 2x \sin x - \int 2 \sin x \, \mathrm{d}x$

$= 2x \sin x + 2 \cos x + \quad C$

$\therefore I = -x^2 \cos x + 2x \sin x + 2 \cos x + \quad k$

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Integration
Exercise L, Question 8

Question:

(a) Find $\int x \sin 2x \, \mathrm{d}x$.

(b) Given that $y = 0$ at $x = \frac{\pi}{4}$, solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x \sin 2x \cos^2 y$.

Solution:

(a) $I = \int x \sin 2x \, \mathrm{d}x$

$$u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x \quad \Rightarrow \quad v = \frac{-1}{2} \cos 2x$$

$$\therefore I = -\frac{1}{2} x \cos 2x - \int \frac{-1}{2} \cos 2x \, \mathrm{d}x$$

$$= -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = x \sin 2x \cos^2 y$

$$\Rightarrow \quad \int \sec^2 y \, \mathrm{d}y = \int x \sin 2x \, \mathrm{d}x$$

$$\Rightarrow \quad \tan y = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$$

$$y = 0, x = \frac{\pi}{4} \quad \Rightarrow \quad 0 = 0 + \frac{1}{4} + C \quad \Rightarrow \quad C = -\frac{1}{4}$$

$$\therefore \tan y = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x - \frac{1}{4}$$

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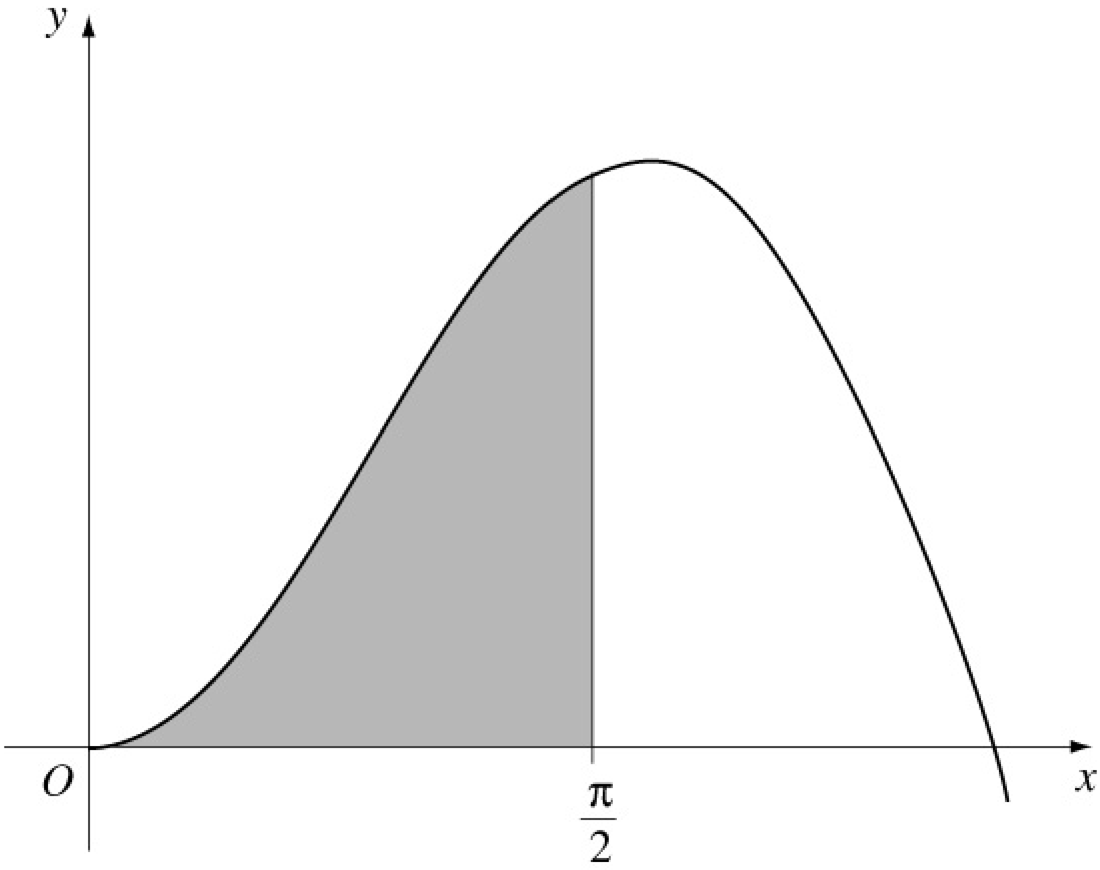
Integration

Exercise L, Question 9

Question:

(a) Find $\int x \cos 2x \, dx$.

(b) This diagram shows part of the curve with equation $y = 2x^{\frac{1}{2}} \sin x$. The shaded region in the diagram is bounded by the curve, the x-axis and the line with equation $x = \frac{\pi}{2}$. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution. Using calculus, calculate the volume of the solid of revolution formed, giving your answer in terms of π.



Solution:

(a) $I = \int x \cos 2x \, dx$

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x \quad \Rightarrow \quad v = \frac{1}{2}\sin 2x$$

$$\therefore I = \frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,\mathrm{d}x$$

$$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$$

(b) $V = \pi \int_0^{\frac{\pi}{2}} y^2\,\mathrm{d}x = \pi \int_0^{\frac{\pi}{2}} 4x\sin^2 x\,\mathrm{d}x$

$$\cos 2A = 1 - 2\sin^2 A \quad \Rightarrow \quad 2\sin^2 x = 1 - \cos 2x$$

$$\therefore V = \pi \int_0^{\frac{\pi}{2}} 2x \left(1 - \cos 2x \right) \mathrm{d}x$$

$$= \pi \int_0^{\frac{\pi}{2}} 2x\,\mathrm{d}x - 2\pi \int_0^{\frac{\pi}{2}} x\cos 2x\,\mathrm{d}x$$

$$= \left[\pi x^2 \right]_0^{\frac{\pi}{2}} - 2\pi \left[\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^3}{4} - 2\pi \left[\left(\frac{\pi}{4}\sin\pi + \frac{1}{4}\cos\pi \right) - \left(0 + \frac{1}{4} \right) \right]$$

$$= \frac{\pi^3}{4} + \pi$$

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Integration

Exercise L, Question 10

Question:

A curve has equation $y = f(x)$ and passes through the point with coordinates $(0, -1)$. Given that $f'(x) = \frac{1}{2}e^{2x} - 6x$,

(a) use integration to obtain an expression for f(x),

(b) show that there is a root α of the equation $f'(x) = 0$, such that $1.41 < \alpha < 1.43$.

Solution:

(a) $f'(x) = \frac{1}{2}e^{2x} - 6x$

$$\Rightarrow \quad f(x) = \frac{1}{4}e^{2x} - 3x^2 + C$$

$$f(0) = -1 \quad \Rightarrow \quad -1 = \frac{1}{4} - 0 + C \quad \Rightarrow \quad C = -\frac{5}{4}$$

$$\therefore f(x) = \frac{1}{4}e^{2x} - 3x^2 - \frac{5}{4}$$

(b) $f'(1.41) = -0.07... < 0$

$f'(1.43) = +0.15... > 0$

Change of sign $\therefore$ root in interval (1.41, 1.43).

Solutionbank
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Integration
Exercise L, Question 11

Question:

$$\mathrm{f}\left(x\right)=16x^{\frac{1}{2}}-\frac{2}{x},\ x>0.$$

(a) Solve the equation $\mathrm{f}\left(x\right)=0$.

(b) Find $\int \mathrm{f}\left(x\right)\,\mathrm{d}x$.

(c) Evaluate $\int_1^4 \mathrm{f}\left(x\right)\,\mathrm{d}x$, giving your answer in the form $p+q\ln r$, where p, q and r are rational numbers.

Solution:

(a) $\mathrm{f}\left(x\right)=0 \quad\Rightarrow\quad 16x^{\frac{1}{2}}=\frac{2}{x}$

$$\Rightarrow\quad 16x^{\frac{3}{2}}=2$$

$$\Rightarrow\quad x^{\frac{3}{2}}=\frac{1}{8}$$

$$\Rightarrow\quad x=\left(\sqrt[3]{\frac{1}{8}}\right)^2=\frac{1}{4}$$

(b) $\int\left(16x^{\frac{1}{2}}-\frac{2}{x}\right)\mathrm{d}x=\frac{16x^{\frac{3}{2}}}{\frac{3}{2}}-2\ln\left|x\right|+C$

$$=\frac{32}{3}x^{\frac{3}{2}}-2\ln\left|x\right|+C$$

(c) $\int_1^4\mathrm{f}\left(x\right)\mathrm{d}x=\left[\frac{32}{3}x^{\frac{3}{2}}-2\ln\left|x\right|\right]_1^4$

$$=\left(\frac{32}{3}\times 2^3-2\ln 4\right)-\left(\frac{32}{3}-0\right)$$

$$= \frac{224}{3} - 2\ln 4$$

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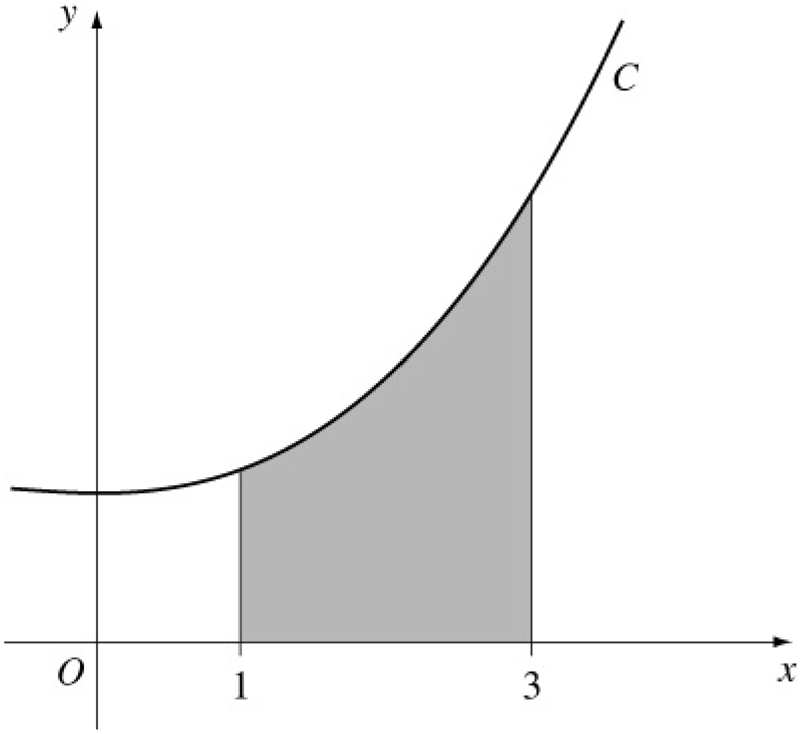
Integration
Exercise L, Question 12

Question:

Shown is part of a curve C with equation $y = x^2 + 3$. The shaded region is bounded by C, the x-axis and the lines with equations $x = 1$ and $x = 3$. The shaded region is rotated through 360° about the x-axis.

Using calculus, calculate the volume of the solid generated. Give your answer as an exact multiple of π.



Solution:

$$V = \pi \int_1^3 y^2 \, dx = \pi \int_1^3 \left(x^2 + 3 \right)^2 dx$$

$$= \pi \int_1^3 \left(x^4 + 6x^2 + 9 \right) \, dx$$

$$= \pi \left[\frac{1}{5}x^5 + 2x^3 + 9x \right]_1^3$$

$$= \pi \left[\left(\frac{243}{5} + 54 + 27 \right) - \left(\frac{1}{5} + 2 + 9 \right) \right]$$

$$= \pi \left(\frac{242}{5} + 81 - 11 \right)$$

$= 118.4\pi$

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Integration
Exercise L, Question 13

Question:

(a) Find $\int x\left(x^{2}+3\right)^{5}\mathrm{d}x$

(b) Show that $\int_{1}^{\mathrm{e}}\frac{1}{x^{2}}\ln x\,\mathrm{d}x=1-\frac{2}{\mathrm{e}}$

(c) Given that $p>1$, show that $\int_{1}^{p}\frac{1}{\left(x+1\right)\left(2x-1\right)}\mathrm{d}x=\frac{1}{3}\ln\frac{4p-2}{p+1}$

Solution:

(a) Let $y=\left(x^{2}+3\right)^{6}$

$$\Rightarrow\quad\frac{\mathrm{d}y}{\mathrm{d}x}=6\left(x^{2}+3\right)^{5}\times 2x$$

$$\therefore\quad\int x\left(x^{2}+3\right)^{5}\mathrm{d}x=\frac{1}{12}\left(x^{2}+3\right)^{6}+\quad C$$

(b) $I=\int_{1}^{\mathrm{e}}\frac{1}{x^{2}}\ln x\,\mathrm{d}x$

$$u=\ln x\quad\Rightarrow\quad\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{1}{x^{2}}\quad\Rightarrow\quad v=-\frac{1}{x}$$

$$\therefore I=\left[-\frac{1}{x}\ln x\right]_{1}^{\mathrm{e}}-\int_{1}^{\mathrm{e}}\left(-\frac{1}{x^{2}}\right)\mathrm{d}x$$

$$=\left(-\frac{1}{\mathrm{e}}\right)-\left(0\right)+\left[-\frac{1}{x}\right]_{1}^{\mathrm{e}}$$

$$=-\frac{1}{\mathrm{e}}+\left(-\frac{1}{\mathrm{e}}\right)-\left(-1\right)$$

$$=1-\frac{2}{\mathrm{e}}$$

(c) $\frac{1}{(x+1)(2x-1)} \equiv \frac{A}{x+1} + \frac{B}{2x-1}$

$$\Rightarrow \quad 1 \equiv A(2x-1) + B(x+1)$$

$$x = \frac{1}{2} \quad \Rightarrow \quad 1 = \frac{3}{2}B \quad \Rightarrow \quad B = \frac{2}{3}$$

$$x = -1 \quad \Rightarrow \quad 1 = -3A \quad \Rightarrow \quad A = -\frac{1}{3}$$

$$\therefore \int_1^p \frac{1}{(x+1)(2x-1)} \mathrm{d}x = \int_1^p \left(\frac{\frac{2}{3}}{2x-1} + \frac{-\frac{1}{3}}{x+1} \right) \mathrm{d}x$$

$$= \left[\frac{1}{3} \ln |2x-1| - \frac{1}{3} \ln |x+1| \right]_1^p$$

$$= \left[\frac{1}{3} \ln \left| \frac{2x-1}{x+1} \right| \right]_1^p$$

$$= \left[\frac{1}{3} \ln \left(\frac{2p-1}{p+1} \right) \right] - \left(\frac{1}{3} \ln \frac{1}{2} \right)$$

$$= \frac{1}{3} \ln \left(\frac{4p-2}{p+1} \right)$$

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Integration

Exercise L, Question 14

Question:

$$\text{f}\left(x\right) \equiv \frac{5x^2-8x+1}{2x\left(x-1\right)^2} \equiv \frac{A}{x}+\frac{B}{x-1}+\frac{C}{\left(x-1\right)^2}$$

(a) Find the values of the constants *A*, *B* and *C*.

(b) Hence find $\int \text{f}\left(x\right)\,\text{d}x$.

(c) Hence show that $\int_4^9 \text{f}\left(x\right)\,\text{d}x = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$

Solution:

(a) $\text{f}\left(x\right) \equiv \frac{5x^2-8x+1}{2x\left(x-1\right)^2} \equiv \frac{A}{x}+\frac{B}{x-1}+\frac{C}{\left(x-1\right)^2}$

$\Rightarrow \quad 5x^2-8x+1 \equiv 2A\left(x-1\right)^2+2Bx\left(x-1\right)+2Cx$

$x=0 \quad \Rightarrow \quad 1=2A \quad \Rightarrow \quad A=\frac{1}{2}$

$x=1 \quad \Rightarrow \quad -2=2C \quad \Rightarrow \quad C=-1$

Coefficients of x^2: $\quad 5=2A+2B \quad \Rightarrow \quad B=2$

(b) $\int \text{f}\left(x\right)\text{d}x = \int\left(\frac{\frac{1}{2}}{x}+\frac{2}{x-1}-\frac{1}{\left(x-1\right)^2}\right)\text{d}x$

$= \frac{1}{2}\ln\left|x\right| + 2\ln\left|x-1\right| + \frac{1}{x-1} + \quad C$

(c) $\int_4^9 \text{f}\left(x\right)\text{d}x = \left[\frac{1}{2}\ln\left|x\right| + 2\ln\left|x-1\right| + \frac{1}{x-1}\right]_4^9$

$= \left[\ln\left|\sqrt{x}\left(x-1\right)^2\right| + \frac{1}{x-1}\right]_4^9$

$$= \left[\ln \left(3 \times 64 \right) + \frac{1}{8} \right] - \left[\ln \left(2 \times 9 \right) + \frac{1}{3} \right]$$

$$= \ln \left(\frac{3 \times 64}{2 \times 9} \right) + \frac{1}{8} - \frac{1}{3}$$

$$= \ln \frac{32}{3} - \frac{5}{24}$$

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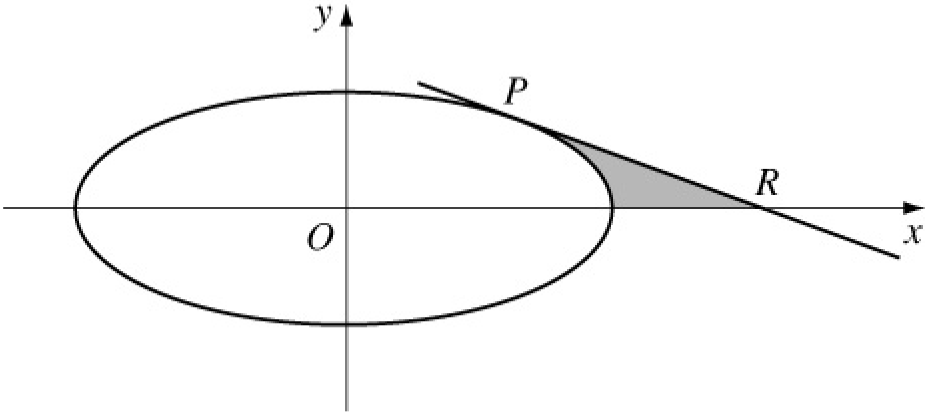
Integration
Exercise L, Question 15

Question:

The curve shown has parametric equations
$x = 5\cos\theta, y = 4\sin\theta, 0 \leq \theta < 2\pi.$



(a) Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$.

(b) Find an equation of the tangent to the curve at the point P.

(c) Find the coordinates of the point R where this tangent meets the x-axis.
The shaded region is bounded by the tangent PR, the curve and the x-axis.

(d) Find the area of the shaded region, leaving your answer in terms of π.

Ⓔ

Solution:

(a) $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = -\frac{4\cos\theta}{5\sin\theta}$

$\therefore$ gradient of tangent at $P = -\frac{4}{5}$

(b) $P = \left(\frac{5}{\sqrt{2}}, \frac{4}{\sqrt{2}}\right)$

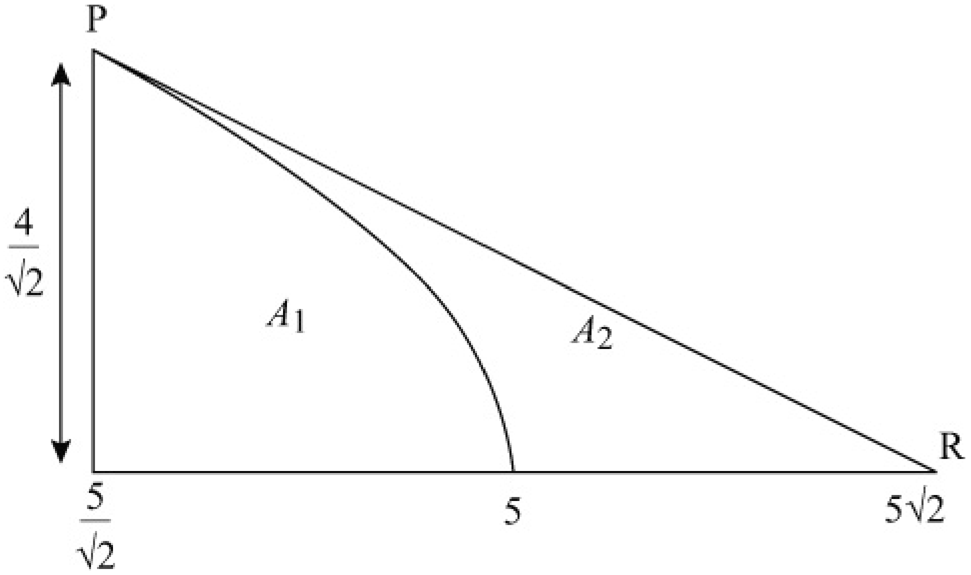
$\therefore$ equation of tangent is

$$y - \frac{4}{\sqrt{2}} = -\frac{4}{5}\left(x - \frac{5}{\sqrt{2}}\right) \quad \text{or} \quad y - 2\sqrt{2} = -\frac{4}{5}\left(x - \frac{5}{\sqrt{2}}\right)$$

(c) At R, $y = 0 \quad \Rightarrow \quad x = \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} = 5\sqrt{2}$

$\therefore R$ is $(5\sqrt{2}, 0)$

(d)



$$A_1 + A_2 = \frac{1}{2} \times \left(5\sqrt{2} - \frac{5}{\sqrt{2}}\right) \times \frac{4}{\sqrt{2}} = \frac{1}{2} \times \frac{5}{\sqrt{2}} \times \frac{4}{\sqrt{2}} = 5$$

$$A_1 = \int y \, dx = \int_{\frac{\pi}{4}}^{0} 4\sin\theta \times \left(-5\sin\theta\right) d\theta$$

$$= 10\int_0^{\frac{\pi}{4}} \left(1 - \cos 2\theta\right) d\theta$$

$$= \left[10\theta - 5\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{5\pi}{2} - 5$$

$$\therefore A_2 = 5 - A_1 = 5 - \left(\frac{5\pi}{2} - 5\right) = 10 - 2.5\pi$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 16

Question:

(a) Obtain the general solution of the differential equation

$$\frac{dy}{dx} = xy^2, y > 0.$$

(b) Given also that $y = 1$ at $x = 1$, show that

$$y = \frac{2}{3 - x^2}, \ - \sqrt{3} < x < \sqrt{3}$$

is a particular solution of the differential equation.

The curve C has equation $y = \frac{2}{3 - x^2}, x \neq - \sqrt{3}, x \neq \sqrt{3}$

(c) Write down the gradient of C at the point (1, 1).

(d) Deduce that the line which is a tangent to C at the point (1, 1) has equation $y = x$.

(e) Find the coordinates of the point where the line $y = x$ again meets the curve C.

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Solution:

(a) $\frac{dy}{dx} = xy^2$

$$\Rightarrow \int \frac{1}{y^2} dy = \int x \, dx$$

$$\Rightarrow - \frac{1}{y} = \frac{x^2}{2} + C$$

or $y = \frac{-2}{x^2 + k} \quad \left(k = 2C \right)$

(b) $y = 1, x = 1 \quad \Rightarrow \quad 1 = \frac{-2}{1 + k} \quad \Rightarrow \quad k = -3$

$\therefore y = \frac{2}{3 - x^2}$

for $x^2 \neq 3$ and $y > 0$, i.e. $- \sqrt{3} < x < \sqrt{3}$

(c) When $x = 1$, $y = 1$ $\frac{dy}{dx}$ is 1

(d) Equation of tangent is $y - 1 = 1 (x - 1)$, i.e. $y = x$.

(e) $x = \frac{2}{3 - x^2} \Rightarrow - x^3 + 3x = 2$ or $x^3 - 3x + 2 = 0$

$\Rightarrow (x - 1)^2 (x + 2) = 0$

$\therefore y = x$ meets curve at $(- 2 , - 2)$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

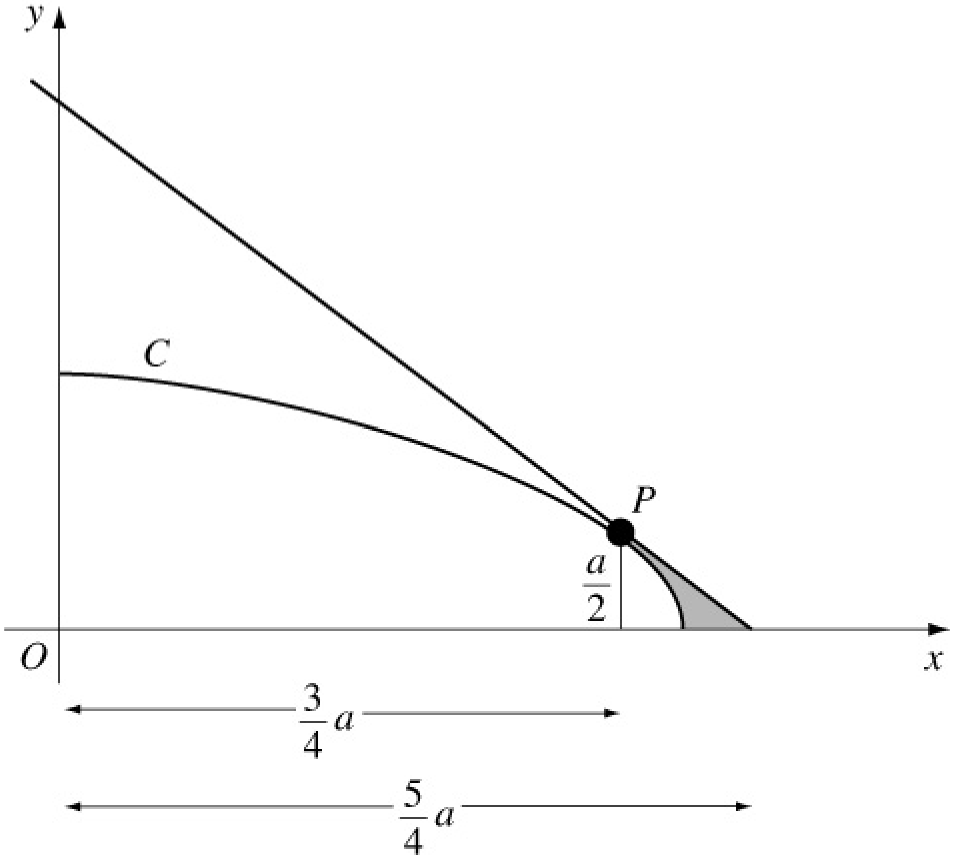
Exercise L, Question 17

Question:

The diagram shows the curve C with parametric equations

$$x = a\sin^2 t,\ y = a\cos t,\ 0 \leq t \leq \frac{1}{2}\pi,$$

where a is a positive constant. The point P lies on C and has coordinates $\left(\frac{3}{4}a,\ \frac{1}{2}a \right)$.



(a) Find $\frac{dy}{dx}$, giving your answer in terms of t.

(b) Find an equation of the tangent to C at P.

(c) Show that a cartesian equation of C is $y^2 = a^2 - ax$.
The shaded region is bounded by C, the tangent at P and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid of

revolution.

(d) Use calculus to calculate the volume of the solid revolution formed, giving your answer in the form $k\pi a^3$, where k is an exact fraction.

Solution:

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-a\sin t}{2a\sin t\cos t} = -\frac{1}{2}\sec t$

(b) P is $\left(\frac{3}{4}a, \frac{1}{2}a \right)$, so $\cos t = \frac{1}{2}$

$$\Rightarrow \quad M = -\frac{1}{2 \times \frac{1}{2}} = -1$$

$\therefore$ tangent is $y - \frac{1}{2}a = -1\left(x - \frac{3}{4}a \right)$

or $\quad y = -x + \frac{5}{4}a$

(c) $\sin^2 t + \cos^2 t = 1 \quad \Rightarrow \quad \frac{x}{a} + \frac{y^2}{a^2} = 1$

or $\quad y^2 = a^2 - ax$

(d) volume $=$ cone $- \pi \int_{\frac{3}{4}a}^{a} y^2\,\mathrm{d}x$

$$\text{cone} = \frac{1}{3}\pi \left(\frac{1}{2}a \right)^2 \left(\frac{5}{4}a - \frac{3}{4}a \right) = \frac{\pi a^3}{24}$$

$$\pi \int_{\frac{3}{4}a}^{a} y^2\,\mathrm{d}x = \pi \left[a^2x - \frac{a}{2}x^2 \right]_{\frac{3}{4}a}^{a}$$

$$= \pi \left[\left(a^3 - \frac{a^3}{2} \right) - \left(\frac{3}{4}a^3 - \frac{9}{32}a^3 \right) \right] = \frac{\pi a^3}{32}$$

$$\therefore \text{ Volume } = \pi \left(\frac{a^3}{24} - \frac{a^3}{32} \right) = \frac{\pi a^3}{96}$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 18

Question:

(a) Using the substitution $u = 1 + 2x$, or otherwise, find

$$\int \frac{4x}{(1+2x)^2} \mathrm{d}x,\ x > -\frac{1}{2},$$

(b) Given that $y = \frac{\pi}{4}$ when $x = 0$, solve the differential equation

$$(1+2x)^2 \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\sin^2 y}$$

Solution:

(a) $I = \int \frac{4x}{(1+2x)^2} \mathrm{d}x$

$u = 1 + 2x$

$\Rightarrow \quad \frac{\mathrm{d}u}{2} = \mathrm{d}x$ and $4x = 2(u-1)$

$\therefore I = \int \frac{2(u-1)}{u^2} \times \frac{\mathrm{d}u}{2}$

$= \int \left(\frac{1}{u} - u^{-2} \right) \mathrm{d}u$

$= \ln |u| + \frac{1}{u} + C$

$= \ln |1+2x| + \frac{1}{1+2x} + C$

(b) $(1+2x)^2 \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{\sin^2 y}$

$\Rightarrow \quad \int \sin^2 y \, \mathrm{d}y = \int \frac{x}{(1+2x)^2} \mathrm{d}x$

$\Rightarrow \quad \int 4 \sin^2 y \, \mathrm{d}y = \int \frac{4x}{(1+2x)^2} \mathrm{d}x$

$\Rightarrow \quad \int (2 - 2\cos 2y) \, \mathrm{d}y = I$

$$\Rightarrow \quad 2y - \sin 2y = \ln | 1 + 2x | \quad + \frac{1}{1 + 2x} + \quad C$$

$$x = 0, y = \frac{\pi}{4} \quad \Rightarrow \quad \frac{\pi}{2} - 1 = \ln 1 + 1 + \quad C$$

$$\Rightarrow \quad C = \frac{\pi}{2} - 2$$

$$\therefore 2y - \sin 2y = \ln | 1 + 2x | \quad + \frac{1}{1 + 2x} + \frac{\pi}{2} - 2$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 19

Question:

The diagram shows the curve with equation $y = xe^{2x}$, $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

The finite region R_1 bounded by the curve, the x-axis and the line $x = -\frac{1}{2}$ has area A_1.

The finite region R_2 bounded by the curve, the x-axis and the line $x = \frac{1}{2}$ has area A_2.

(a) Find the exact values of A_1 and A_2 by integration.

(b) Show that $A_1 : A_2 = (e - 2) : e$.

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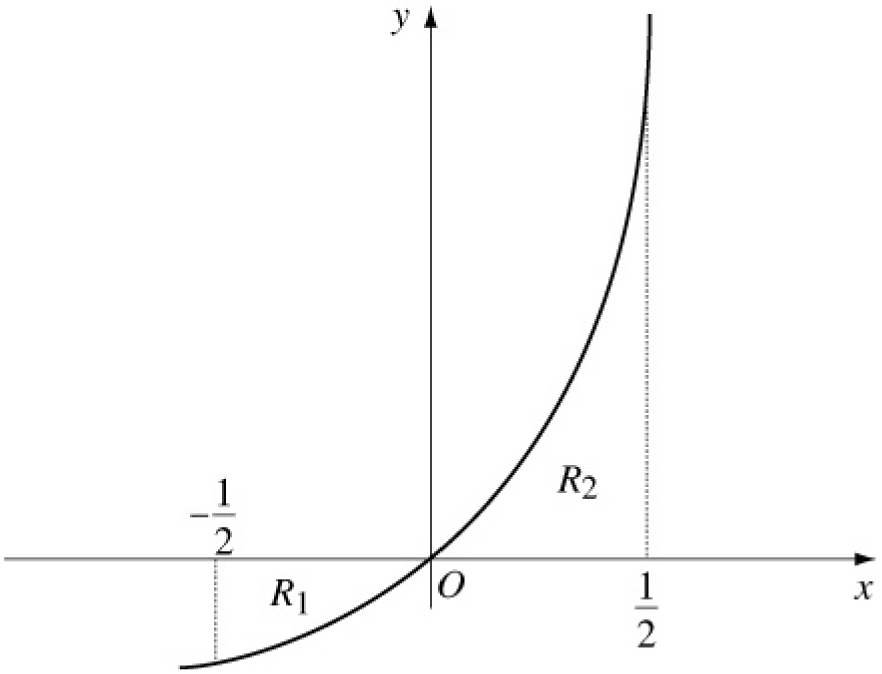

Solution:

(a) $\int xe^{2x} dx$

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \quad \Rightarrow \quad v = \frac{1}{2}\mathrm{e}^{2x}$$

$$\therefore \int x\mathrm{e}^{2x}\,\mathrm{d}x = \frac{1}{2}x\mathrm{e}^{2x} - \int \frac{1}{2}\mathrm{e}^{2x}\,\mathrm{d}x = \frac{1}{2}x\mathrm{e}^{2x} - \frac{1}{4}\mathrm{e}^{2x} + C$$

$$A_1 = -\left[\frac{1}{2}x\mathrm{e}^{2x} - \frac{1}{4}\mathrm{e}^{2x}\right]_{-\frac{1}{2}}^{0}$$

$$= -\left[\left(0 - \frac{1}{4}\right) - \left(-\frac{1}{4}\mathrm{e}^{-1} - \frac{1}{4}\mathrm{e}^{-1}\right)\right]$$

$$= \frac{1}{4}\left(1 - 2\mathrm{e}^{-1}\right)$$

$$A_2 = \left[\frac{1}{2}x\mathrm{e}^{2x} - \frac{1}{4}\mathrm{e}^{2x}\right]_{0}^{\frac{1}{2}}$$

$$= \left(\frac{1}{4}\mathrm{e}^{1} - \frac{1}{4}\mathrm{e}^{1}\right) - \left(0 - \frac{1}{4}\right)$$

$$= \frac{1}{4}$$

(b) $$\frac{A_1}{A_2} = \frac{\frac{1}{4}\left(1 - 2\mathrm{e}^{-1}\right)}{\frac{1}{4}} = 1 - 2\mathrm{e}^{-1} = \frac{\mathrm{e} - 2}{\mathrm{e}}$$

$$\therefore A_1 : A_2 = \left(\mathrm{e} - 2\right) : \mathrm{e}$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 20

Question:

Find $\int x^2 e^{-x} dx$.

Given that $y = 0$ at $x = 0$, solve the differential equation $\frac{dy}{dx} = x^2 e^{3y - x}$.

Solution:

$$I = \int x^2 e^{-x} dx$$

$$u = x^2 \quad \Rightarrow \quad \frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$$

$$\therefore I = -x^2 e^{-x} - \int (-e^{-x}) \times 2x\,dx$$

$$= -x^2 e^{-x} + \int 2x\,e^{-x} dx$$

$$J = \int 2xe^{-x} dx$$

$$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$$

$$\frac{dv}{dx} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$$

$$\therefore J = -2xe^{-x} - \int (-e^{-x}) \times 2\,dx$$

$$= -2xe^{-x} - 2e^{-x} + k$$

$$\therefore I = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$\frac{dy}{dx} = x^2 e^{3y - x} = x^2 e^{-x} e^{3y}$$

$$\Rightarrow \quad \int e^{-3y} dy = \int x^2 e^{-x} dx$$

$$\Rightarrow \quad -\frac{1}{3} e^{-3y} = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$x = 0, y = 0 \quad \Rightarrow \quad -\frac{1}{3} = -2 + C \quad \Rightarrow \quad C = \frac{5}{3}$$

$$\therefore \frac{1}{3} e^{-3y} = x^2 e^{-x} + 2xe^{-x} + 2e^{-x} - \frac{5}{3}$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 21

Question:

The curve with equation $y = e^{3x} + 1$ meets the line $y = 8$ at the point $(h, 8)$.

(a) Find h, giving your answer in terms of natural logarithms.

(b) Show that the area of the finite region enclosed by the curve with equation $y = e^{3x} + 1$, the x-axis, the y-axis and the line $x = h$ is $2 + \frac{1}{3}\ln 7$.

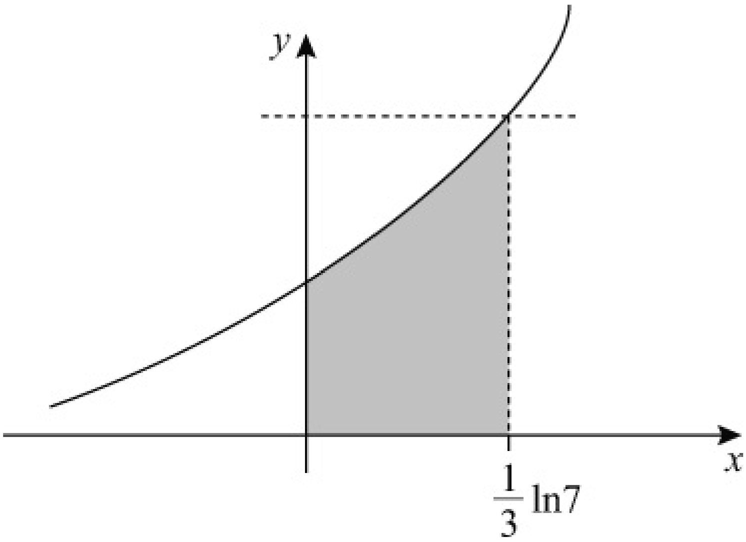
E

Solution:

(a) $8 = e^{3x} + 1 \quad \Rightarrow \quad 7 = e^{3x}$

$\therefore x = \frac{1}{3}\ln 7$, i.e. $h = \frac{1}{3}\ln 7$

(b)



$$\begin{aligned} \text{Area} &= \int_0^{\frac{1}{3}\ln 7} y \, dx \\ &= \int_0^{\frac{1}{3}\ln 7} \left(e^{3x} + 1 \right) dx \\ &= \left[\frac{1}{3}e^{3x} + x \right]_0^{\frac{1}{3}\ln 7} \end{aligned}$$

$$= \left(\frac{1}{3}e^{\ln 7} + \frac{1}{3}\ln 7 \right) - \left(\frac{1}{3} + 0 \right)$$

$$= \frac{1}{3}\left(7 + \ln 7 \right) - \frac{1}{3}$$

$$= \frac{1}{3}\left(6 + \ln 7 \right)$$

$$= 2 + \frac{1}{3}\ln 7$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 22

Question:

(a) Given that

$$\frac{x^2}{x^2-1} \equiv A + \frac{B}{x-1} + \frac{C}{x+1},$$

find the values of the constants A, B and C.

(b) Given that $x = 2$ at $t = 1$, solve the differential equation

$$\frac{dx}{dt} = 2 - \frac{2}{x^2}, x > 1.$$

You need not simplify your final answer. E

Solution:

(a) $\frac{x^2}{x^2-1} \equiv A + \frac{B}{x-1} + \frac{C}{x+1}$

$$\Rightarrow \quad x^2 \equiv A(x-1)(x+1) + B(x+1) + C(x-1)$$

$$x = 1 \quad \Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$$

$$x = -1 \quad \Rightarrow \quad 1 = -2C \quad \Rightarrow \quad C = -\frac{1}{2}.$$

Coefficients of x^2: $\quad 1 = A \quad \Rightarrow \quad A = 1$

(b) $\frac{dx}{dt} = 2\frac{(x^2-1)}{x^2}$

$$\Rightarrow \quad \int \frac{x^2}{x^2-1} dx = \int 2\,dt$$

$$\Rightarrow \quad \int \left(1 + \frac{\left(\frac{1}{2}\right)}{x-1} - \frac{\left(\frac{1}{2}\right)}{x+1} \right) dx = 2t$$

$$\Rightarrow \quad x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = 2t + C$$

$$x = 2, t = 1 \quad \Rightarrow \quad 2 + \frac{1}{2}\ln\frac{1}{3} = 2 + \quad C \quad \Rightarrow \quad C = \frac{1}{2}\ln\frac{1}{3}$$

$$\therefore x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = 2t + \frac{1}{2}\ln\frac{1}{3}$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 23

Question:

The curve C is given by the equations

$x = 2t, y = t^2,$

where t is a parameter.

(a) Find an equation of the normal to C at the point P on C where $t = 3$.
The normal meets the y-axis at the point B. The finite region R is bounded by the part of the curve C between the origin O and P, and the lines OB and OP.

(b) Show the region R, together with its boundaries, in a sketch.
The region R is rotated through 2π about the y-axis to form a solid S.

(c) Using integration, and explaining each step in your method, find the volume of S, giving your answer in terms of π.

Solution:

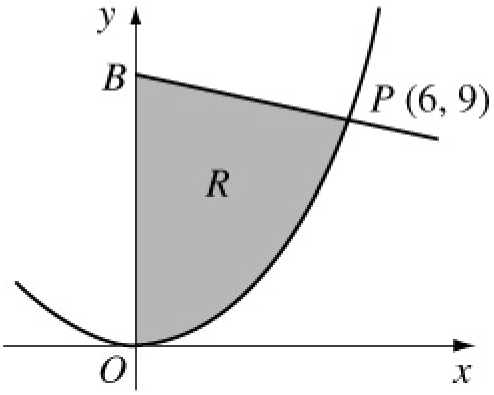
(a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2t}{2} = t.$

$\therefore$ at P (6, 9) gradient of normal is $-\frac{1}{3}$

$\therefore$ equation of normal is $y - 9 = -\frac{1}{3}\left(x - 6\right)$ or $y = -\frac{1}{3}x + 11$

(b) $x = 2t, y = t^2 \quad \Rightarrow \quad y = \frac{x^2}{4}$

B is (0, 11)



(c) volume $= \text{cone} + \pi \int_0^9 x^2 \, \mathrm{d}y$

cone $= \frac{1}{3}\pi \times 6^2 \times 2 = 24\pi$

$\pi \int_0^9 x^2 \, \mathrm{d}y = \pi \int_{t=0}^{t=3} 4t^2 \times 2t \, \mathrm{d}t = \pi \int_0^3 8t^3 \, \mathrm{d}t$

$= \pi \left[2t^4 \right]_0^3 = \pi \times 2 \times 81 = 162\pi$

$\therefore$ Volume of $S = 186\pi$

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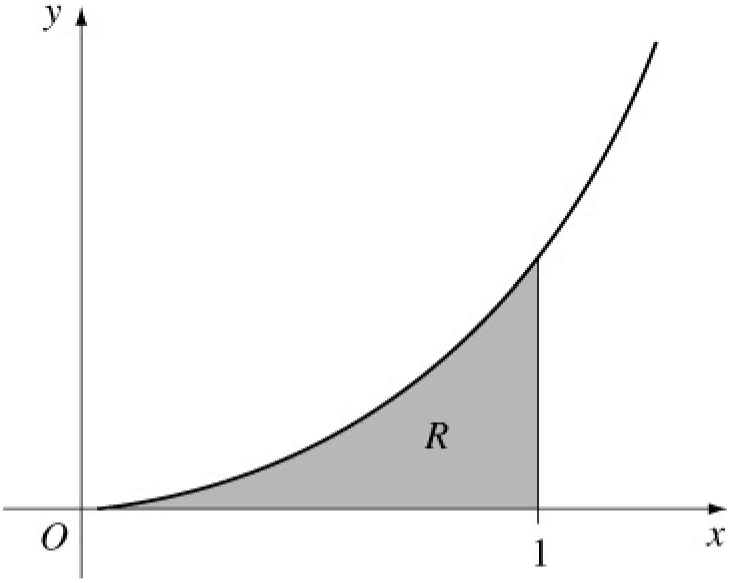
Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 24

Question:

Shown is part of the curve with equation $y = e^{2x} - e^{-x}$. The shaded region R is bounded by the curve, the x-axis and the line with equation $x = 1$.

Use calculus to find the area of R, giving your answer in terms of e. E



Solution:

$$\text{Area} = \int_0^1 \left(e^{2x} - e^{-x} \right) dx$$

$$= \left[\frac{1}{2}e^{2x} + e^{-x} \right]_0^1$$

$$= \left(\frac{1}{2}e^2 + e^{-1} \right) - \left(\frac{1}{2} + 1 \right)$$

$$= \frac{1}{2} \left(e^2 + \frac{2}{e} - 3 \right)$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 25

Question:

(a) Given that $2y = x - \sin x \cos x$, show that $\frac{dy}{dx} = \sin^2 x$.

(b) Hence find $\int \sin^2 x \, dx$.

(c) Hence, using integration by parts, find $\int x \sin^2 x \, dx$. E

Solution:

(a) $2y = x - \sin x \cos x$

$$\Rightarrow \quad 2\frac{dy}{dx} = 1 - \left[\cos^2 x + \sin x \left(- \sin x \right) \right] = 1 - \cos^2 x + \sin^2 x$$

$$\therefore \ \frac{dy}{dx} = \sin^2 x \qquad (\text{using } \sin^2 x = 1 - \cos^2 x)$$

(b) $\int \sin^2 x \, dx = y + C_1$

$$= \frac{x}{2} - \frac{1}{2} \sin x \cos x + C_1$$

(c) $\int x \sin^2 x \, dx$

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sin^2 x \Rightarrow v = \left(\text{b} \right)$$

$$\therefore \ \int x \sin^2 x \, dx = \frac{x^2}{2} - \frac{1}{2} x \sin x \cos x - \int \left(\frac{x}{2} - \frac{1}{2} \sin x \cos x \right) dx$$

$$= \frac{x^2}{2} - \frac{1}{2} x \sin x \cos x - \frac{x^2}{4} + \frac{1}{4} \int \sin 2x \, dx$$

$$= \frac{x^2}{4} - \frac{1}{2} x \sin x \cos x - \frac{1}{8} \cos 2x + C_2$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 26

Question:

The rate, in cm^3s^{-1}, at which oil is leaking from an engine sump at any time t seconds is proportional to the volume of oil, $V\text{cm}^3$, in the sump at that instant. At time $t = 0$, $V = A$.

(a) By forming and integrating a differential equation, show that

$V = A\text{e}^{-kt}$

where k is a positive constant.

(b) Sketch a graph to show the relation between V and t.

Given further that $V = \frac{1}{2}A$ at $t = T$,

(c) show that $kT = \ln 2$. E

Solution:

(a) $\frac{\text{d}v}{\text{d}t} = -k \quad V$

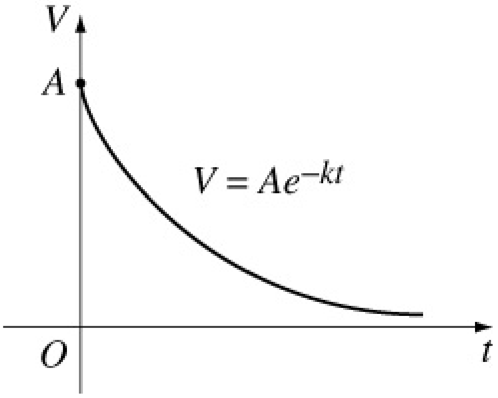
$\Rightarrow \quad \int \frac{1}{V}\text{d}V = \int -k\text{d}t$

$\Rightarrow \quad \ln|V| \quad = -kt + \quad C$

$\Rightarrow \quad V = A_1\text{e}^{-kt}$

$t = 0, V = A \quad \Rightarrow \quad V = A\text{e}^{-kt} \qquad (A_1 = A)$

(b)



(c) $t = T, V = \frac{1}{2}A \quad \Rightarrow \quad \frac{1}{2}A = A\mathrm{e}^{-kT}$

$\Rightarrow \quad -\ln 2 = -kT$

$\Rightarrow \quad kT = \ln 2$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 27

Question:

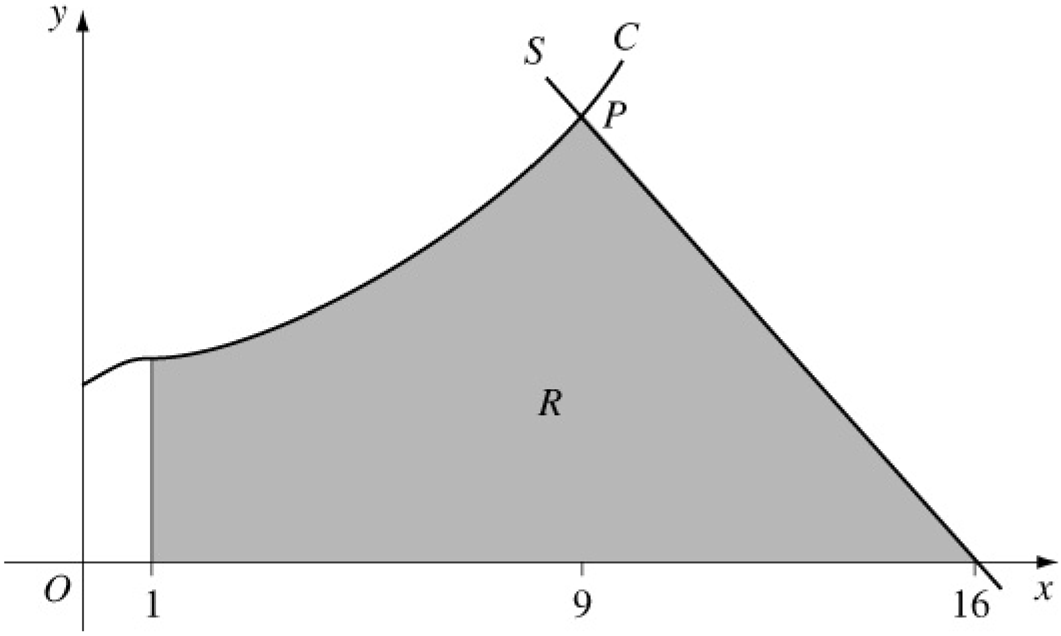
This graph shows part of the curve *C* with parametric equations

$$x = \left(t + 1 \right)^{2}, y = \frac{1}{2}t^3 + 3, t \geq -1.$$

P is the point on the curve where $t = 2$. The line *S* is the normal to *C* at *P*.

(a) Find an equation of *S*.
The shaded region *R* is bounded by *C*, *S*, the *x*-axis and the line with equation $x = 1$.

(b) Using integration and showing all your working, find the area of *R*. E



Solution:

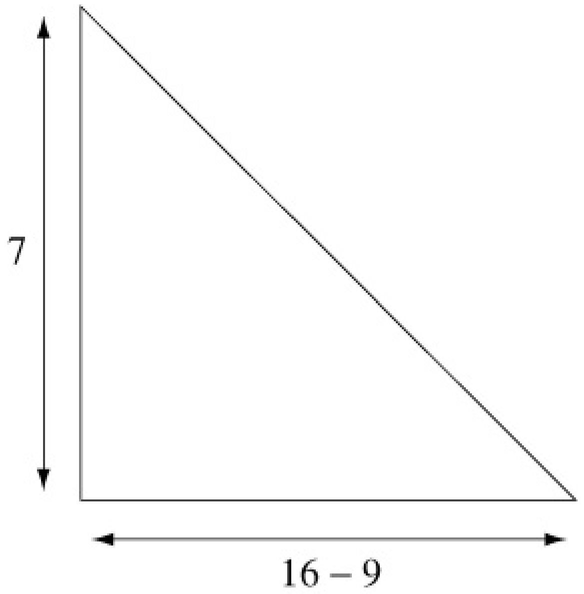
(a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\frac{3}{2}t^2}{2 \left(t + 1 \right)} = \frac{3t^2}{4 \left(t + 1 \right)}$

At *P* (9, 7) gradient of normal is $- \frac{4 \times 3}{3 \times 2^2} = -1$

$\therefore$ equation of line *S* is $y - 7 = -1 \left(x - 9 \right)$

i.e. $y = -x + 16$ or $y + x = 16$

(b) Area $= \int_{x=1}^{x=9} y \, dx +$ area of triangle shown below



$$\int_{x=1}^{x=9} y \, dx = \int_{t=0}^{t=2} \left(\frac{1}{2}t^3 + 3 \right) \quad 2 \left(t + 1 \right) dt$$

$$= \int_0^2 (t^4 + t^3 + 6t + 6) \, dt$$

$$= \left[\frac{1}{5}t^5 + \frac{1}{4}t^4 + \frac{6t^2}{2} + 6t \right]_0^2$$

$$= \left(\frac{32}{5} + \frac{16}{4} + 3 \times 4 + 6 \times 2 \right) - \left(0 \right)$$

$$= 34.4$$

$$\therefore \text{ Area } = 34.4 + \frac{1}{2} \times 7^2 = 58.9$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 28

Question:

Shown is part of the curve C with parametric equations

$x = t^2, y = \sin 2t, t \geq 0.$

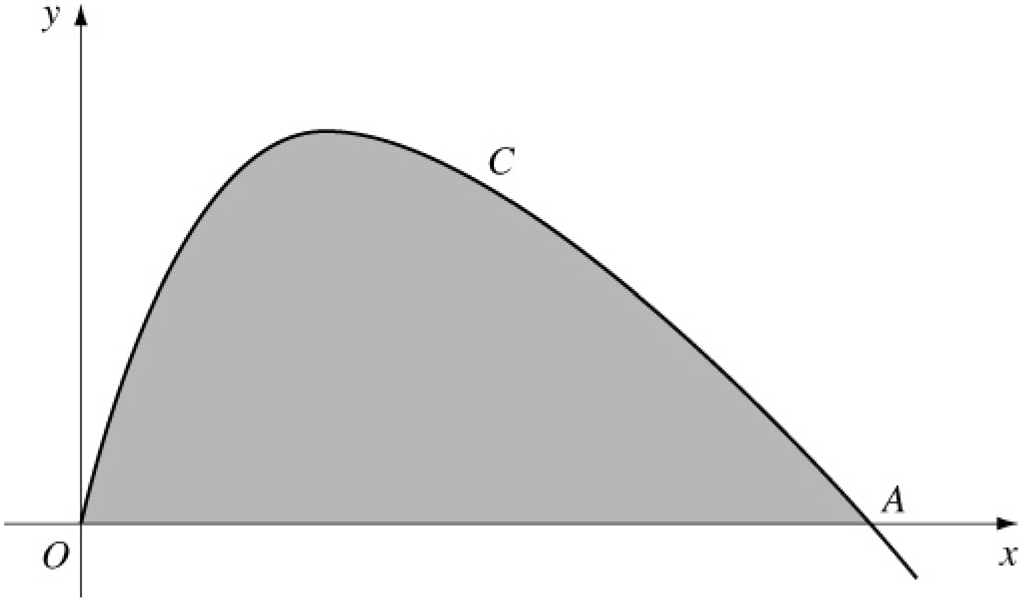
The point A is an intersection of C with the x-axis.

(a) Find, in terms of π, the x-coordinate of A.

(b) Find $\frac{dy}{dx}$ in terms of t, $t > 0$.

(c) Show that an equation of the tangent to C at A is $4x + 2\pi y = \pi^2$.
The shaded region is bounded by C and the x-axis.

(d) Use calculus to find, in terms of π, the area of the shaded region. **E**



Solution:

(a) At A, $y = 0 \quad \Rightarrow \quad \sin 2t = 0 \quad \Rightarrow \quad 2t = 0 \text{ or } \pi \quad \Rightarrow \quad t = \frac{\pi}{2}$

$\therefore A \text{ is } \left(\left(\frac{\pi}{2} \right)^2, 0 \right) \text{ or } \left(\frac{\pi^2}{4}, 0 \right)$

(b) $\frac{dy}{dx} = \frac{2\cos 2t}{2t} = \frac{\cos 2t}{t}$

(c) Gradient of tangent at A is $\frac{\cos\pi}{\left(\frac{\pi}{2}\right)} = -\frac{1}{\left(\frac{\pi}{2}\right)} = -\frac{2}{\pi}$

$\therefore$ equation of tangent is $y - 0 = -\frac{2}{\pi}\left(x - \frac{\pi^2}{4}\right)$

$\Rightarrow \quad \pi y = -2x + \frac{2\pi^2}{4}$

or $2\pi y + 4x = \pi^2$

(d) Area $= \int y\,dx = \int_{t=0}^{t=\frac{\pi}{2}} \sin 2t \times 2t\,dt$

$u = t \quad \Rightarrow \quad \frac{du}{dt} = 1$

$\frac{dv}{dt} = 2\sin 2t \Rightarrow v = -\cos 2t$

$\therefore$ Area $= \left[-t\cos 2t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\left(-\cos 2t\right)dt$

$= \left(+\frac{\pi}{2}\right) - \left(0\right) + \left[\frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 29

Question:

Showing your method clearly in each case, find

(a) $\int \sin^2 x \cos x \, dx$,

(b) $\int x \ln x \, dx$.

Using the substitution $t^2 = x + 1$, where $x > -1$, $t > 0$,

(c) Find $\int \frac{x}{\sqrt{x+1}} dx$.

(d) Hence evaluate $\int_0^3 \frac{x}{\sqrt{x+1}} dx$.

Solution:

(a) Let $y = \sin^3 x \quad \Rightarrow \quad \frac{dy}{dx} = 3 \sin^2 x \cos x$

$$\therefore \int \sin^2 x \cos x \, dx = \frac{1}{3} \sin^3 x + C$$

(b) $\int x \ln x \, dx$

$$u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x \quad \Rightarrow \quad v = \frac{1}{2} x^2$$

$$\therefore \int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \times \frac{1}{x} dx$$

$$= \frac{1}{2} x^2 \ln x - \frac{x^2}{4} + C$$

(c) $t^2 = x + 1 \quad \Rightarrow \quad 2t \, dt = dx$

$$\therefore I = \int \frac{x}{\sqrt{x+1}} dx$$

$$= \int \frac{t^2 - 1}{t} \times 2t \, dt$$

$$= \int (2t^2 - 2) \, dt$$

$$= \frac{2}{3}t^3 - 2t + C$$

$$= \frac{2}{3} (x + 1)^{\frac{3}{2}} - 2\sqrt{x + 1} + C$$

$$= \frac{2}{3}\sqrt{x + 1} \left(x - 2 \right) + C$$

(d) $\int_0^3 \frac{x}{\sqrt{x + 1}} dx = \left[\frac{2}{3} \left(x - 2 \right) \sqrt{x + 1} \right]_0^3$

$$= \left(\frac{2}{3} \times 2 \right) - \left(- \frac{4}{3} \right) = \frac{8}{3}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 30

Question:

(a) Using the substitution $u = 1 + 2x^2$, find $\int x (1 + 2x^2)^5 \mathrm{d}x$.

(b) Given that $y = \frac{\pi}{8}$ at $x = 0$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x (1 + 2x^2)^5 \cos^2 2y.$$

Solution:

(a) $u = 1 + 2x^2 \quad \Rightarrow \quad \mathrm{d}u = 4x\,\mathrm{d}x \quad \Rightarrow \quad x\,\mathrm{d}x = \frac{\mathrm{d}u}{4}$

$$\text{So } \int x (1 + 2x^2)^5 \mathrm{d}x = \int \frac{u^5}{4} \mathrm{d}u = \frac{u^6}{24} + C_1 = \frac{(1 + 2x^2)^6}{24} + C_1$$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = x (1 + 2x^2)^5 \cos^2 2y$

$$\Rightarrow \quad \int \sec^2 2y\,\mathrm{d}y = \int x (1 + 2x^2)^5 \mathrm{d}x$$

$$\Rightarrow \quad \frac{1}{2} \tan 2y = \frac{(1 + 2x^2)^6}{24} + C_2$$

$$y = \frac{\pi}{8}, x = 0 \quad \Rightarrow \quad \frac{1}{2} = \frac{1}{24} + C_2 \quad \Rightarrow \quad C_2 = \frac{11}{24}$$

$$\therefore \tan 2y = \frac{(1 + 2x^2)^6}{12} + \frac{11}{12}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 31

Question:

Find $\int x^2 \ln 2x \, dx$.

Solution:

$$I = \int x^2 \ln 2x \, dx$$

$$u = \ln 2x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^2 \quad \Rightarrow \quad v = \frac{x^3}{3}$$

$$\therefore I = \frac{x^3}{3} \ln 2x - \int \frac{x^3}{3} \times \frac{1}{x} \, dx$$

$$= \frac{x^3}{3} \ln 2x - \int \frac{x^2}{3} dx$$

$$= \frac{x^3}{3} \ln 2x - \frac{x^3}{9} + \quad C$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 32

Question:

Obtain the solution of

$$x\left(x+2\right)\frac{dy}{dx}=y\ ,\ y>0,\ x>0,$$

for which $y=2$ at $x=2$, giving your answer in the form $y^2=f\left(x\right)$.

Solution:

$$x\left(x+2\right)\frac{dy}{dx}=y$$

$$\Rightarrow\quad\int\frac{1}{y}dy=\int\frac{1}{x\left(x+2\right)}dx$$

$$\frac{1}{x\left(x+2\right)}\equiv\frac{A}{x}+\frac{B}{x+2}$$

$$\Rightarrow\quad 1\equiv A\left(x+2\right)+Bx$$

$$x=0\quad\Rightarrow\quad 1=2A\quad\Rightarrow\quad A=\frac{1}{2}$$

$$x=-2\quad\Rightarrow\quad 1=-2B\quad\Rightarrow\quad B=-\frac{1}{2}$$

$$\text{So }\ln y=\int\left(\frac{\left(\frac{1}{2}\right)}{x}-\frac{\left(\frac{1}{2}\right)}{x+2}\right)dx$$

$$=\frac{1}{2}\ln\left|x\right|-\frac{1}{2}\ln\left|x+2\right|+C$$

$$\therefore y=\sqrt{\frac{kx}{x+2}}\qquad\left(C=\frac{1}{2}\ln k\right)$$

$$x=2,y=2\quad\Rightarrow\quad 2=\sqrt{\frac{2k}{4}}\quad\Rightarrow\quad 4\times 2=k$$

$$\therefore y=\sqrt{\frac{8x}{x+2}}\quad\text{or}\quad y^2=\frac{8x}{x+2}$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 33

Question:

(a) Use integration by parts to show that

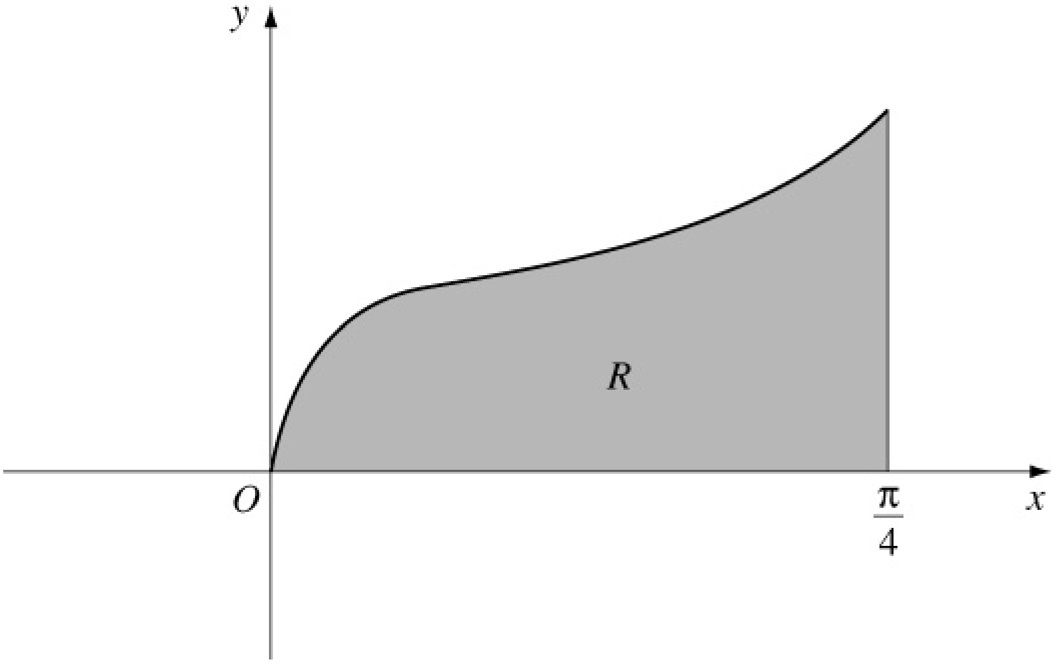
$$\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx = \frac{1}{4}\pi - \frac{1}{2}\ln 2.$$

The finite region R, bounded by the curve with equation $y = x^{\frac{1}{2}}\sec x$, the line $x = \frac{\pi}{4}$ and the x-axis is shown. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid of revolution generated.

(c) Find the gradient of the curve with equation $y = x^{\frac{1}{2}}\sec x$ at the point where $x = \frac{\pi}{4}$.

E



Solution:

(a) $I = \int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$

$$u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x \quad \Rightarrow \quad v = \tan x$$

$$\therefore I = \left[x \tan x \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, \mathrm{d}x$$

$$= \left(\frac{\pi}{4} \right) - \left(0 \right) - \left[\ln | \sec x | \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - \left[\left(\ln \sqrt{2} \right) - \left(\ln 1 \right) \right]$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$

(b) $V = \pi \int_0^{\frac{\pi}{4}} y^2 \, \mathrm{d}x = \pi \int_0^{\frac{\pi}{4}} x \sec^2 x \, \mathrm{d}x$

Using (a) $V = \frac{\pi^2}{4} - \frac{\pi}{2} \ln 2 = 1.38$ (3 s.f.)

(c) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} x^{-\frac{1}{2}} \sec x + x^{\frac{1}{2}} \sec x \tan x$

At $x = \frac{\pi}{4}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} \times \frac{2}{\sqrt{\pi}} \times \sqrt{2} + \frac{\sqrt{\pi}}{2} \times \sqrt{2} \times 1 = \sqrt{\frac{2}{\pi}} + \sqrt{\frac{\pi}{2}} = 2.05$ (3 s.f.)

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 34

Question:

Part of the design of a stained glass window is shown. The two loops enclose an area of blue glass. The remaining area within the rectangle *ABCD* is red glass. The loops are described by the curve with parametric equations

$x = 3\cos t, y = 9\sin 2t, 0 \leq t < 2\pi.$

(a) Find the cartesian equation of the curve in the form $y^2 = f(x)$.

(b) Show that the shaded area enclosed by the curve and the *x*-axis, is given by

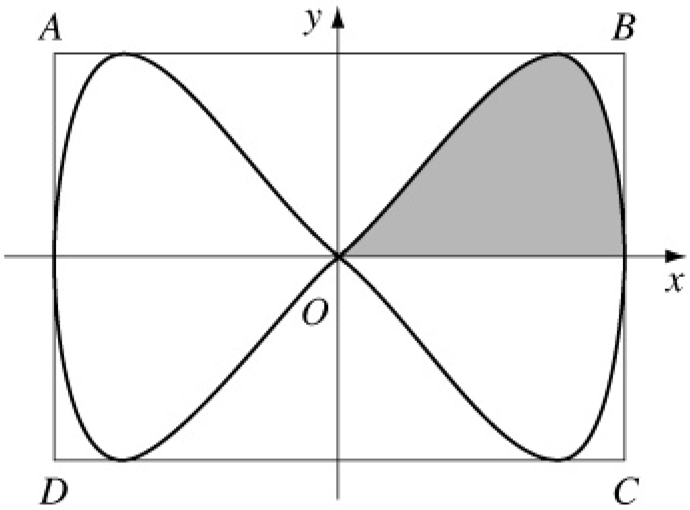
$\int_0^{\frac{\pi}{2}} A\sin 2t\sin t\,dt$, stating the value of the constant *A*.

(c) Find the value of this integral.

The sides of the rectangle *ABCD* are the tangents to the curve that are parallel to the coordinate axes. Given that 1 unit on each axis represents 1 cm,

(d) find the total area of the red glass.

E



Solution:

(a) $x = 3\cos t$

$y = 9\sin 2t \quad \Rightarrow \quad y = 18\cos t\sin t$

$\Rightarrow \quad y = 6x\sin t$

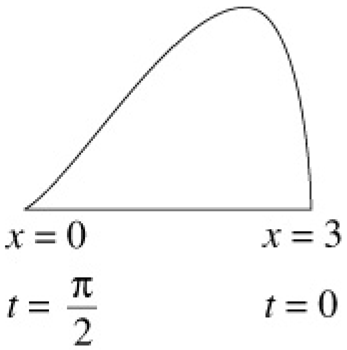
$$\therefore \cos t = \frac{x}{3},\ \sin t = \frac{y}{6x}$$

$$\cos^2 t + \sin^2 t = 1 \quad \Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{36x^2} = 1$$

i.e. $4x^4 + y^2 = 36x^2$

or $y^2 = 4x^2 (9 - x^2)$

(b)



Area $= \int y \, dx$

$$= \int_{t = \frac{\pi}{2}}^{t = 0} 9 \sin 2t \times \left(- 3 \sin t \right) dt$$

$$= 27 \int_0^{\frac{\pi}{2}} \sin 2t \sin t \, dt$$

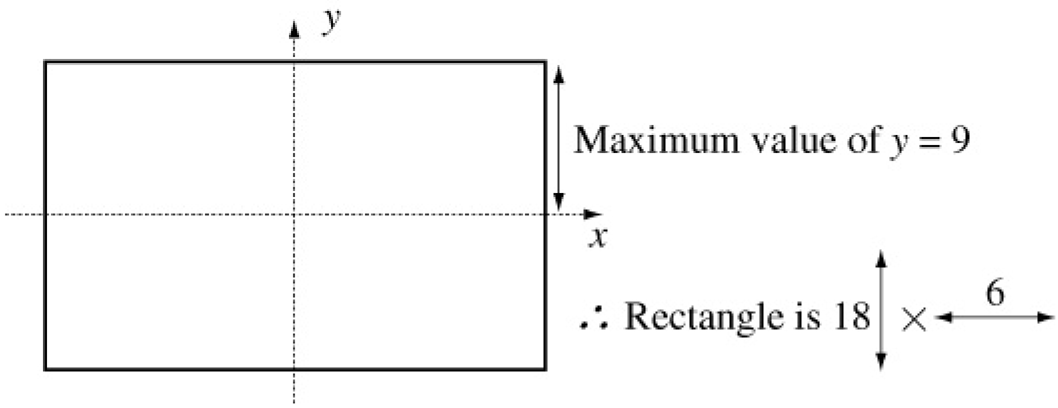
(c) $27 \int_0^{\frac{\pi}{2}} \sin 2t \sin t \, dt = 54 \int_0^{\frac{\pi}{2}} \sin^2 t \cos t \, dt$

$$= \left[\frac{54 \sin^3 t}{3} \right]_0^{\frac{\pi}{2}}$$

$= (18 \times 1) - (0)$

$= 18$

(d) Area of blue glass is 18 × 4 = 72



$$\begin{aligned}\text{Area of rectangle } &= 108\\ \therefore\ \text{Area of red glass } &= 108 - 72 = 36\,\text{cm}^2\end{aligned}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Exam style paper

Exercise A, Question 1

Question:

Use the binomial theorem to expand $\frac{1}{(2+x)^2}$, $|x| < 2$, in ascending powers of x, as far as the term in x^3, giving each coefficient as a simplified fraction. (6)

Solution:

$$(2+x)^{-2} = 2^{-2}\left(1+\frac{x}{2}\right)^{-2}$$

$$= 2^{-2}\left[1+\left(-2\right)\left(\frac{x}{2}\right)+\frac{(-2)(-3)}{1\times 2}\left(\frac{x}{2}\right)^2+\frac{(-2)(-3)(-4)}{1\times 2\times 3}\left(\frac{x}{2}\right)^3+\ldots\right]$$

$$= 2^{-2}\left(1-x+\frac{3}{4}x^2-\frac{1}{2}x^3+\ldots\right)$$

$$= \frac{1}{4}-\frac{x}{4}+\frac{3x^2}{16}-\frac{x^3}{8}+\ldots$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Exam style paper
Exercise A, Question 2

Question:

The curve C has equation
$x^2 + 2y^2 - 4x - 6yx + 3 = 0$
Find the gradient of C at the point (1, 3). (7)

Solution:

$x^2 + 2y^2 - 4x - 6yx + 3 = 0$
Differentiate with respect to x:

$$2x + 4y\frac{dy}{dx} - 4 - \left(6x\frac{dy}{dx} + 6y\right) = 0$$

At the point (1, 3), $x = 1$ and $y = 3$.

$$\therefore 2 + 12\frac{dy}{dx} - 4 - \left(6\frac{dy}{dx} + 18\right) = 0$$

$$\therefore 6\frac{dy}{dx} - 20 = 0$$

$$\therefore \frac{dy}{dx} = \frac{20}{6} = \frac{10}{3}$$

$\therefore$ the gradient of C at (1, 3) is $\frac{10}{3}$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

Exam style paper

Exercise A, Question 3

Question:

Use the substitution $u = 5x + 3$, to find an exact value for

$$\int_0^3 \frac{10x}{(5x+3)^3} dx \qquad (9)$$

Solution:

$$u = 5x + 3$$

$$\therefore \frac{du}{dx} = 5 \text{ and } x = \frac{u-3}{5}$$

$$\therefore \int \frac{10x}{(5x+3)^3} dx = \int \frac{2(u-3)}{u^3} \frac{du}{5}$$

$$= \frac{2}{5} \int \frac{u-3}{u^3} du$$

$$= \frac{2}{5} \int \frac{u}{u^3} - \frac{3}{u^3} du$$

$$= \frac{2}{5} \int u^{-2} - 3u^{-3} \ du$$

$$= \frac{2}{5} \left[-u^{-1} + \frac{3}{2} u^{-2} \right]$$

Change the limits: $x = 0 \Rightarrow u = 3$ and $x = 3 \Rightarrow u = 18$

$$\therefore \text{Integral} = \frac{2}{5} \left[-\frac{1}{18} + \frac{3}{2 \times 18^2} - \left(-\frac{1}{3} + \frac{3}{2 \times 3^2} \right) \right] = \frac{5}{108}$$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Exam style paper
Exercise A, Question 4

Question:

(a) Find the values of A and B for which

$$\frac{1}{(2x+1)(x-2)} \equiv \frac{A}{2x+1} + \frac{B}{x-2} \qquad (3)$$

(b) Hence find $\int \frac{1}{(2x+1)(x-2)} \mathrm{d}x$, giving your answer in the form $y = \ln f(x)$. (4)

(c) Hence, or otherwise, obtain the solution of

$$\left(2x+1\right)\left(x-2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 10y \text{ , } y > 0, x > 2$$

for which $y = 1$ at $x = 3$, giving your answer in the form $y = f(x)$. (5)

Solution:

(a) $\frac{1}{(2x+1)(x-2)} \equiv \frac{A}{(2x+1)} + \frac{B}{(x-2)} \equiv \frac{A(x-2)+B(2x+1)}{(2x+1)(x-2)}$

$\therefore A(x-2) + B(2x+1) \equiv 1$

Substitute $x = 2$, then $5B = 1 \quad \Rightarrow \quad B = \frac{1}{5}$

Substitute $x = -\frac{1}{2}$, then $-\frac{5}{2}A = 1 \quad \Rightarrow \quad A = -\frac{2}{5}$

(b) $\therefore$ Integral $= \int \frac{-\frac{2}{5}}{2x+1} + \frac{\frac{1}{5}}{x-2} \mathrm{d}x$

$$= -\frac{1}{5}\ln\left|2x+1\right| + \frac{1}{5}\ln\left|x-2\right| + C$$

$$= \ln\left[k\left(\frac{|x-2|}{|2x+1|}\right)^{\frac{1}{5}}\right]$$

(c) Separate the variables to give

$$\int \frac{\mathrm{d}y}{y} = \int \frac{10\ \mathrm{d}x}{(2x+1)(x-2)}$$

$\therefore \ln y = 2 \ln | x - 2 | - 2 \ln | 2x + 1 | + C$

$y = 1$ when $x = 3 \quad \Rightarrow \quad C = 2 \ln 7 = \ln 49$

$\therefore y = 49 \left(\frac{| x - 2 |}{| 2x + 1 |} \right) 2$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Exam style paper

Exercise A, Question 5

Question:

A population grows in such a way that the rate of change of the population P at time t in days is proportional to P.

(a) Write down a differential equation relating P and t. (2)

(b) Show, by solving this equation or by differentiation, that the general solution of this equation may be written as $P = Ak^t$, where A and k are positive constants. (5)

Initially the population is 8 million and 7 days later it has grown to 8.5 million.

(c) Find the size of the population after a further 28 days. (5)

Solution:

(a) $\frac{dP}{dt} \propto P$

$\therefore \frac{dP}{dt} = m'P$

(b) $\int \frac{dP}{P} = \int m \, dt$

$\therefore \ln P = mt + C$

$\therefore P = e^{mt + C}$

$= Ae^{mt}$ where $A = e^C$

$= Ak^t$ where $k = e^m$

(c) When $t = 0$, $P = 8$ $\therefore A = 8$

When $t = 7$, $P = 8.5$ $\therefore 8.5 = 8k^7$

$\therefore k^7 = \frac{8.5}{8}$

When $t = 35$,

$P = 8k^{35}$

$= 8 (k^7)^5$

$= 8 \left(\frac{8.5}{8} \right) 5$

= 10.8 million (to 3 s.f.)

Solutionbank

Edexcel AS and A Level Modular Mathematics

Exam style paper
Exercise A, Question 6

Question:

Referred to an origin *O* the points *A* and *B* have position vectors i − 5j − 7k and 10i + 10j + 5k respectively. *P* is a point on the line *AB*.

(a) Find a vector equation for the line passing through *A* and *B*. (3)

(b) Find the position vector of point *P* such that *OP* is perpendicular to *AB*. (5)

(c) Find the area of triangle *OAB*. (4)

(d) Find the ratio in which *P* divides the line *AB*. (2)

Solution:

(a) *AB* = 9i + 15j + 12k (or *BA* = − 9i − 15j − 12k)

∴ the line may be written

$$r = \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 9 \\ 15 \\ 12 \end{pmatrix} \quad \text{or} \quad r = \begin{pmatrix} 10 \\ 10 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} \quad \text{or equivalent}$$

(b) $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} . \begin{pmatrix} +1+9\lambda \\ -5+15\lambda \\ -7+12\lambda \end{pmatrix} = 0$

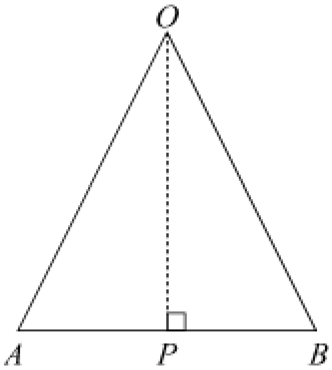
∴ $+3+27\lambda - 25 + 75\lambda - 28 + 48\lambda = 0$

∴ $150\lambda - 50 = 0$

∴ $\lambda = \frac{1}{3}$

∴ the point *P* has position vector $\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

(c) $| OP | = 5$ and $| AB | = \sqrt{9^2 + 15^2 + 12^2} = 15\sqrt{2}$



Area of △ *OAB* = $\frac{1}{2}$ × base × height = $\frac{1}{2}$ × 15 √ 2 × 5 = $\frac{1}{2}$ × 75 √ 2

(d) $AP = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ and $PB = \begin{pmatrix} 10 \\ 10 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ 10 \\ 8 \end{pmatrix}$

∴ *PB* = 2*AP*

i.e. *P* divides *AB* in the ratio 1 : 2.

Solutionbank
Edexcel AS and A Level Modular Mathematics

Exam style paper
Exercise A, Question 7

Question:

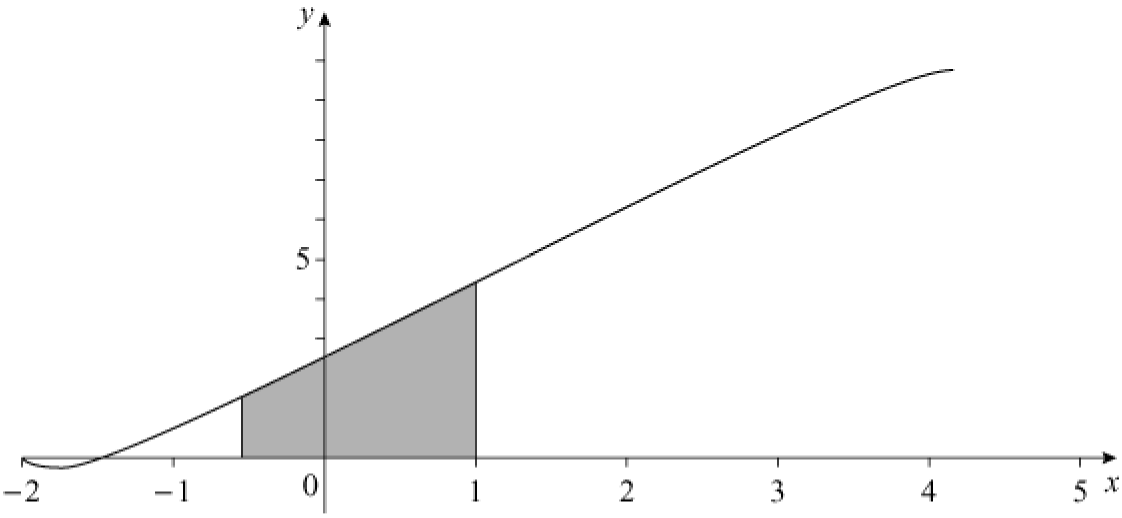
The curve C, shown has parametric equations
$x = 1 - 3 \cos t, y = 3t - 2 \sin 2t, 0 < t < \pi.$

(a) Find the gradient of the curve at the point P where $t = \frac{\pi}{6}$. (4)

(b) Show that the area of the finite region beneath the curve, between the lines $x = -\frac{1}{2}$, $x = 1$ and the x-axis, shown shaded in the diagram, is given by the integral

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 9t \sin t \, dt - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 12 \sin^2 t \cos t \, dt.$$ (4)

(c) Hence, by integration, find an exact value for this area. (7)



Solution:

(a) $x = 1 - 3 \cos t, y = 3t - 2 \sin 2t$

$\frac{dx}{dt} = 3 \sin t$ and $\frac{dy}{dt} = 3 - 4 \cos 2t$

$\therefore \frac{dy}{dx} = \frac{3 - 4 \cos 2t}{3 \sin t}$

When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{3-2}{\left(\frac{3}{2}\right)} = \frac{2}{3}$

(b) The area shown is given by $\int_{t_1}^{t_2} 2y\frac{dx}{dt}dt$

Where t_1 is value of parameter when $x = -\frac{1}{2}$

and t_2 is value of parameter when $x = 1$

i.e. $1 - 3\cos t_1 = -\frac{1}{2}$

$\therefore \cos t_1 = \frac{1}{2}$

$\therefore t_1 = \frac{\pi}{3}$

Also $1 - 3\cos t_2 = 1$

$\therefore \cos t_2 = 0$

$\therefore t_2 = \frac{\pi}{2}$

The area is given by

$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(3t - 2\sin 2t\right) \times 3\sin t\, dt$

$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 9t\sin t\, dt - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 6 \times 2\sin t\cos t\sin t\, dt$ Using the double angle formula

$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 9t\sin t\, dt - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 12\sin^2 t\cos t\, dt$

(c) Area $= \left[-9t\cos t\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 9\cos t\, dt - \left[4\sin^3 t\right]_{\frac{\pi}{2}}^{\frac{\pi}{3}}$

$= \left[-9t\cos t + 9\sin t - 4\sin^3 t\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \left(9 - 4\right) - \left(-\frac{3\pi}{2} + \frac{9\sqrt{3}}{2} - 4 \times \frac{3\sqrt{3}}{8}\right)$

$= 5 - 3\sqrt{3} + \frac{3\pi}{2}$

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 1

Question:

Express $\frac{2x-1}{(x-1)(2x-3)}$ in partial fractions. ***E***

Solution:

$$\frac{2x-1}{(x-1)(2x-3)} \equiv \frac{A}{x-1}+\frac{B}{2x-3}$$

Set $\frac{2x-1}{(x-1)(2x-3)}$ identical to $\frac{A}{x-1}+\frac{B}{2x-3}$.

$$\equiv \frac{A(2x-3)+B(x-1)}{(x-1)(2x-3)}$$

Use a common denominator and add the two fractions.

Compare numerators of fractions

$2x-1 \equiv A(2x-3)+B(x-1)*$

Because the fractions are equivalent, the numerators are also.

Put $x=1$ in equation *

$\therefore 1=-A+0 \Rightarrow A=-1$

To find A, substitute $x=1$.

Put $x=1\frac{1}{2}$ in equation *

$\therefore 2=0+\frac{1}{2}B \Rightarrow B=4$

To find B, substitute $x=1\frac{1}{2}$.

So $\frac{2x-1}{(x-1)(2x-3)} \equiv \frac{-1}{x-1}+\frac{4}{2x-3}$

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 2

Question:

It is given that $f(x) = \dfrac{3x+7}{(x+1)(x+2)(x+3)}$.

Express f(x) as the sum of three partial fractions. *E*

Solution:

$$\frac{3x+7}{(x+1)(x+2)(x+3)}$$

$$\equiv \frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3}$$

$$\equiv \frac{A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)}{(x+1)(x+2)(x+3)}$$

Set $\dfrac{3x+7}{(x+1)(x+2)(x+3)}$ identical to $\dfrac{A}{x+1}+\dfrac{B}{x+2}+\dfrac{C}{x+3}$.

Use a common denominator and add the three fractions.

Compare numerators of fractions

$$3x+7 \equiv A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)$$

Now put the numerators equal to each other.

Put $x=-2$ in equation
$1=0-B+0 \Rightarrow B=-1$

To find B, substitute $x=-2$.

Put $x=-3$ in equation
$-2=0+0+2C \Rightarrow C=-1$

To find C, substitute $x=-3$.

Put $x=-1$ in equation
$4=2A \Rightarrow A=2$

To find A, substitute $x=-1$.

So $\dfrac{3x+7}{(x+1)(x+2)(x+3)} \equiv \dfrac{2}{(x+1)}-\dfrac{1}{(x+2)}-\dfrac{1}{(x+3)}$

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 3

Question:

Given that $f(x)=\frac{2}{(2-x)(1+x)^2}$, express f($x$) in the form

$\frac{A}{(2-x)}+\frac{B}{(1+x)}+\frac{C}{(1+x)^2}$. ***E***

Solution:

$$\frac{2}{(2-x)(1+x)^2} \equiv \frac{A}{2-x}+\frac{B}{(1+x)}+\frac{C}{(1+x)^2}$$

You need denominators of $(2-x),(1+x)$ and $(1+x)^2$.

$$\equiv \frac{A(1+x)^2+B(2-x)(1+x)+C(2-x)}{(2-x)(1+x)^2}$$

Add the three fractions.

Compare numerators of fractions

$$2 \equiv A(1+x)^2+B(2-x)(1+x)+C(2-x)$$

Set the numerators equal.

Put $x=2$

$$2 = A\times 9+0+0$$

To find A substitute $x=2$.

$$\therefore A = \frac{2}{9}$$

Put $x=-1$

$$2 = 0+0+3C$$

$$\therefore C = \frac{2}{3}$$

To find C substitute $x=-1$.

$$\therefore 2 \equiv \frac{2}{9}(1+x)^2+B(2-x)(1+x)+\frac{2}{3}(2-x)$$

$$2 \equiv \frac{2}{9}+\frac{4}{9}x+\frac{2}{9}x^2+2B+Bx-Bx^2$$
$$+\frac{4}{3}-\frac{2}{3}x$$

Equate terms in x^2 on both sides

$$0=\frac{2}{9}x^2-Bx^2 \qquad \therefore B=\frac{2}{9}$$

Equate terms in x^2 to find B.

$$\therefore \frac{2}{(2-x)(1+x)^2}=\frac{2}{9(2-x)}+\frac{2}{9(1+x)}+\frac{2}{3(1+x)^2}$$

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 4

Question:

$$\frac{14x^2+13x+2}{(x+1)(2x+1)^2} \equiv$$

$$\frac{A}{x+1}+\frac{B}{2x+1}+\frac{C}{(2x+1)^2}.$$

Find the values of the constants A, B and C. ***E***

Solution:

$$\frac{14x^2+13x+2}{(x+1)(2x+1)^2} = \frac{A}{x+1}+\frac{B}{2x+1}+\frac{C}{(2x+1)^2}$$

You need denominators of $(x+1), (2x+1)$ and $(2x+1)^2$.

$$= \frac{A(2x+1)^2+B(x+1)(2x+1)+C(x+1)}{(x+1)(2x+1)^2}$$

Add the three fractions.

Compare numerators of fractions

$$14x^2+13x+2 = A(2x+1)^2+B(x+1)(2x+1)+C(x+1)$$

Set the numerators equal.

Put $x=-1$

$$\therefore 3 = A+0+0 \Rightarrow A=3$$

To find A set $x=-1$.

Put $x=-\frac{1}{2}$

$$\therefore \frac{14}{4}-\frac{13}{2}+2=\frac{1}{2}C \Rightarrow C=-2$$

To find C set $x=-\frac{1}{2}$.

$$\therefore 14x^2+13x+2 = 3(2x+1)^2+B(x+1)(2x+1)-2(x+1)$$

Compare coefficients of x^2:

Equate terms in x^2.
$14x^2 = 3.2^2x^2+2Bx^2$

$$14 = 12+2B \Rightarrow B=1$$

Solve equation to find B.

Check constant term

$$2 = 3+1-2$$

$$\therefore \frac{14x^2+13x+2}{(x+1)(2x+1)^2} = \frac{3}{x+1}+\frac{1}{2x+1}-\frac{2}{(2x+1)^2}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 5

Question:

$$f(x)=\frac{x^2+6x+7}{(x+2)(x+3)}, x\in\mathbb{R}.$$

Given that $f(x)=A+\frac{B}{(x+2)}+\frac{C}{(x+3)}$ find the values of A, B and C. *E*

Solution:

$$f(x)=\frac{x^2+6x+7}{(x+2)(x+3)}=\frac{x^2+6x+7}{x^2+5x+6}$$

This is an improper fraction so multiply out the denominator.

$$\begin{array}{r}
1\text{ rem}(x+1)\\
x^2+5x+6\,\big)\,\overline{x^2+6x+7}\\
\underline{x^2+5x+6}\\
x+1
\end{array}$$

Then divide the denominator into the numerator.

It goes in 1 time with a remainder of $(x+1)$.

$$\therefore f(x) = 1+\frac{x+1}{(x+2)(x+3)}$$

Write f(x) as a mixed fraction.

$$= 1+\frac{B}{x+2}+\frac{C}{x+3}$$

The denominators must be $(x+2)$ and $(x+3)$.

$$= 1+\frac{B(x+3)+C(x+2)}{(x+2)(x+3)}$$

Add the two fractions.

$$\therefore A = 1$$

Compare numerators of remainder term

Set the numerators equal.

$x+1=B(x+3)+C(x+2)$

Put $x=-2$

$-1=B\Rightarrow B=-1$

Substitute $x=-2$ to find B.

Put $x=-3$

$-2 = -C\Rightarrow C=2$

Substitute $x=-3$ to find C.

$$\therefore f(x) = 1-\frac{1}{(x+2)}+\frac{2}{(x+3)}$$

Write out the full solution.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 6

Question:

Given that $\mathrm{f}(x)=\frac{11-5x^2}{(x+1)(2-x)}$, find constants A and B such that

$\mathrm{f}(x)=5+\frac{A}{(x+1)}+\frac{B}{(2-x)}$. **E**

Solution:

6

$$\mathrm{f}(x) = \frac{11-5x^2}{(x+1)(2-x)}$$

This is an improper fraction so multiply out the denominator.

$$= \frac{11-5x^2}{2+x-x^2}$$

$$= \frac{10+5x-5x^2}{2+x-x^2}+\frac{1-5x}{2+x-x^2}$$

Either divide denominator into numerator to obtain 5 with $(1-5x)$ as remainder or split numerator, as shown.

$$= 5+\frac{A}{(x+1)}+\frac{B}{(2-x)}$$

where

$$\frac{1-5x}{(x+1)(2-x)} \equiv \frac{A}{x+1}+\frac{B}{2-x}$$

Use partial fractions with denominators $(x+1)$ and $(2-x)$.

$$\equiv \frac{A(2-x)+B(x+1)}{(x+1)(2-x)}$$

Add the two fractions.

$\therefore 1-5x=A(2-x)+B(x+1)$

Set the numerators equal.

Put $x=2$

$-9=3B \Rightarrow B=-3$

Substitute $x=2$ to find B.

Put $x=-1$

$6=3A \Rightarrow A=2$

Substitute $x=1$ to find B.

$\therefore \mathrm{f}(x)=5+\frac{2}{x+1}-\frac{3}{2-x}$

Write out full solution.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 7

Question:

$$\mathrm{f}(x)=\frac{9-3x-12x^2}{(1-x)(1+2x)}.$$

Given that $\mathrm{f}(x)=A+\frac{B}{(1-x)}+\frac{C}{(1+2x)}$, find the values of the constants A, B and C. ***E***

Solution:

$$\mathrm{f}(x) = \frac{9-3x-12x^2}{(1-x)(1+2x)}$$

Multiply out the denominator.

$$= \frac{9-3x-12x^2}{1+x-2x^2}$$

Numerator is split into two parts. The first is a multiple of the denominator. The second is the remainder.

$$= \frac{6+6x-12x^2}{1+x-2x^2}+\frac{3-9x}{1+x-2x^2}$$

This could be done by long division to give $6+\frac{3-9x}{1+x-2x^2}$.

$$= 6+\frac{B}{1-x}+\frac{C}{1+2x}$$

where

$$\frac{3-9x}{(1-x)(1+2x)} = \frac{B}{1-x}+\frac{C}{1+2x}$$

Use partial fractions with denominators $(1-x)$ and $(1+2x)$.

$$= \frac{B(1+2x)+C(1-x)}{(1-x)(1+2x)}$$

Add the two fractions.

$$\therefore 3-9x \equiv B(1+2x)+C(1-x)$$

Set the numerators equal.

Put $x=-\frac{1}{2}$

$$\therefore 7\frac{1}{2}=1\frac{1}{2}C \Rightarrow C=5$$

Substitute $x=-\frac{1}{2}$ to find C.

Put $x=1$

$$\therefore -6=3B \Rightarrow B=-2.$$

Substitute $x=-1$ to find A.

So

$$\mathrm{f}(x)=6-\frac{2}{1-x}+\frac{5}{1+2x}$$

Write out the full solution.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 8

Question:

Use the Binomial theorem to expand $\sqrt{(4-9x)}, |x| < \frac{4}{9}$, in ascending powers of x, as far as the term in x^3, simplifying each term. *E*

Solution:

$$\sqrt{(4-9x)} = (4-9x)^{\frac{1}{2}}$$

Write in index form.

$$= \left[4(1-\frac{9}{4}x)\right]^{\frac{1}{2}}$$

Take out a factor of 4.

$$= 4^{\frac{1}{2}}\left(1-\frac{9}{4}x\right)^{\frac{1}{2}}$$

Write $4^{\frac{1}{2}}$ as $\sqrt{4} = 2$.

$$= 2\left(1-\frac{9}{4}x\right)^{\frac{1}{2}}$$

Expand $\left(1-\frac{9x}{4}\right)^{\frac{1}{2}}$ using the binomial expansion with $n = \frac{1}{2}$ and $x = \frac{-9x}{4}$.

$$= 2\left[1-\frac{9}{8}x+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-9x}{4}\right)^2}{2!}+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-9}{4}x\right)^3}{3!}+\cdots\right]$$

$$= 2\left[1-\frac{9}{8}x-\frac{81}{128}x^2-\frac{729}{1024}x^3\cdots\right]$$

Simplify coefficients.

$$= 2-\frac{9}{4}x-\frac{81}{64}x^2-\frac{729}{512}x^3\cdots$$

Multiply by the 2.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 9

Question:

$\mathrm{f}(x) = (2-5x)^{-2}, |x| < \frac{2}{5}$.

Find the binomial expansion of f(x), in ascending powers of x, as far as the term in x^3, giving each coefficient as a simplified fraction. *E*

Solution:

$$\begin{aligned}
\mathrm{f}(x) &= (2-5x)^{-2} \\
&= \left[2\left(1-\frac{5}{2}x\right)\right]^{-2}
\end{aligned}$$

Take out a factor of 2.

$$\begin{aligned}
&= 2^{-2}\left(1-\frac{5}{2}x\right)^{-2} \\
&= \frac{1}{4}\left(1-\frac{5}{2}x\right)^{-2}
\end{aligned}$$

Write 2^{-2} as $\frac{1}{4}$.

$$\begin{aligned}
&= \frac{1}{4}\left[1+(-2)\left(\frac{-5x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{-5x}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{-5x}{2}\right)^3+\right] \\
&= \frac{1}{4}\left[1+5x+\frac{75x^2}{4}+\frac{125x^3}{2}+\cdots\right] \\
&= \frac{1}{4}+\frac{5}{4}x+\frac{75}{16}x^2+\frac{125}{8}x^3+\cdots
\end{aligned}$$

Expand $\left(1-\frac{5}{2}x\right)^{-2}$ using binomial expansion with $n=-2$ and $x=\frac{-5}{2}x$.

Simplify the coefficients.

Multiply by $\frac{1}{4}$.

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Review Exercise

Exercise A, Question 10

Question:

$f(x) = (3+2x)^{-3}, |x| < \frac{3}{2}$.

Find the binomial expansion of $f(x)$, in ascending powers of x, as far as the term in x^3. Give each coefficient as a simplified fraction. *E*

Solution:

$$f(x) = (3+2x)^{-3}$$

$$\left[3\left(1+\frac{2x}{3}\right)\right]^{-3}$$

Take out a factor of 3.

$$= 3^{-3}\left(1+\frac{2x}{3}\right)^{-3}$$

Write 3^{-3} as $\frac{1}{27}$.

$$= \frac{1}{27}\left(1+(-3)\left(\frac{2x}{3}\right)+\frac{(-3)(-4)}{2!}\left(\frac{2x}{3}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(\frac{2x}{3}\right)^3+\cdots\right)$$

Expand $\left(1+\frac{2x}{3}\right)^{-3}$ using the binomial expansion with $n=-3$ and $x=\frac{2x}{3}$.

$$= \frac{1}{27}\left(1-2x+\frac{48x^2}{18}-\frac{480x^3}{162}+\cdots\right)$$

Simplify coefficients.

$$= \frac{1}{27}-\frac{2x}{27}+\frac{8x^2}{81}-\frac{80x^3}{729}+$$

Multiply by $\frac{1}{27}$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 11

Question:

$$\mathrm{f}(x)=\frac{1}{\sqrt{(1-x)}}-\sqrt{(1+x)},-1<x<1$$

Find the series expansion of $\mathrm{f}(x)$ in ascending powers of x up to and including the term in x^3. ***E***

Solution:

$$\mathrm{f}(x) = \frac{1}{\sqrt{(1-x)}}-\sqrt{(1+x)}$$

$$= (1-x)^{-\frac{1}{2}}-(1+x)^{\frac{1}{2}}$$

Write each expression in index form.

$$= \left[1+\left(\frac{-1}{2}\right)(-x)+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(-x)^2}{2}\right.$$

$$\left.+\ \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(-x)^3}{3!}\cdots\right]$$

Replace n by $\frac{1}{2}$ and x by $-x$ in binomial expansion.

$$-\ \left[1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)x^2}{2}+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^3}{3!}+\right]$$

Replace n by $-\frac{1}{2}$ in binomial expansion.

$$= \left[1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5x^3}{16}+\cdots\right]$$

$$-\left[1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots\right]$$

Simplify the terms in both series expansions.

$$= \frac{4}{8}x^2+\frac{4}{16}x^3+\cdots$$

Collect terms.

$$= \frac{1}{2}x^2+\frac{1}{4}x^3+\cdots$$

Simplify answer.

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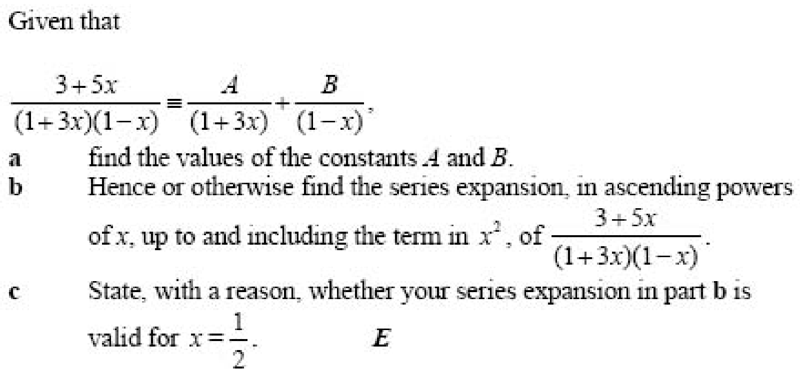
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 12

Question:



Given that

$$\frac{3+5x}{(1+3x)(1-x)} \equiv \frac{A}{(1+3x)} + \frac{B}{(1-x)},$$

a find the values of the constants A and B.

b Hence or otherwise find the series expansion, in ascending powers of x, up to and including the term in x^2, of $\frac{3+5x}{(1+3x)(1-x)}$.

c State, with a reason, whether your series expansion in part **b** is valid for $x=\frac{1}{2}$. ***E***

Solution:

a

$$\frac{3+5x}{(1+3x)(1-x)} \equiv \frac{A}{1+3x}+\frac{B}{1-x}$$

The denominators must be $(1+3x)$ and $(1-x)$.

$$\equiv \frac{A(1-x)+B(1+3x)}{(1+3x)(1-x)}$$

Add the fractions.

$$\therefore 3+5x \equiv A(1-x)+B(1+3x)$$

Set the numerators equal.

Put $x=1$

$$\therefore 8=4B \Rightarrow B=2$$

Set $x=1$ to find B.

Put $x=-\frac{1}{3}$

$$\therefore 3-\frac{5}{3}=\frac{4}{3}A \Rightarrow A=1$$

Set $x=\frac{-1}{3}$ to find A.

b

$$\therefore \frac{3+5x}{(1+3x)(1-x)} = \frac{1}{1+3x}+\frac{2}{1-x}$$

$$= (1+3x)^{-1}+2(1-x)^{-1}$$

Write in index form.

Expand using binomial theorem:

$$= \left[1+(-1)(3x)+\frac{(-1)(-2)}{1\times 2}(3x)^2+\cdots\right]$$

Expand $(1+3x)^{-1}$ using the binomial expansion with $n=-1$ and $x=3x$.

$$+\ 2\left[1+(-1)(-x)+\frac{(-1)(-2)(-x)^2}{1\times 2}+\cdots\right]$$

Expand $2(1-x)^{-1}$ using the binomial expansion with $n=-1$ and $x=(-x)$.

$$= \left[1-3x+9x^2\quad+\cdots\right]+2\left(1+x+x^2+\right)$$

Simplify each expression.

$$= 3-x+11x^2-\cdots$$

Collect the terms.

c Not valid when $x=\frac{1}{2}$, as expansion of $(1+3x)^{-1}$ is valid for $|3x|<1$ only.

Terms are $(3x),(3x)^2\cdots$ and when $x=\frac{1}{2}, 3x>1$ and the terms get larger.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 13

Question:

$\mathrm{f}(x)=\frac{3x-1}{(1-2x)^2}, |x|<\frac{1}{2}$.

Given that, for $x \neq \frac{1}{2}, \frac{3x-1}{(1-2x)^2} \equiv \frac{A}{(1-2x)}+\frac{B}{(1-2x)^2}$, where A and B are constants,

a find the values of A and B.

b Hence or otherwise find the series expansion of f(x), in ascending powers of x, up to and including the term in x^3, simplifying each term. ***E***

Solution:

a

$$\frac{3x-1}{(1-2x)^2} \equiv \frac{A}{(1-2x)}+\frac{B}{(1-2x)^2}$$

$$\equiv \frac{A(1-2x)+B}{(1-2x)^2}$$

Add the fractions.

$$\therefore 3x-1 \equiv A(1-2x)+B$$

Set the numerators equal.

Put $x=\frac{1}{2}$

$$\frac{1}{2} = B$$

Set $x=\frac{1}{2}$ to find B.

$$\therefore 3x-1 \equiv A(1-2x)+\frac{1}{2}$$

Compare coefficients of x

$$3=-2A \Rightarrow A=-\frac{3}{2}$$

As expressions are identical equate terms in x and put coefficients equal.

[check constant term $-1=-\frac{3}{2}+\frac{1}{2}$]

$$\therefore \frac{3x-1}{(1-2x)^2}=-\frac{3}{2}(1-2x)^{-1}+\frac{1}{2}(1-2x)^{-2}$$

Write in index form.

b Use binomial expansions:

$$= -\frac{3}{2}\left[1+(-1)(-2x)+\frac{(-1)(-2)}{2!}(-2x)^2+\frac{(-1)(-2)(-3)(-2x)^3}{3!}+\cdots\right]$$

$$+\frac{1}{2}\left[1+(-2)(-2x)+\frac{(-2)(-3)}{2!}(-2x)^2+\frac{(-2)(-3)(-4)(-2x)^3}{3!}+\cdots\right]$$

Expand $-\frac{3}{2}(1-2x)^{-1}$ using the binomial expansion with $n=-1$ and $x=-2x$.

Expand $\frac{1}{2}(1-2x)^{-2}$ using the binomial expansion with $n=-2$ and $x=-2x$.

$$= -\frac{3}{2}\left[1+2x+4x^2+8x^3+\cdots\right]+\frac{1}{2}\left[1+4x+12x^2+32x^3+\cdots\right]$$

Simplify each expression.

$$= -1-x+0x^2+4x^3+\cdots$$

Collect the terms.

$$= -1-x+4x^3+\cdots$$

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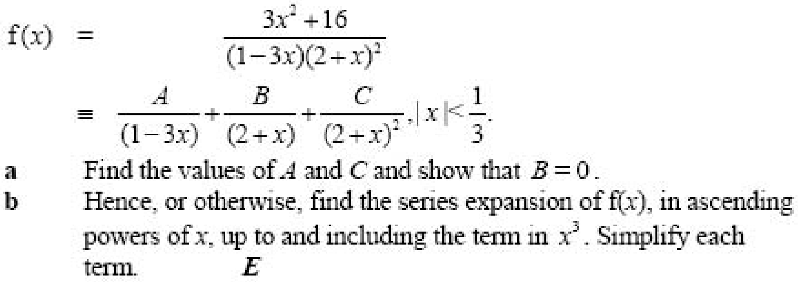
Review Exercise

Exercise A, Question 14

Question:



$$\text{f}(x) = \frac{3x^2+16}{(1-3x)(2+x)^2}$$

$$\equiv \frac{A}{(1-3x)} + \frac{B}{(2+x)} + \frac{C}{(2+x)^2}, |x| < \frac{1}{3}.$$

a Find the values of A and C and show that $B = 0$.

b Hence, or otherwise, find the series expansion of f(x), in ascending powers of x, up to and including the term in x^3. Simplify each term. ***E***

Solution:

a

$$\frac{3x^2+16}{(1-3x)(2+x)^2} \equiv \frac{A}{(1-3x)}+\frac{B}{(2+x)}+\frac{C}{(2+x)^2}$$

$$\equiv \frac{A(2+x)^2+B(1-3x)(2+x)+C(1-3x)}{(1-3x)(2+x)^2}$$

Add the fractions.

$$\therefore 3x^2+16 \equiv A(2+x)^2+B(1-3x)(2+x)+C(1-3x)$$

Set the numerators equal.

Put $x=-2$

$28=7C \Rightarrow C=4$

Set $x=-2$ to find C.

Put $x=\frac{1}{3}$

$16\frac{1}{3}=\frac{49}{9}A \Rightarrow A=3$

Set $x=\frac{1}{3}$ to find A.

$\therefore 3x^2+16 \equiv 3(2+x)^2+B(1-3x)(2+x)+4(1-3x)$

Compare x^2 terms.

$3=3-3B \Rightarrow B=0.$

Compare constants.

$16=12+2B+4 \Rightarrow B=0$

Equate coefficients of terms in x^2
or
equate constant terms to find B.

b

$$\frac{3x^2+16}{(1-3x)(2+x)^2} \equiv \frac{3}{(1-3x)}+\frac{4}{(2+x)^2}$$

$$= 3(1-3x)^{-1}+4(2+x)^{-2}$$

Write in index form.

$$= 3(1-3x)^{-1}+4\times 2^{-2}\left(1+\frac{x}{2}\right)^{-2}$$

Take out a factor of 2 so $(2+x)^{-2}=\left[2\left(1+\frac{x}{2}\right)\right]^{-2}$.

$$= 3\left[1+(-1)(-3x)+\frac{(-1)(-2)(-3x)^2}{2!}+\frac{(-1)(-2)(-3)(-3x)^3}{3!}+\right]$$

$$+ \frac{4}{4}\left[1+(-2)\left(\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$

Expand $3(1-3x)^{-1}$ using the binomial expansion with $n=-1$ and $x=-3x$.

$$= 3\left[1+3x+9x^2+27x^3+\cdots\right]+\left[1-x+\frac{3x^2}{4}\frac{-4x^3}{8}+\cdots\right]$$

Expand $4\times 2^{-2}\left(1+\frac{x}{2}\right)^{-2}$ using binomial expansion with $n=-2$ and $x=\frac{x}{2}$.

$$= 4+8x+\frac{111x^2}{4}+\frac{161}{2}x^3+\cdots$$

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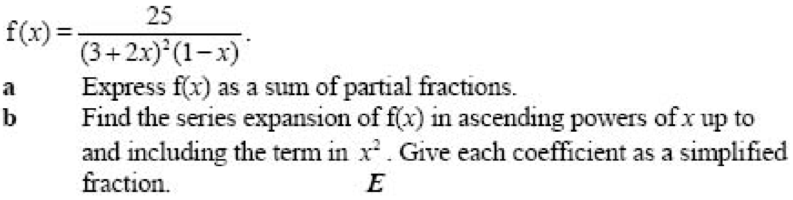
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 15

Question:



$$f(x) = \frac{25}{(3+2x)^2(1-x)}.$$

a Express $f(x)$ as a sum of partial fractions.

b Find the series expansion of $f(x)$ in ascending powers of x up to and including the term in x^2. Give each coefficient as a simplified fraction. *E*

Solution:

a

$$\frac{25}{(3+2x)^2(1-x)} \equiv \frac{A}{(1-x)}+\frac{B}{(3+2x)}+\frac{C}{(3+2x)^2}$$

The denominators must be $(1-x), (3+2x)$ and $(3+2x)^2$.

$$\equiv \frac{A(3+2x)^2+B(1-x)(3+2x)+C(1-x)}{(1-x)(3+2x)^2}$$

Add the fractions.

$$\therefore 25 \equiv A(3+2x)^2+B(1-x)(3+2x)+C(1-x) *$$

Set the numerators equal.

Put $x=1$ in *

$$\therefore 25 = A\times 5^2+0+0$$

$$\therefore A = 1$$

Set $x=1$ to find A.

Put $x=-\frac{3}{2}$ in *

$$\therefore 25 = C\times\left(1-(-\frac{3}{2})\right)=C\times 2\frac{1}{2}$$

$$\therefore C = 10$$

Set $x=-\frac{3}{2}$ to find C.

Compare coefficients of x^2 in *

$0=4A-2B$

As $A=1, B=2$

Equate coefficients of terms in x^2 to find B.

$$\therefore \frac{25}{(3+2x)^2(1-x)}\equiv\frac{1}{(1-x)}+\frac{2}{(3+2x)}+\frac{10}{(3+2x)^2}$$

b

$$\text{RHS} = (1-x)^{-1}+2(3+2x)^{-1}+10(3+2x)^{-2}$$

Write the right hand side in index form.

$$(1-x)^{-1} = 1+(-1)(-x)+\frac{(-1)(-2)(-x)^2}{2!}+\cdots$$

$$= 1+x+x^2+\cdots$$

Expand $(1-x)^{-1}$ using the binomial expansion with $n=-1$ and $x=(-x)$.

$$2(3+2x)^{-1} = 2\left[3\left(1+\frac{2x}{3}\right)\right]^{-1}$$

Take out a factor of 3.

$$= 2\times 3^{-1}\left(1+\frac{2x}{3}\right)^{-1}$$

Expand $2\times 3^{-1}\left(1+\frac{2x}{3}\right)^{-1}$ using the binomial expansion with $n=-1$ and $x=\left(\frac{2x}{3}\right)$.

$$= \frac{2}{3}\left(1+(-1)\left(\frac{2x}{3}\right)+\frac{(-1)(-2)}{2!}\left(\frac{2x}{3}\right)^2\right.$$

$$= \frac{2}{3}\left(1-\frac{2}{3}x+\frac{4}{9}x^2+\cdots\right)$$

$$= \frac{2}{3}-\frac{4}{9}x+\frac{8}{27}x^2\cdots$$

$$
\begin{aligned}
10(3+2x)^{-2} &= 10\left[3\left(1+\frac{2x}{3}\right)\right]^{-2} \\
&= 10\times 3^{-2}\left(1+\frac{2x}{3}\right)^{-2} \\
&= \frac{10}{9}\left(1+(-2)\left(\frac{2x}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{2x}{3}\right)^2+\cdots\right) \\
&= \frac{10}{9}\left(1-\frac{4x}{3}+\frac{4x^2}{3}\cdots\right) \\
&= \frac{10}{9}-\frac{40x}{27}+\frac{40x^2}{27}\cdots
\end{aligned}
$$

Take out a factor of 3.

Expand $10\times 3^{-2}\left(1+\frac{2x}{3}\right)^{-2}$ using the binomial expansion with $n=-2$ and $x=\frac{2x}{3}$.

Adding these series expansions gives

$$\left(1+\frac{2}{3}+\frac{10}{9}\right)+\left(1-\frac{4}{9}-\frac{40}{27}\right)x+\left(1+\frac{8}{27}+\frac{40}{27}\right)x^2$$

Add the three series expansions and collect and simplify the coefficients.

$$=\frac{25}{9}+\frac{-25}{27}x+\frac{25}{9}x^2+\cdots$$

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 16

Question:

When $(1+ax)^n$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 45 respectively.

a Find the value of a and the value of n.

b Find the coefficient of x^3.

c Find the set of values of x for which the expansion is valid.

E**[adapted]**

Solution:

a $$(1+ax)^n \equiv 1+nax+\frac{n(n-1)}{2}a^2x^2+\cdots$$

$\therefore na=-6$ and ……(1)

$\frac{n(n-1)}{2}a^2=45$ ……(2)

Set coefficient of x, from binomial theorem, equal to 6 and set coefficient of x^2 equal to 45.

From (1) $a=\frac{-6}{n}$, substitute into equation (2).

$$\therefore \frac{n(n-1)}{2}\times\frac{36}{n^2}=45$$

$$\therefore 36n^2-36n=90n^2$$

$$\therefore -36n=54n^2$$

Eliminate a from the simultaneous equations to obtain equation in one variable n.

$$\Rightarrow n=0 \text{ or } n=\frac{-36}{54}=\frac{-2}{3}$$

Solve to find non-zero value for n.

Substitute into equation (1) to give $a=9$.

Check solutions in equation (2).

b Coefficient of $x^3=\frac{n(n-1)(n-2)}{3!}a^3$

$$=\frac{-\frac{2}{3}\times-\frac{5}{3}\times-\frac{8}{3}\times 9^3}{3!}$$

$$=\frac{-80\times 27}{6}$$

$$=-360$$

Substitute values found for n and a into the binomial expansion to give the coefficient of x^3.

c $|x|<\frac{1}{a}$, so $\frac{-1}{9}<x<\frac{1}{9}$

The terms in the expansion are $(9x)$, $(9x)^2$, $(9x)^3$ and so $|9x|<1$.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 17

Question:

a Find the binomial expansion of $\sqrt{(1-x)}$, in ascending powers of x up to and including the term in x^3.

b By substituting a suitable value for x in this expansion, find an approximation to $\sqrt{0.9}$, giving your answer to 6 decimal places.

Solution:

a $\sqrt{(1-x)} = (1-x)^{\frac{1}{2}}$

Write the expression in index form.

$$= 1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!}(-x)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)}{3!}(-x)^3+\cdots$$

$$= 1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3\cdots$$

Replace n by $\frac{1}{2}$ and x by $-x$ in the binomial expansion. Simplify the terms.

b Let $(1-x)=0.9$ and solve

Put $x=0.1$ into expansion

This is valid as $|x|<1$.

$$\sqrt{0.9} = 1-0.05-\frac{1}{8}\times 0.01-\frac{1}{16}\times 0.001$$

$$= 1-0.05-0.00125-0.0000625$$

$$= 0.948688 \text{ (6 d.p.)}$$

This gives an estimate for $\sqrt{0.9}$. You would need to calculate further terms to give increased accuracy.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 18

Question:

In the binomial expansion, in ascending powers of x, of $(1+ax)^n$, where a and n are constants, the coefficient of x is 15. The coefficients of x^2 and of x^3 are equal.

a Find the value of a and the value of n.

b Find the coefficient of x^3.

Solution:

a

$$(1+ax)^n = 1+nax+\frac{n(n-1)}{2}a^2x^2+\cdots$$
$$+\frac{n(n-1)(n-2)a^3x^3}{6}+\cdots$$

As coefficient of x is 15

$$na=15 \qquad \ldots\ldots (1)$$

Set the coefficient of x from the binomial theorem equal to 15 and set the coefficients of x^2 and x^3 as equal to each other.

As coefficient of x^2 and x^3 are equal:

$$\frac{n(n-1)}{2}a^2=\frac{n(n-1)(n-2)a^3}{6}$$

and $\therefore (n-2)a=3 \qquad \ldots\ldots (2)$

Divide both sides of the equation by $\frac{n(n-1)}{6}a^2$.

Subtract equation on (2) from equation (1)

$$2a=12 \Rightarrow a=6$$

Substitute into equation (1)

$$\therefore n=\frac{15}{6}=\frac{5}{2}.$$

Solve equations (1) and (2) as simultaneous equations and check your answer.

b Coefficient of x^3 is

$$\frac{n(n-1)(n-2)}{6}a^3 = \frac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}}{6}\times 6^3$$
$$= \frac{15}{8}\times 36$$
$$= \frac{135}{2}$$
$$= 67.5$$

Substitute the values you have found for a and n into the binomial expansion term for x^3.

[You could also check the term for x^2, which should be equal.]

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 19

Question:

The vectors **u** and **v** are given by

$\mathbf{u} = 5\mathbf{i} - 4\mathbf{j} + s\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + t\mathbf{j} - 3\mathbf{k}$

a Given that the vectors **u** and **v** are perpendicular, find a relation between the scalars s and t.

b Given instead that the vectors **u** and **v** are parallel, find the values of the scalars s and t. ***E***

Solution:

a $\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} 5 \\ -4 \\ s \end{pmatrix} \cdot \begin{pmatrix} 2 \\ t \\ -3 \end{pmatrix}$

Write in column matrix form.

$\therefore \mathbf{u} \cdot \mathbf{v} = 5 \times 2 + (-4) \times t + S \times -3$

Use $\mathbf{u} \cdot \mathbf{v} = \mathrm{u}_1\mathrm{v}_1 + \mathrm{u}_2\mathrm{v}_2 + \mathrm{u}_3\mathrm{v}_3$.

Use the condition $\mathbf{u} \cdot \mathbf{v} = 0$

For perpendicular vectors the scalar product is zero.

$\therefore 10 - 4t - 3s = 0$

or $3s + 4t = 10$

Simplify your answers.

b As **u** and **v** are parallel

$\mathbf{v} = \lambda \mathbf{u}$ where λ is constant.

$\therefore 2\mathbf{i} + t\mathbf{j} - 3\mathbf{k} = \lambda(5\mathbf{i} - 4\mathbf{j} + s\mathbf{k})$

For parallel vectors one vector is a multiple of the other.

Compare coefficients of **i**, **j** and **k**.

$$5\lambda = 2 \Rightarrow \lambda = \frac{2}{5}$$

Equate the coefficients of x, y and z.

$$t = -4\lambda \Rightarrow t = -\frac{8}{5} = -1.6$$

$$\lambda s = -3 \Rightarrow s = -3 \div \frac{2}{5}$$

$$= \frac{-15}{2} = -7.5$$

Solve to find the values of s and t.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

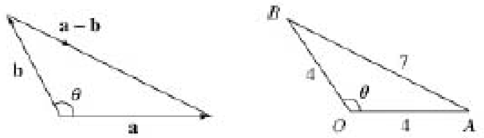
Review Exercise

Exercise A, Question 20

Question:

Find the angle between the vectors **a** and **b** given that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 4$ and $|\mathbf{a} - \mathbf{b}| = 7$. *E* [adapted]

Solution:



Use the triangle law and draw two triangles. One shows vectors. The other shows the magnitudes of the vectors.

Use the cosine rule on ΔOAB.

$$
\begin{aligned}
7^2 &= 4^2 + 4^2 - 2 \times 4 \times 4 \times \cos\theta \\
\therefore 49 &= 16 + 16 - 32\cos\theta \\
\Rightarrow 32\cos\theta &= -17 \\
\therefore \cos\theta &= \frac{-17}{32} \\
\therefore \theta &= 122^\circ (3\text{ s.f.})
\end{aligned}
$$

use the cosine rule to find $\cos\theta$.

The cosine is negative, so the angle is obtuse.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

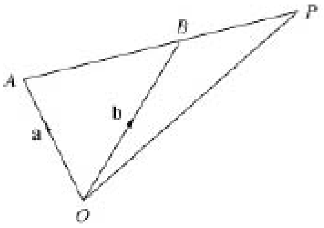
Review Exercise

Exercise A, Question 21

Question:

The position vectors of the points A and B relative to an origin O are $3\mathbf{i}+2\mathbf{j}-\mathbf{k}, 5\mathbf{i}+6\mathbf{j}+5\mathbf{k}$, respectively. Find the position vector of the point P which lies on AB produced such that $AP=3BP$. **E [adapted]**

Solution:



Draw a sketch to help you to see the triangles used later.

$$\overrightarrow{AB} = \mathbf{b}-\mathbf{a}$$
$$= 2\mathbf{i}+4\mathbf{j}+6\mathbf{k} \quad \ldots\ldots (1)$$

Use the triangle law to give $\overrightarrow{AB}$.

As $\overrightarrow{AP}=3\overrightarrow{BP}$

and $\overrightarrow{AP}=\overrightarrow{AB}+\overrightarrow{BP}$

$\therefore \overrightarrow{AB}+\overrightarrow{BP}=3\overrightarrow{BP}$

$\therefore \overrightarrow{AB}=2\overrightarrow{BP}$

$\therefore$ From (1) $\overrightarrow{BP}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

As A, B and P are points on the same line, one vector is a multiple of the other.

$$\therefore \overrightarrow{OP} = \overrightarrow{OB}+\overrightarrow{BP}$$
$$= 6\mathbf{i}+8\mathbf{j}+8\mathbf{k}$$

Use the triangle law to give $\overrightarrow{OP}$.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 22

Question:

The points A and B have coordinates $(2t, 10, 1)$ and $(3t, 2t, 5)$ respectively.

a Find $|\overrightarrow{AB}|$.

b By differentiating $\left|\overrightarrow{AB}\right|^2$, find the value of t for which $|\overrightarrow{AB}|$ is a minimum.

c Find the minimum value of $|\overrightarrow{AB}|$.

Solution:

$$\mathbf{a}=\begin{pmatrix}2t\\10\\1\end{pmatrix} \text{ and } \mathbf{b}=\begin{pmatrix}3t\\2t\\5\end{pmatrix}$$

Write down the position vectors of A and B.

a

$$\therefore \overrightarrow{AB} = \mathbf{b}-\mathbf{a}=\begin{pmatrix}t\\2t-10\\4\end{pmatrix}$$

Use $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$.

$$\therefore |\overrightarrow{AB}| = \sqrt{t^2+(2t-10)^2+4^2}$$
$$= \sqrt{5t^2-40t+116}$$

Use the vector magnitude formula.

b $|\overrightarrow{AB}|^2=5t^2-40t+116$

Call this p and differentiate.

Differentiating with respect to t gives

$$\frac{\mathrm{d}p}{\mathrm{d}t}=10t-40$$

So

$$10t-40 = 0$$
$$t = 4$$

Use the fact that $\frac{\mathrm{d}p}{\mathrm{d}t}=0$ at a minimum.

$$\frac{\mathrm{d}^2p}{\mathrm{d}t^2}=10\text{, positive, } \therefore \text{ minimum}$$

Use the fact that if the second derivative is positive, the value is a minimum.

c

$$|\overrightarrow{AB}| = \sqrt{5t^2-40t+116}$$
$$= \sqrt{80-160+116}$$
$$= \sqrt{36}$$
$$= 6$$

Substitute $t=4$ back into $|\overrightarrow{AB}|$.

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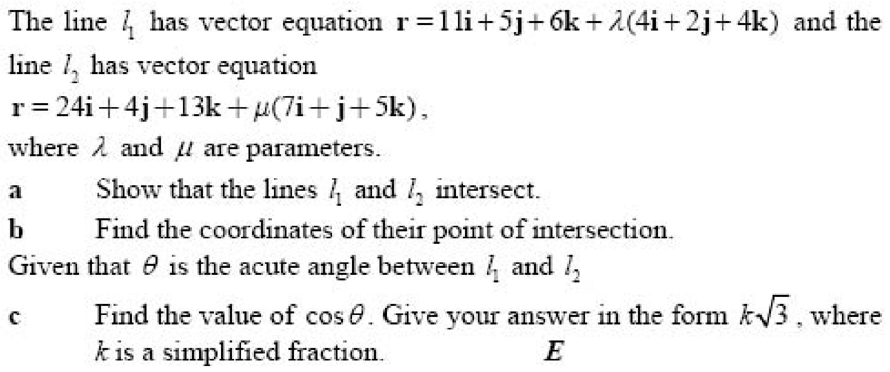
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 23

Question:



The line l_1 has vector equation $\mathbf{r} = 11\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} + \lambda(4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$ and the line l_2 has vector equation

$\mathbf{r} = 24\mathbf{i} + 4\mathbf{j} + 13\mathbf{k} + \mu(7\mathbf{i} + \mathbf{j} + 5\mathbf{k})$,

where λ and μ are parameters.

a Show that the lines l_1 and l_2 intersect.

b Find the coordinates of their point of intersection.

Given that θ is the acute angle between l_1 and l_2

c Find the value of $\cos\theta$. Give your answer in the form $k\sqrt{3}$, where k is a simplified fraction. ***E***

Solution:

a Assuming that the lines do intersect:

$$\begin{pmatrix}11+4\lambda\\5+2\lambda\\6+4\lambda\end{pmatrix}=\begin{pmatrix}24+7\mu\\4+\mu\\13+5\mu\end{pmatrix} \quad *$$

You can write the equations of the lines in column vector form and put them equal.

Rearranging gives:

$$\begin{aligned}4\lambda-7\mu &= 13 \qquad (1)\\ 2\lambda-\mu &= -1 \qquad (2)\\ 4\lambda-5\mu &= 7 \qquad (3)\end{aligned}$$

Equate the x, y and z components.

Solve these simultaneous equations.
(1) − (3) gives

$$\begin{aligned}-2\mu &= 6\\ \therefore \mu &= -3\end{aligned}$$

Substitute into (1) to give

$$4\lambda+21=13 \Rightarrow \lambda=-2$$

Solve equations (1) and (3) simultaneously.

As this solution satisfies all three equations, the lines *do* meet.

y components must also be equal so $\mu=-3, \lambda=-2$ must

b Substituting into * gives the coordinates of the point of intersection

$$\begin{pmatrix}11-8\\5-4\\6-8\end{pmatrix}=\begin{pmatrix}24-21\\4-3\\13-15\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$$

Substituting λ or μ will give the point of intersection.

$\therefore (3, 1, -2)$ is point of intersection.

The directions of the lines are
$4\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ and $7\mathbf{i}+\mathbf{j}+5\mathbf{k}$

c

$$\begin{aligned}\cos\theta &= \frac{(4\mathbf{i}+2\mathbf{j}+4\mathbf{k})\cdot(7\mathbf{i}+\mathbf{j}+5\mathbf{k})}{\sqrt{4^2+2^2+4^2}\sqrt{7^2+1^2+5^2}}\\ &= \frac{28+2+20}{\sqrt{36}\sqrt{75}}\\ &= \frac{50}{6\times5\sqrt{3}}\\ &= \frac{5}{3\sqrt{3}}\\ &= \frac{5}{9}\sqrt{3}\end{aligned}$$

Use $\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$, where **a** and **b** are the direction vectors of the lines.

Simplify the surds.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 24

Question:

The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and the line l_2 has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

a Show that l_1 and l_2 do not meet.

A is the point on l_1 where $\lambda = 1$ and B is the point on l_2 where $\mu = 2$.

b Find the cosine of the acute angle between AB and l_1. ***E***

Solution:

a Assume that the lines do meet:

$$\begin{pmatrix}1+\lambda\\0+\lambda\\-1\end{pmatrix}=\begin{pmatrix}1+2\mu\\3+\mu\\6-\mu\end{pmatrix}$$

Put the right hand sides of the equations of the two lines equal.

$$\begin{aligned}\text{So}\quad 2\mu-\lambda &= 0 &&(1)\\ \mu-\lambda &= -3 &&(2)\\ -1 &= 6-\mu &&(3)\end{aligned}$$

Equate the x, y and z components.

Solve equation (3) to give $\mu=7$ substitute into equation (1) to give $\lambda=14$.

Solve equations (1) and (3) simultaneously.

Check in equation (2) $7-14\neq-3$ to find a contradiction.

This implies that no values for λ and μ satisfy all three equations simultaneously

$\therefore$ The lines do not meet.

The values $\mu=7,\lambda=14$ do not satisfy equation (2) and so y components are not equal.

b A is the point $(2,1,-1)$ and

Substitute $\lambda=1$ into equation of line l_1.

B is the point (5, 5, 4)

Substitute $\mu=2$ into equation of line l_2.

$$\text{So } \overrightarrow{AB}=\begin{pmatrix}3\\4\\5\end{pmatrix}$$

Use $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$.

$$\text{Direction of } l_1 \text{ is } \begin{pmatrix}1\\1\\0\end{pmatrix}$$

Obtain the direction of the line l_1 from the equation of l_1.

$$\begin{aligned}\therefore \cos\theta &= \frac{\begin{pmatrix}3\\4\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\sqrt{3^2+4^2+5^2}\sqrt{1^2+1^2+0^2}}=\frac{3+4+0}{\sqrt{50}\sqrt{2}}\\ &= \frac{7}{10}\end{aligned}$$

Use $\cos\theta=\dfrac{\mathbf{c}\cdot\mathbf{d}}{|\mathbf{c}||\mathbf{d}|}$ where $\mathbf{c}$ is the vector $\overrightarrow{AB}$ and $\mathbf{d}$ is the direction of the line l_1.

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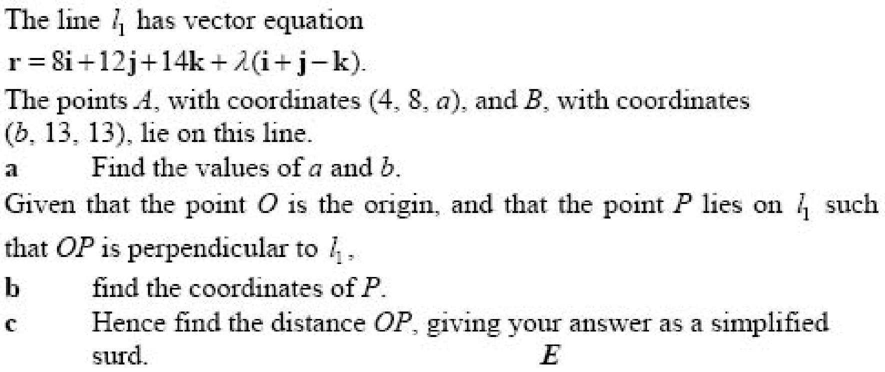
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 25

Question:



The line l_1 has vector equation

$\mathbf{r} = 8\mathbf{i} + 12\mathbf{j} + 14\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

The points A, with coordinates $(4, 8, a)$, and B, with coordinates $(b, 13, 13)$, lie on this line.

a Find the values of a and b.

Given that the point O is the origin, and that the point P lies on l_1 such that OP is perpendicular to l_1,

b find the coordinates of P.

c Hence find the distance OP, giving your answer as a simplified surd. ***E***

Solution:

a $\mathbf{r} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

You can write the line equation in this form.

The position vector of A $\begin{pmatrix} 4 \\ 8 \\ a \end{pmatrix} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

As A lies on the line equate position vectors.

use $4 = 8+\lambda$ or $8 = 12+\lambda$

$\therefore \lambda = -4$

Use the x or y coordinates to find λ.

Substitute to give $a = 14-\lambda = 18$

Find a using the value of λ.

The position vector of B $\begin{pmatrix} b \\ 13 \\ 13 \end{pmatrix} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

Also B lies on the line.

Use $13 = 12+\lambda$ or $13 = 14-\lambda$

Use the y or z coordinates to find λ.

$\therefore \lambda = 1$

Substitute to give $b = 8+\lambda = 9$

Find b using this value of λ.

b Direction $\overline{OP}$ is $\begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

and direction of l_1 is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

This is obtained from the equation of l_1.

These are perpendicular

$\therefore \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$

Use the condition for perpendicular lines, $\mathbf{c} \cdot \mathbf{d} = 0$.

$\therefore 8+\lambda+12+\lambda-(14-\lambda) = 0$

$\therefore 3\lambda + 6 = 0$

$\Rightarrow \lambda = -2$

$\therefore$ Point P is at (6, 10, 16)

Substitute the value of λ into the line equation to give the coordinates of P.

c

Distance OP $= \sqrt{6^2+10^2+16^2}$

Use the formula for magnitude of a vector.

$= \sqrt{392}$

$= 14\sqrt{2}$

Simplify the surd using $\sqrt{392} = \sqrt{196}\sqrt{2}$.

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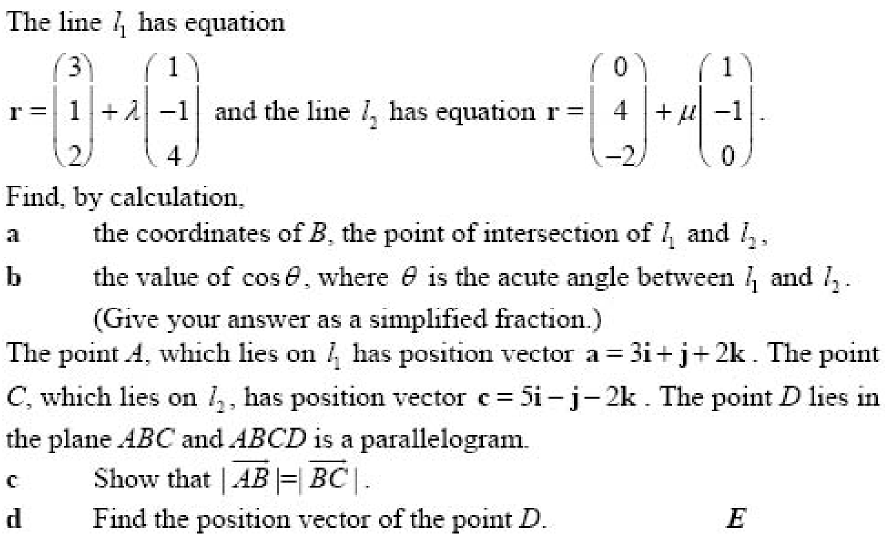
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 26

Question:



The line l_1 has equation

$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ and the line l_2 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Find, by calculation,

a the coordinates of B, the point of intersection of l_1 and l_2,

b the value of $\cos\theta$, where θ is the acute angle between l_1 and l_2. (Give your answer as a simplified fraction.)

The point A, which lies on l_1 has position vector $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. The point C, which lies on l_2, has position vector $\mathbf{c} = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. The point D lies in the plane ABC and $ABCD$ is a parallelogram.

c Show that $|\overrightarrow{AB}| = |\overrightarrow{BC}|$.

d Find the position vector of the point D. ***E***

Solution:

a As the two lines meet:

$$\begin{pmatrix} 3+\lambda \\ 1-\lambda \\ 2+4\lambda \end{pmatrix} = \begin{pmatrix} \mu \\ 4-\mu \\ -2 \end{pmatrix}$$

You can write the equations in this form.

$$\begin{aligned} \therefore \mu-\lambda &= 3 && (1) \\ -\mu+\lambda &= -3 && (2) \\ 4\lambda &= -4 && (3) \end{aligned}$$

Put the x, y and z components equal and rearrange.

Use equation (3) to find λ.

Solve equation (3) to give $\lambda=-1$
Substitute this value into equation (1)
Then $\mu=2$

Substitute λ into equation (1) or (2) to find μ.

$\therefore$ Point of intersection is $(2,2,-2)$

Use λ or μ in the line equations to find the coordinates of B.

b The directions of the lines are $\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

The directions of the lines are from the line equation.

$$\begin{aligned} \therefore \cos\theta &= \frac{1\times1+(-1)\times(-1)+4\times0}{\sqrt{1^2+(-1)^2+4^2}\sqrt{1^2+(-1)^2}} \\ &= \frac{2}{\sqrt{18}\sqrt{2}} \\ &= \frac{1}{3} \end{aligned}$$

Use $\cos\theta = \dfrac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$ where **x** and **y** are the directions of the lines.

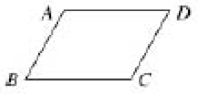
c

$$\begin{aligned} \overrightarrow{AB} &= \mathbf{b}-\mathbf{a} = -\mathbf{i}+\mathbf{j}-4\mathbf{k} \\ \overrightarrow{BC} &= \mathbf{c}-\mathbf{b} = 3\mathbf{i}-3\mathbf{j} \\ |\overrightarrow{AB}| &= \sqrt{((-1)^2+1^2+(-4)^2)} = \sqrt{18} = 3\sqrt{2} \\ |\overrightarrow{BC}| &= \sqrt{(3^2+(-3)^2)} = \sqrt{18} = 3\sqrt{2} \\ \therefore |\overrightarrow{AB}| &= |\overrightarrow{BC}| \end{aligned}$$

Use the triangle law to find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

Use the formula for the magnitude of a vector.

d



Draw a diagram and label the vertices of the parallelogram as A, B, C and D in a cyclic order.

$$\begin{aligned} \overrightarrow{BA} &= \overrightarrow{CD} \\ \overrightarrow{BA} &= \mathbf{i}-\mathbf{j}+4\mathbf{k} = \overrightarrow{CD} \end{aligned}$$

Use the fact that opposite sides are equal and parallel.

$$\overrightarrow{OD} = \overrightarrow{OC}+\overrightarrow{CD}$$

Also $= (5\mathbf{i}-\mathbf{j}-2\mathbf{k})+(\mathbf{i}-\mathbf{j}+4\mathbf{k})$
$= (6\mathbf{i}-2\mathbf{j}+2\mathbf{k})$

Use the triangle law.

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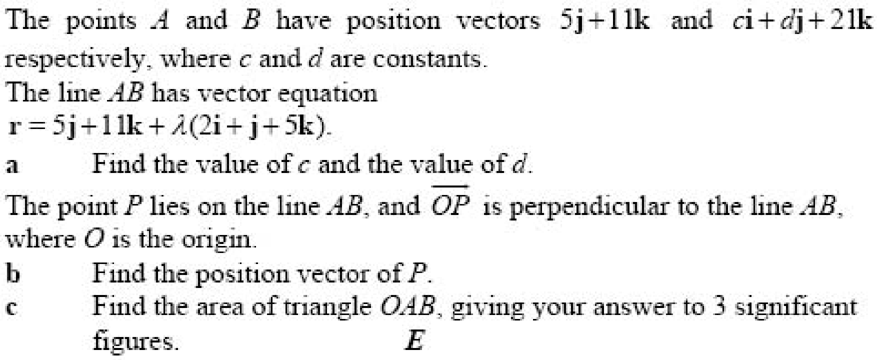
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 27

Question:



The points A and B have position vectors $5\mathbf{j}+11\mathbf{k}$ and $c\mathbf{i}+d\mathbf{j}+21\mathbf{k}$ respectively, where c and d are constants.

The line AB has vector equation

$\mathbf{r}=5\mathbf{j}+11\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}+5\mathbf{k})$.

a Find the value of c and the value of d.

The point P lies on the line AB, and $\overrightarrow{OP}$ is perpendicular to the line AB, where O is the origin.

b Find the position vector of P.

c Find the area of triangle OAB, giving your answer to 3 significant figures. E

Solution:

$$\mathbf{r}=\begin{pmatrix}2\lambda\\ \lambda+5\\ 5\lambda+11\end{pmatrix}$$

You can write the line equation in this form.

a As B lies on the line

$2\lambda=c,(\lambda+5)=d,5\lambda+11=21$

$\therefore$ Solving $5\lambda+11=21,\lambda=2$

and substituting into other equations gives $c=4,d=7$.

Use the z coordinate to find the value of λ.

Find c and d using the value of λ.

b $\begin{pmatrix}2\lambda\\ \lambda+5\\ 5\lambda+11\end{pmatrix}\cdot\begin{pmatrix}2\\1\\5\end{pmatrix}=0$

use $\overrightarrow{OP}\cdot\mathbf{y}=0$ where y is the direction of the line and is obtained from the equation of the line.

$\therefore 2(2\lambda)+1(\lambda+5)+5(5\lambda+11)=0$

$\therefore 30\lambda+60=0$

$\therefore \lambda=-2$

$\therefore$ P has position vector $\begin{pmatrix}-4\\3\\1\end{pmatrix}$

Substitute $\lambda=-2$ into the equation of the line.

c Area of $\Delta OAB=\frac{1}{2}|\overrightarrow{OA}|\cdot|\overrightarrow{OB}|\sin B\hat{O}A$

and $\cos B\hat{O}A=\frac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}|\cdot|\overrightarrow{OB}|}$

Use area of triangle is $\frac{1}{2}ab\sin C$.

$\overrightarrow{OA}=\begin{pmatrix}0\\5\\11\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}4\\7\\21\end{pmatrix}$

$$\therefore\cos B\hat{O}A=\frac{0\times4+5\times7+11\times21}{\sqrt{(0^2+5^2+11^2)}\sqrt{(4^2+7^2+21^2)}}$$

Use the scalar product to find the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$$=\frac{266}{\sqrt{146}\sqrt{506}}$$

$$=0.9787\text{ (4 s.f.)}$$

$\therefore B\hat{O}A = 11.86$ (4 s.f.)

$\therefore$ Area $=$ 27.9 (3 s.f.)

Substitute $\sqrt{146},\sqrt{506}$ and angle 11.86° into the formula for area of a triangle.

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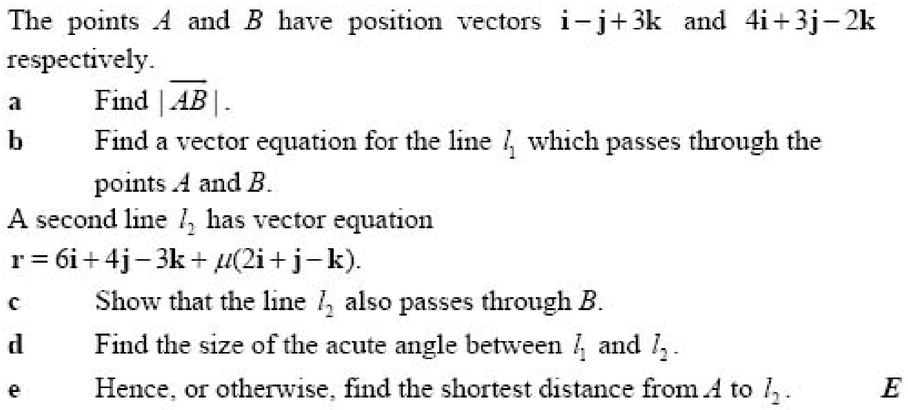
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 28

Question:



The points A and B have position vectors $\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $4\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ respectively.

a Find $|\overline{AB}|$.

b Find a vector equation for the line l_1 which passes through the points A and B.

A second line l_2 has vector equation

$\mathbf{r}=6\mathbf{i}+4\mathbf{j}-3\mathbf{k}+\mu(2\mathbf{i}+\mathbf{j}-\mathbf{k})$.

c Show that the line l_2 also passes through B.

d Find the size of the acute angle between l_1 and l_2.

e Hence, or otherwise, find the shortest distance from A to l_2. ***E***

Solution:

a $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}, \mathbf{b} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$$\therefore \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$
$$= 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$$

Use the triangle law.

$$\therefore |\overrightarrow{AB}| = \sqrt{3^2 + 4^2 + (-5)^2}$$
$$= \sqrt{50} \text{ or } 5\sqrt{2} \text{ or } 7.07$$

Use the formula for the magnitude of a vector.

b $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

or

$\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \mu(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

There are other forms of this equation, but these two are the simplest.

c If $\mathbf{r} = 6\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k})$ passes through $4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$$\text{then} \quad 6 + 2\mu = 4$$
$$4 + \mu = 3$$
$$-3 - \mu = -2$$

Equate *x*, *y* and *z* components.

As $\mu = -1$ satisfies all three equations, the line passes through *B* as required.

Solve for μ and check that μ satisfies *all three* equations.

d The lines have directions $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$

If the angle between the lines is θ then

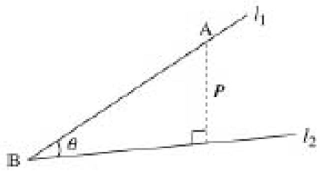
Use $\cos\theta = \frac{\mathbf{c} \cdot \mathbf{d}}{|\mathbf{c}||\mathbf{d}|}$ where **c** and **d** are the directions of the lines.

$$\cos\theta = \frac{(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k})}{|3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}||2\mathbf{i} + \mathbf{j} - \mathbf{k}|}$$
$$= \frac{3 \times 2 + 4 \times 1 + (-5) \times (-1)}{\sqrt{50}\sqrt{2^2 + 1^2 + (-1)^2}}$$
$$= \frac{15}{\sqrt{50}\sqrt{6}}$$

$$\therefore \cos\theta = \frac{\sqrt{3}}{2}$$
$$\text{and } \theta = 30^\circ$$

This answer is acute. If your answer is obtuse, subtract it from 180°.

e



Draw a diagram showing l_1, l_2 with common point *B*.

The shortest distance from point *A* to the line l_2 is

The shortest distance is the perpendicular distance.

$$|\overrightarrow{AB}|\sin\theta = 5\sqrt{2} \times \frac{1}{2}$$

Use trigonometry $\sin\theta = \frac{P}{|\overrightarrow{AB}|}$.

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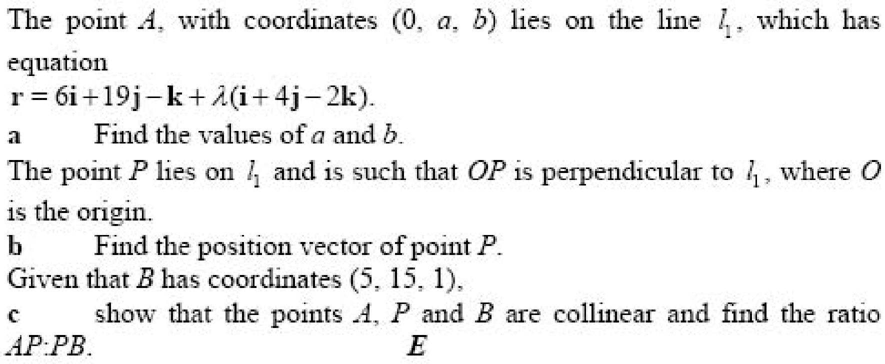
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 29

Question:



The point A, with coordinates $(0, a, b)$ lies on the line l_1, which has equation

$\mathbf{r} = 6\mathbf{i} + 19\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$.

a Find the values of a and b.

The point P lies on l_1 and is such that OP is perpendicular to l_1, where O is the origin.

b Find the position vector of point P.

Given that B has coordinates $(5, 15, 1)$,

c show that the points A, P and B are collinear and find the ratio AP:PB. ***E***

Solution:

a $$\mathbf{r} = \begin{pmatrix} 6+\lambda \\ 19+4\lambda \\ -1-2\lambda \end{pmatrix}$$

You can write the equation in the form.

$$\begin{aligned} \therefore 6+\lambda &= 0 && (1) \\ 19+4\lambda &= a && (2) \\ -1-2\lambda &= b && (3) \end{aligned}$$

Equate *x*, *y* and *z* coordinates of *A* to those of the line.

From equation (1) $\lambda = -6$

Find λ from equation (1).

Substituting this value for λ into equation (2) gives $a = -5$

Substituting $\lambda = -6$ into equation (3) gives $b = 11$

Find *a* and *b* using this value of λ.

b $$\overrightarrow{OP} = \begin{pmatrix} 6+\lambda \\ 19+4\lambda \\ -1-2\lambda \end{pmatrix}$$ and l_1 is in direction $\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

The directions of $\overrightarrow{OP}$ and of l_1 are obtained from the equation of the line.

As $\overrightarrow{OP}$ is perpendicular to l_1,

$$\begin{pmatrix} 6+\lambda \\ 19+4\lambda \\ -1-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = 0$$

Use condition for perpendicular lines, $\mathbf{c} \cdot \mathbf{d} = 0$.

$\therefore 6+\lambda+4(19+4\lambda)-2(-1-2\lambda) = 0$

i.e. $84+21\lambda = 0 \Rightarrow \lambda = -4$

$\therefore \overrightarrow{OP} = 2\mathbf{i}+3\mathbf{j}+7\mathbf{k}$

Substitute $\lambda = -4$ back into vector $\overrightarrow{OP}$.

c $\overrightarrow{OA} = -5\mathbf{j}+11\mathbf{k}$

$\therefore \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = 2\mathbf{i}+8\mathbf{j}-4\mathbf{k}$

Find $\overrightarrow{AP}$ using triangle law.

$\overrightarrow{OB} = 5\mathbf{i}+15\mathbf{j}+\mathbf{k}$

$\therefore \overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = 3\mathbf{i}+12\mathbf{j}-6\mathbf{k}$

Find $\overrightarrow{PB}$ using triangle law.

$\therefore \overrightarrow{AP} = \frac{2}{3}\overrightarrow{PB} \Rightarrow$ vectors are in the same direction, and as they have a point in mmon they are collinear.

Ratio

$$\begin{aligned} \overrightarrow{AP} : \overrightarrow{PB} &= \frac{2}{3}\overrightarrow{PB} : \overrightarrow{PB} \\ &= \frac{2}{3} : 1 \\ &= 2 : 3 \end{aligned}$$

Note that each of these vectors is a multiple of $\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ and so one is a multiple of the other.

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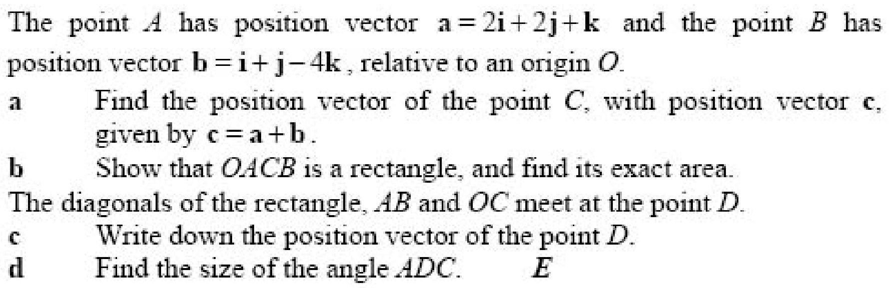
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 30

Question:



The point A has position vector $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and the point B has position vector $\mathbf{b} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$, relative to an origin O.

a Find the position vector of the point C, with position vector $\mathbf{c}$, given by $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

b Show that $OACB$ is a rectangle, and find its exact area.

The diagonals of the rectangle, AB and OC meet at the point D.

c Write down the position vector of the point D.

d Find the size of the angle ADC. *E*

Solution:

a

$$\begin{aligned} \mathbf{c} &= \mathbf{a}+\mathbf{b} \\ &= (2\mathbf{i}+2\mathbf{j}+\mathbf{k})+(\mathbf{i}+\mathbf{j}-4\mathbf{k}) \\ &= 3\mathbf{i}+3\mathbf{j}-3\mathbf{k} \end{aligned}$$

b As $\overline{OA}=\overline{BC}$ and $\overline{OB}=\overline{AC}$ $OACB$ is a parallelogram.

As $\mathbf{a}\cdot\mathbf{b}=2+2-4=0$

$\mathbf{a}$ is perpendicular to $\mathbf{b}$

$\therefore OACB$ is a parallelogram with all of its angles right angles i.e., it is a rectangle

Opposite sides are equal and parallel.

Adjacent sides are perpendicular.

$$\begin{aligned} \text{Its area} &= |\mathbf{a}|\times|\mathbf{b}| \\ &= \sqrt{2^2+2^2+1^2}\times\sqrt{1^2+1^2+(-4)^2} \\ &= 3\times3\sqrt{2} \\ &= 9\sqrt{2} \end{aligned}$$

Use the formula for magnitude of a vector.

c The diagonals bisect each other.

D is the mid-point of OC, and of AB.

$$\therefore \mathbf{d}=\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}-\frac{3}{2}\mathbf{k}$$

d

$$\begin{aligned} \overrightarrow{AD} &= \mathbf{d}-\mathbf{a}=-\frac{1}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}-\frac{5}{2}\mathbf{k} \\ \overrightarrow{CD} &= \mathbf{d}-\mathbf{c}=-\frac{3}{2}\mathbf{i}-\frac{3}{2}\mathbf{j}+\frac{3}{2}\mathbf{k} \end{aligned}$$

Use the triangle law to find $\overrightarrow{AD}$ and $\overrightarrow{CD}$, or $\overrightarrow{DA}$ and $\overrightarrow{DC}$.

$$\begin{aligned} \therefore \cos A\hat{D}C &= \frac{\overrightarrow{AD}\cdot\overrightarrow{CD}}{|\overrightarrow{AD}|\cdot|\overrightarrow{CD}|} \\ &= \frac{\frac{3}{4}+\frac{3}{4}-\frac{15}{4}}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{25}{4}}\sqrt{\frac{9}{4}+\frac{9}{4}+\frac{9}{4}}} \\ &= \frac{-\frac{9}{4}}{\frac{27}{4}} \\ &= \frac{-1}{3} \end{aligned}$$

Use the formula $\cos\theta=\dfrac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$ with $x=\overrightarrow{AD}$ and $y=\overrightarrow{CD}$.

As $\cos A\hat{D}C$ is negative, angle ADC is obtuse.

$$\therefore A\hat{D}C = 109.5^\circ\ (1\text{ d.p.})$$

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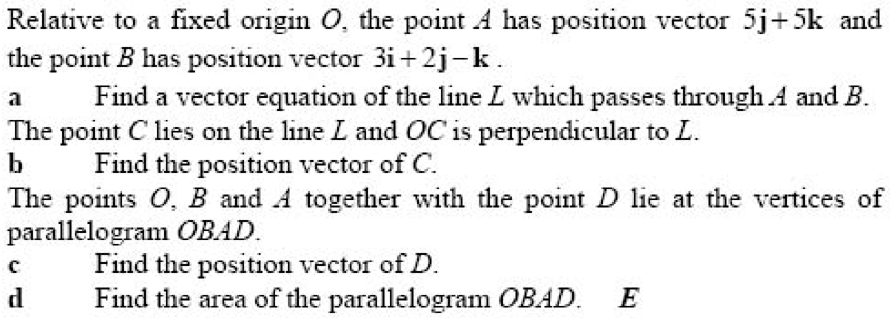
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 31

Question:



Relative to a fixed origin O, the point A has position vector $5\mathbf{j}+5\mathbf{k}$ and the point B has position vector $3\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

a Find a vector equation of the line L which passes through A and B.

The point C lies on the line L and OC is perpendicular to L.

b Find the position vector of C.

The points O, B and A together with the point D lie at the vertices of parallelogram $OBAD$.

c Find the position vector of D.

d Find the area of the parallelogram $OBAD$. ***E***

Solution:

a $\mathbf{a} = \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

The position vectors can be written in this form.

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a},$$

Using $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$

Use the triangle law.

$$= \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix}$$

The direction of the line is any multiple of $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

You might have found $\overrightarrow{BA}$ and used $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

An equation of the line is

$$\mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \text{ or } \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

You only need *one* form of the equation.

b C lies on the line

$$\therefore OC = \begin{pmatrix} \lambda \\ 5-\lambda \\ 5-2\lambda \end{pmatrix} \text{ or } \begin{pmatrix} 3+\mu \\ 2-\mu \\ -1-2\mu \end{pmatrix}$$

You obtain your directions from the equation of line l.

The direction of L is $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

Use the condition for perpendicular vectors $\mathbf{x} \cdot \mathbf{y} = 0$.

As OC is perpendicular to L

$$\begin{pmatrix} \lambda \\ 5-\lambda \\ 5-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 0$$

$$\therefore \lambda - (5-\lambda) - 2(5-2\lambda) = 0$$

i.e.,

$$6\lambda - 15 = 0$$

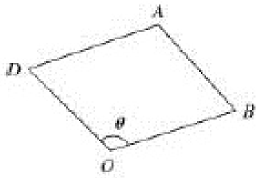
$$\therefore \lambda = \frac{15}{6} = \frac{5}{2}$$

You could have used $\begin{pmatrix} 3+\mu \\ 2-\mu \\ -1-2\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ to obtain $\mu = \frac{-1}{2}$.

$$\therefore \overrightarrow{OC} = \begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \\ 0 \end{pmatrix}$$

Substitute your value of λ (or μ) to obtain the answer.

c



Draw a sketch of parallelogram *OBAD*, labelling vertices in order.

$$\begin{aligned}\overrightarrow{OD} &= \overrightarrow{BA}\\ &= -\overrightarrow{AB}\text{(found in a)}\\ &= -3\mathbf{i}+3\mathbf{j}+6\mathbf{k}\end{aligned}$$

Opposite sides of a parallelogram are equal and parallel.

This is the position vector of *D*.

d Area of parallelogram *OBAD*

$$\begin{aligned}&= 2\times\text{Area of}\Delta\ OBD\\ &= |\overrightarrow{OB}|\times|\overrightarrow{OD}|\times\sin\theta \quad *\end{aligned}$$

where θ is angle between $\overrightarrow{OB}$ and $\overrightarrow{OD}$.

use $\cos\theta=\dfrac{\mathbf{b}\cdot\mathbf{d}}{|\mathbf{b}||\mathbf{d}|}$ to find θ.

Think through the method that you will use before you begin.

$$\begin{aligned}\mathbf{b}\cdot\mathbf{d} &= \begin{pmatrix}3\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}-3\\3\\6\end{pmatrix}\\ &= -3\times3+2\times3+(-1)\times6\\ &= -9\end{aligned}$$

Use the formula for scalar product.

$$\begin{aligned}|\mathbf{b}| &= \sqrt{(3^2+2^2+(-1)^2)}\\ &= \sqrt{14}\end{aligned}$$

Use the formula for magnitude of a vector.

$$\begin{aligned}|\mathbf{d}| &= \sqrt{(-3)^2+3^2+6^2}\\ &= \sqrt{54}\end{aligned}$$

$$\begin{aligned}\therefore\cos\theta &= \frac{-9}{\sqrt{14}\sqrt{54}}(\neq-0.327\cdots)\\ \therefore\theta &= 109.1^\circ(4\text{ s.f.})\\ \therefore\text{Area} &= \sqrt{14}\times\sqrt{54}\times\sin109.1^\circ\\ &= 26.0\ (3\text{ s.f.})\end{aligned}$$

You could use a calculator in this question.

Give the answer to 3 significant figures if you use a calculator.
The exact answer is $15\sqrt{3}$ and is easier to obtain using further mathematics techniques i.e. vector product.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 32

Question:

Find the gradient of the curve

$3x^3 - 2x^2y + y^3 = 17$

at the point with coordinates (2, 1). ***E***

Solution:

$3x^3 - 2x^2y + y^3 = 17$

Differentiate with respect to x:

This is an implicit differentiation as you cannot make y the subject of the formula.

$$9x^2 - \left[2x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 4xy\right]$$

$$+3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Use the product rule to differentiate the $2x^2y$ term.

Use the chain rule to differentiate y^3.

Differentiate 17 to give 0.

Substitute $x = 2, y = 1$

$$36 - \left[8\frac{\mathrm{d}y}{\mathrm{d}x} + 8\right] + 3\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\therefore 28 - 5\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{28}{5} = 5\frac{3}{5}$$

You can make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject of the formula before substituting $x = 2, y = 1$ but the algebra is more difficult.

You would get $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9x^2 - 4xy}{2x^2 - 3y^2}$.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 33

Question:

A curve has equation

$x^2 + 2xy - 3y^2 + 16 = 0.$

Find the coordinates of the points on the curve where $\frac{dy}{dx} = 0$. ***E***

Solution:

$x^2 + 2xy - 3y^2 + 16 = 0$ …… (1)

Differentiate with respect to x.

Use implicit differentiation as it is awkward to make y the subject of the formula.

$$2x + \left[2x\frac{dy}{dx} + 2y\right] - 6y\frac{dy}{dx} + 0 = 0$$

Use the product rule to differentiate the $2xy$ term.

Use the chain rule to differentiate $-3y^2$.

Differentiate 16 to give 0.

Put $\frac{dy}{dx} = 0$

$\therefore 2x + 0 + 2y - 0 = 0$

i.e. $2(x + y) = 0$

Find the relationship between x and y when $\frac{dy}{dx} = 0$.

$\therefore x = -y$ …… (2)

Substitute this into equation (1)

$y^2 - 2y^2 - 3y^2 + 16 = 0$

Solve equations (1) and (2) as simultaneous equation.

$$\begin{aligned} \therefore 4y^2 &= 16 \\ \therefore y &= \pm 2 \\ \therefore x &= \mp 2 \end{aligned}$$

The points at which $\frac{dy}{dx} = 0$ are $(-2, 2)$ and $(2, -2)$.

Match corresponding values for x and y to give the required coordinates.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 34

Question:

A curve C is described by the equation

$3x^2 - 2y^2 + 2x - 3y + 5 = 0.$

Find an equation of the normal to C at the point (0, 1), giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. ***E***

Solution:

$3x^2 - 2y^2 + 2x - 3y + 5 = 0$

Differentiate with respect to x

Then

Use implicit differentiation.

$$6x - 4y\frac{dy}{dx} + 2 - 3\frac{dy}{dx} + 0 = 0$$

Use the chain rule to differentiate $-2y^2$ and $-3y$.

Substitute $x = 0, y = 1$ then

$$-4\frac{dy}{dx} + 2 - 3\frac{dy}{dx} = 0$$

You could make $\frac{dy}{dx}$ the subject of the formula before substituting $x = 0, y = 1$.

In this case $\frac{dy}{dx} = \frac{6x+2}{3+4y}$.

$$\therefore 7\frac{dy}{dx} = 2$$

$$\text{i.e.} \frac{dy}{dx} = \frac{2}{7}$$

The gradient of the normal to C at (0,1) is $\frac{-7}{2}$

Use the result that $mm^1 = -1$ for perpendicular lines.

$\therefore$ Equation of the normal is $y - 1 = \frac{-7}{2}(x - 0)$

i.e. $y = \frac{-7}{2}x + 1$

This could be obtained directly from $y = mx + c$.

or $7x + 2y - 2 = 0$

give the answer in the form required by the question.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 35

Question:

A curve C is described by the equation

$$3x^2+4y^2-2x+6xy-5=0.$$

Find an equation of the tangent to C at the point $(1,-2)$, giving your answer in the form $ax+by+c=0$, where a, b and c are integers. ***E***

Solution:

$$3x^2+4y^2-2x+6xy-5=0$$

Use implicit differentiation.

Differentiate with respect to x

$$6x+8y\frac{\mathrm{d}y}{\mathrm{d}x}-2+\left[6x\frac{\mathrm{d}y}{\mathrm{d}x}+6y\right]-0=0$$

Use the chain rule to differentiate $4y^2$.

Use the product rule to differentiate $6xy$.

Substitute $x=1$, $y=-2$

Then

$$6-16\frac{\mathrm{d}y}{\mathrm{d}x}-2+6\frac{\mathrm{d}y}{\mathrm{d}x}-12=0$$

If you rearranged you would get $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2-6x-6y}{8y+6x}$.

$$\therefore -8-10\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-8}{10}$$

Gradient of the tangent at $(1,-2)$ is $-\frac{8}{10}$.

$\therefore$ Equation of the tangent is

$$(y+2) = \frac{-8}{10}(x-1)$$

Use the equation on $y-y_1=m(x-x_1)$.

$$\therefore y+2 = \frac{-8}{10}x+\frac{8}{10}$$

$$\therefore 10y+8x+12 = 0$$

$$\text{i.e. } 4x+5y+6 = 0$$

Multiply by 10 and collect the terms as required by the question.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 36

Question:

A set of curves is given by the equation $\sin x + \cos y = 0.5$.

a Use implicit differentiation to find an expression for $\frac{dy}{dx}$.

For $-\pi < x < \pi$ and $-\pi < y < \pi$.

b find the coordinates of the points where $\frac{dy}{dx} = 0$. *E*

Solution:

a $\sin x + \cos y = 0.5$ *

Differentiate with respect to x:

$$\cos x - \sin y \frac{dy}{dx} = 0$$

Use the chain rule to differentiate $\cos y$.

$$\therefore \frac{dy}{dx} = \frac{\cos x}{\sin y}$$

Make $\frac{dy}{dx}$ the subject of the formula.

b When $\frac{dy}{dx} = 0$,

$$\cos x = 0$$

$$\therefore x = \pm\frac{\pi}{2}$$

Give answers in the range $-\pi < x < \pi$.

when $x = \frac{\pi}{2}$ substitute into *

$$1 + \cos y = 0.5$$

$$\therefore \cos y = -0.5$$

$$\therefore y = \frac{2\pi}{3} \text{ or } \frac{-2\pi}{3}.$$

Give answers in the range $-\pi < y < \pi$.

when $x = -\frac{\pi}{2}$ substitute into *

$$-1 + \cos y = 0.5$$

$$\therefore \cos y = 1.5$$

As $\cos y$ cannot be greater than 1 this equation has no solutions.

$\therefore$ Stationary points at $(\frac{\pi}{2}, \frac{2\pi}{3})$ and $(\frac{\pi}{2}, \frac{-2\pi}{3})$ only in the given range.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 37

Question:

a Given that $y = 2^x$, and using the result $2^x = e^{x\ln 2}$, or otherwise, show that $\frac{dy}{dx} = 2^x \ln 2$.

b Find the gradient of the curve with equation $y = 2^{x^2}$ at the point with coordinates (2, 16). ***E***

Solution:

a

$$y = 2^x = e^{x\ln 2}$$

You use the result that if $y = e^{kx}, \frac{dy}{dx} = ke^{kx}$.

$$\frac{dy}{dx} = \ln 2e^{x\ln 2}$$

$$= \ln 2(e^{\ln 2})^x$$

$$= \ln 2 \times 2^x$$

Note that $e^{\ln 2} = 2$.

or

$$y = 2^x$$

$$\ln y = \ln 2^x$$

$$= x\ln 2$$

You could use a different method, by taking logs of both sides and using implicit differentiation.

Differentiate with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \ln 2$$

$$\therefore \frac{dy}{dx} = y\ln 2$$

$$= 2^x \ln 2$$

b

$$y = 2^{x^2}$$

Use the chain rule.

If $y = 2^{f(x)}, \frac{dy}{dx} = f'(x)2^{f(x)}\ln 2$.

$$\frac{dy}{dx} = (2x)2^{x^2}\ln 2$$

When $x = 2$

$$\frac{dy}{dx} = 4 \times 2^4 \ln 2$$

$$= 64\ln 2$$

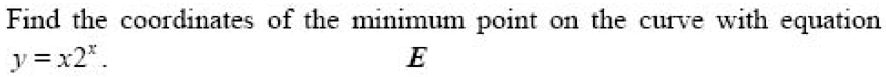
Substitute $x = 2$ into your expression.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 38

Question:



Find the coordinates of the minimum point on the curve with equation $y = x2^x$. ***E***

Solution:

$$y = x2^x \quad *$$

$$\frac{dy}{dx} = x \cdot 2^x \ln 2 + 2^x \times 1$$

$$= 2^x(x\ln 2 + 1)$$

Use the product rule.

At a minimum point,

$$\frac{dy}{dx} = 0$$

Put $\frac{dy}{dx} = 0$ and solve.

$\therefore x\ln 2 + 1 = 0$

i.e. $x = \frac{-1}{\ln 2}$

Substitute into $*$ to give:

$$y = \frac{-1}{\ln 2} \times 2^{\frac{-1}{\ln 2}}$$

$$= \frac{-1}{\ln 2} \times \frac{1}{2^{\frac{1}{\ln 2}}} \dagger$$

Substitute x value into the equation given, to find y.

Let $2^{\frac{1}{\ln 2}} = u$

Take lns of both sides

You may simplify $2^{\frac{1}{\ln 2}} = \text{e}$.

$$\ln 2^{\frac{1}{\ln 2}} = \ln u$$

$$\therefore \frac{1}{\ln 2} \times \ln 2 = \ln u$$

i.e. $\ln u = 1 \Rightarrow u = \text{e}$.

Substitute back into $\dagger$

$$y = \frac{-1}{\text{e}\ln 2}$$

$\therefore$ minimum point is at $\left(\frac{-1}{\ln 2}, \frac{-1}{\text{e}\ln 2}\right)$.

To check that this is indeed a minimum point you would need to find $\frac{d^2y}{dx^2} = 2^x \ln 2(x\ln 2 + 2)$

As this is positive at $x = \frac{-1}{\ln 2}$ the turning point is a minimum.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 39

Question:

The value £V of a car t years after the 1st January 2001 is given by the formula $V = 10000 \times (1.5)^{-t}$.

a Find the value of the car on 1st January 2005.

b Find the value of $\frac{\mathrm{d}V}{\mathrm{d}t}$ when $t = 4$. ***E***

Solution:

a $V = 10000 \times (1.5)^{-t}$

On 1st January 2005, $t = 4$

$$\therefore V = 10\,000 \times (1.5)^{-4}$$
$$= £1975.31 (2 \text{ d.p.})$$

Give your answer to a suitable accuracy.

b

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -10000 \times (1.5)^{-t} \times \ln 1.5$$
$$= -800.92 \text{ (2 d.p.)}$$

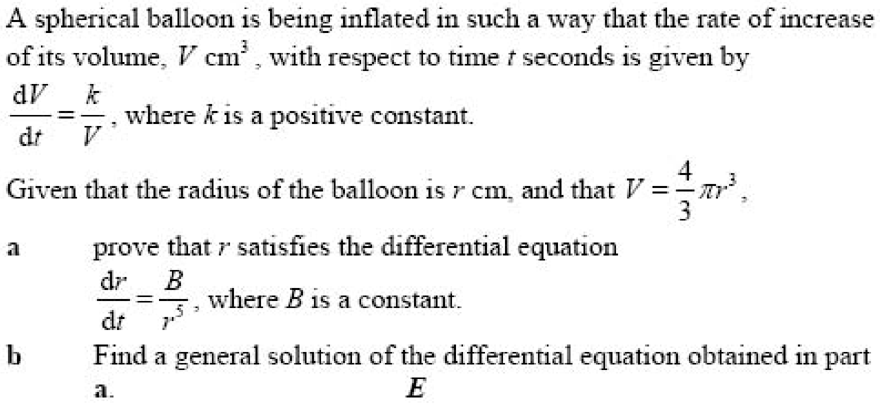
Differentiate and substitute $t = 4$.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 40

Question:



A spherical balloon is being inflated in such a way that the rate of increase of its volume, V cm^3, with respect to time t seconds is given by

$\frac{dV}{dt} = \frac{k}{V}$, where k is a positive constant.

Given that the radius of the balloon is r cm, and that $V = \frac{4}{3}\pi r^3$,

a prove that r satisfies the differential equation
$\frac{dr}{dt} = \frac{B}{r^5}$, where B is a constant.

b Find a general solution of the differential equation obtained in part **a**. ***E***

Solution:

a

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2 \quad *$$

You need to find $\frac{dV}{dr}$ in order to connect $\frac{dr}{dt}$ and $\frac{dV}{dt}$, using the chain rule.

Use the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

Substitute $\frac{dV}{dt} = \frac{k}{V}$ (given) and $\frac{dV}{dr} = 4\pi r^2$ (from $*$) into the chain rule:

$$\therefore \frac{k}{V} = 4\pi r^2 \times \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{k}{V} \div 4\pi r^2$$

$$= \frac{k}{\frac{4}{3}\pi r^3} \times \frac{1}{4\pi r^2}$$

$$= \frac{3k}{16\pi^2 r^5}.$$

Substitute $V = \frac{4}{3}\pi r^3$ and note that $\div \frac{4\pi r^2}{1}$ is the same as $\times$ by $\frac{1}{4\pi r^2}$.

b Separate the variables.

$$\int r^5 dr = \int \frac{3k}{16\pi^2} dt$$

$$\therefore \frac{r^6}{6} = \frac{3k}{16\pi^2} t + A$$

Integrate each side and include constant of integration.

$$\therefore r = \left[\frac{9k}{8\pi^2} t + A'\right]^{\frac{1}{6}}$$

Multiply by 6 and take the sixth root to give r.

$A' = 6A$.

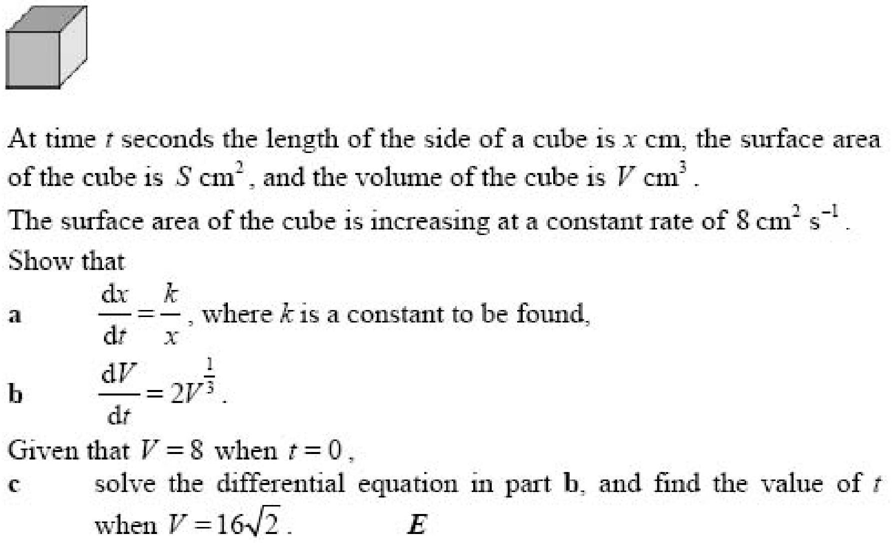
Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 41

Question:



At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3.

The surface area of the cube is increasing at a constant rate of 8 cm^2 s^{-1}.

Show that

a $\frac{dx}{dt} = \frac{k}{x}$, where k is a constant to be found,

b $\frac{dV}{dt} = 2V^{\frac{1}{3}}$.

Given that $V = 8$ when $t = 0$,

c solve the differential equation in part **b**, and find the value of t when $V = 16\sqrt{2}$. ***E***

Solution:

Let S be the surface area

a

$$S = 6x^2$$

The surface is made up of six squares each of area x^2.

$$\therefore \frac{dS}{dx} = 12x$$

Differentiate to give $\frac{dS}{dx}$.

Given that $\frac{dS}{dt} = 8$

This comes from the information that the surface area is increasing at a constant rate of $8\text{ cm}^2 s^{-1}$.

Use $\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt}$

Use the chain rule to connect $\frac{dS}{dt}, \frac{dS}{dx}$ and $\frac{dx}{dt}$.

$$\therefore 8 = 12x \times \frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = \frac{8}{12x}$$

$$= \frac{k}{x} \text{ where } k = \frac{8}{12} = \frac{2}{3}$$

Make $\frac{dx}{dt}$ the subject of the formula

b

$$V = x^3$$

The volume of a cube is x^3.

$$\therefore \frac{dV}{dx} = 3x^2$$

Differentiate to give $\frac{dV}{dx}$.

Use

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

Use the chain rule to correct $\frac{dV}{dt}, \frac{dV}{dx}$ and $\frac{dx}{dt}$.

$$= 3x^2 \frac{k}{x}$$

$$= 3kx = 2x$$

$$= 2V^{\frac{1}{3}}.$$

As $V = x^3$, so $x = V^{1/3}$

c Separate the variables

$$\int \frac{dV}{V^{\frac{1}{3}}} = \int 2\, dt$$

$$\therefore \int V^{-\frac{1}{3}}\, dV = \int 2\, dt$$

Integrate and include an arbitrary constant A.

i.e. $\frac{3}{2}V^{\frac{2}{3}} + A = 2t$

But $V = 8$ when $t = 0$

$$\therefore \frac{3}{2} \times 4 + A = 0$$

i.e. $A = -6$

Use the initial condition to find A.

$\therefore \frac{3}{2}V^{\frac{2}{3}} - 6 = 2t \quad *$

when

$V = 16\sqrt{2}$

Substitute $V = 16\sqrt{2}$ into the solution of the differential equation to find t.

$V = 2^{4\frac{1}{2}}$

$\therefore V^{\frac{2}{3}} = (2^{\frac{9}{2}})^{\frac{2}{3}}$

$= 2^3$

Substitute into *

$\therefore \frac{3}{2} \times 8 - 6 = 2t$

i.e. $t = 3$

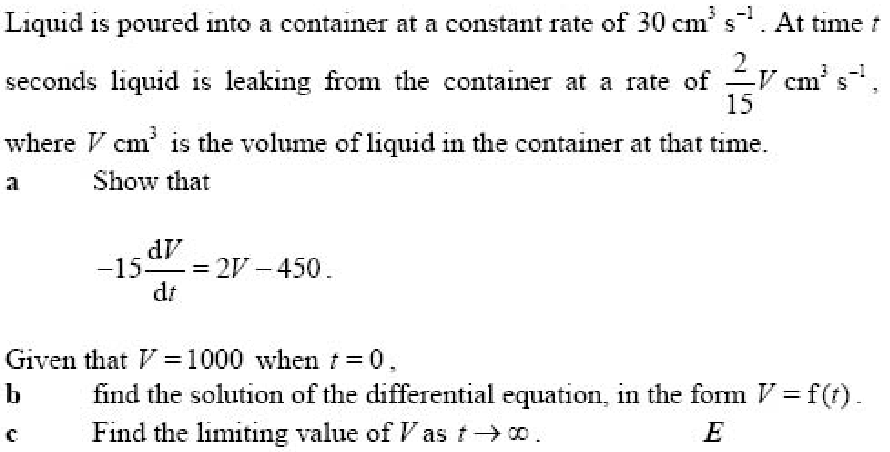
Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 42

Question:



Liquid is poured into a container at a constant rate of 30 $\text{cm}^3\ \text{s}^{-1}$. At time t seconds liquid is leaking from the container at a rate of $\frac{2}{15}V\ \text{cm}^3\ \text{s}^{-1}$, where $V\ \text{cm}^3$ is the volume of liquid in the container at that time.

a Show that

$$-15\frac{\mathrm{d}V}{\mathrm{d}t} = 2V - 450\,.$$

Given that $V = 1000$ when $t = 0$,

b find the solution of the differential equation, in the form $V = \mathrm{f}(t)$.

c Find the limiting value of V as $t \to \infty$. ***E***

Solution:

This is liquid being poured in and increases the volume in the container.

a $\frac{\mathrm{d}V}{\mathrm{d}t} = 30 - \frac{2}{15}V$

Multiply this equation by -15

This is denoting the liquid leaking out, so decreases the volume in the container. So you need a minus sign.

$\therefore -15\frac{\mathrm{d}V}{\mathrm{d}t} = 2V = 450$

This answer was given in the question.

b Separate the variables:

$\int \frac{-15\,\mathrm{d}V}{2V-450} = \int \mathrm{d}t.$

$\therefore -\frac{15}{2}\ln|2V-450| = t + c$

Integrate and include a constant on one side of your equation.

Given that $V = 1000$ when $t = 0$

$$
\begin{aligned}
\therefore \frac{-15}{2}\ln 1550 &= c \\
\therefore \frac{-15}{2}\ln\frac{2V-450}{1550} &= t \\
\therefore \ln\frac{V-225}{775} &= \frac{-2}{15}t \\
\therefore \frac{V-225}{775} &= \mathrm{e}^{-\frac{2}{15}t} \\
\therefore V &= 225 + 775\mathrm{e}^{-\frac{2}{15}t}
\end{aligned}
$$

Use the initial condition to find c.

Collect the two ln terms together

$\frac{-15}{2}[\ln(2V-450) - \ln 1550]$

$= -\frac{15}{2}\ln\frac{2V-450}{1550}.$

Take exponentials of both sides.

Make V the subject of the formula.

c As $t \to \infty, \mathrm{e}^{-\frac{2}{15}t} \to 0$

$\therefore$ limiting value of V is 225

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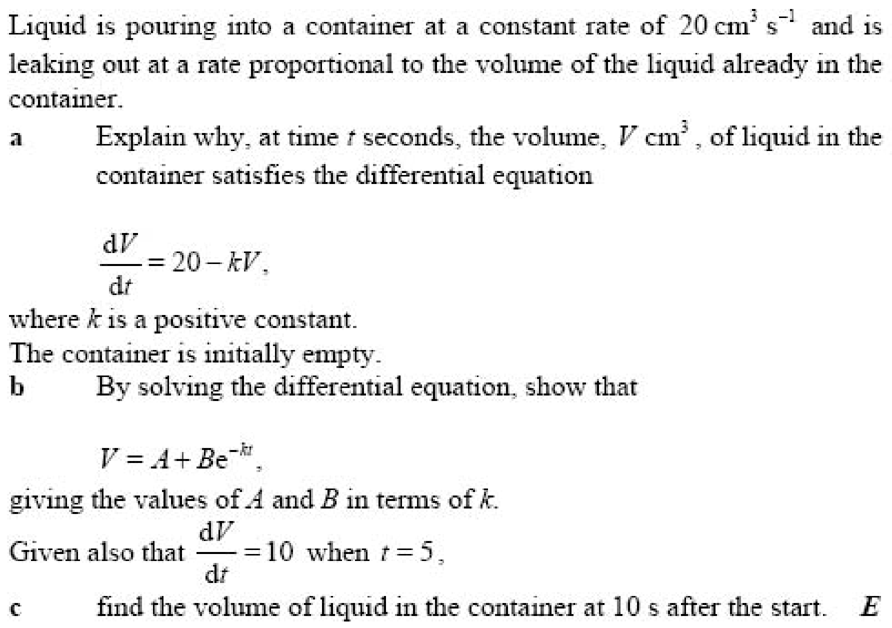
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 43

Question:



Liquid is pouring into a container at a constant rate of $20 \text{ cm}^3 \text{ s}^{-1}$ and is leaking out at a rate proportional to the volume of the liquid already in the container.

a Explain why, at time t seconds, the volume, $V \text{ cm}^3$, of liquid in the container satisfies the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 20 - kV,$$

where k is a positive constant.

The container is initially empty.

b By solving the differential equation, show that

$$V = A + B\mathrm{e}^{-kt},$$

giving the values of A and B in terms of k.

Given also that $\frac{\mathrm{d}V}{\mathrm{d}t} = 10$ when $t = 5$,

c find the volume of liquid in the container at 10 s after the start. ***E***

Solution:

a Rate of change of volume is $\frac{dV}{dt}\text{cm}^3\ \text{s}^{-1}$

Increase is $20\ \text{cm}^3\ \text{s}^{-1}$

Decrease is $kV\ \text{cm}^3\ \text{s}^{-1}$, where k is constant of proportionality.

Explain the minus sign and the function of the constant k.

$$\therefore \frac{dV}{dt} = 20 - kV$$

b Separate the variables:

$$\int \frac{dV}{20-kV} = \int dt$$

$$\therefore -\frac{1}{k}\ln|20-kV| = t + c$$

You need to include a constant of integration c.

When $t = 0, V = 0$

You were told that the container was initially empty i.e. $V = 0$ when $t = 0$.

$$\therefore -\frac{1}{k}\ln 20 \quad = \quad c$$

Use this to find c.

$$\therefore -\frac{1}{k}\ln\frac{20-kV}{20} \quad = \quad t$$

Combine the two in terms together as

$$-\frac{1}{k}(\ln(20-kV) - \ln 20) = -\frac{1}{k}\ln\frac{20-kV}{20}$$

Multiply both sides by $-k$

$$\therefore \ln\frac{20-kV}{20} \quad = \quad -kt$$

$$\therefore \frac{20-kV}{20} \quad = \quad e^{-kt}$$

Take exponentials of each side.

$$\therefore kV \quad = \quad 20 - 20e^{-kt}$$

$$\therefore V \quad = \quad \frac{20}{k} - \frac{20}{k}e^{-kt} \ *$$

Rearrange to give V as the subject of the formula.

i.e. $A = \frac{20}{k}$ and $B = -\frac{20}{k}$

Differentiate the equation *

c

$$\frac{dV}{dt} = 20e^{-kt}$$

Differentiate to give $\frac{dV}{dt}$.

Substitute $\frac{dV}{dt} = 10$ when $t = 5$

$$\therefore 10 \quad = \quad 20\,e^{-5k}$$

use the given information to find k.

$$\therefore e^{-5k} \quad = \quad \frac{1}{2}$$

Taking lns:

$$-5k = \ln\frac{1}{2} \text{ or } 5k = \ln 2$$

$$\therefore k = \frac{1}{5}\ln 2 \text{ or } 0.1386 \text{ (4 d.p.)}$$

Substitute into equation *

$$V = \frac{100}{\ln 2} - \frac{100}{\ln 2}\left(\frac{1}{2}\right)^{\frac{t}{5}}$$

This is the particular solution of the differential equation.

When $t = 10$

$$V = \frac{100}{\ln 2} - \frac{100}{\ln 2} \times \frac{1}{4}$$

$$= \frac{75}{\ln 2}$$

$$= 108.2 \text{ (1 d.p.)}$$

Give V to a suitable accuracy.

or Volume $= 108 \text{ cm}^3$ (3 s.f.)

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 44

Question:

a Express $\frac{2x-1}{(x-1)(2x-3)}$ in partial fractions.

b Given that $x \geq 2$, find the general solution of the differential equation

$$(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y.$$

c Hence find the particular solution of this differential equation that satisfies $y = 10$ at $x = 2$, giving your answer in the form $y = f(x)$.
E

Solution:

a

$$\frac{2x-1}{(x-1)(2x-3)} \equiv \frac{A}{(x-1)} + \frac{B}{(2x-3)}$$

Use denominators $(x-1)$ and $(2x-3)$.

$A = -1$ and $B = 4$

Compare numerators to find A and B (see earlier question 1).

$$\therefore \frac{2x-1}{(x-1)(2x-3)} \equiv \frac{-1}{x-1} + \frac{4}{2x-3} \quad *$$

b $(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y$

Separating the variables.

$$\int \frac{dy}{y} = \int \frac{(2x-1)dx}{(2x-3)(x-1)}$$

Use the partial fractions from part a to split this fraction.

$$\therefore \ln y = \int \frac{-1}{x-1}dx + \int \frac{4}{2x-3}dx$$

These fractions can be integrated to give ln functions.

$$= -\ln|x-1| + 2\ln|2x-3| + c$$

$$\therefore \ln y = -\ln|x-1| + \ln(2x-3)^2 + \ln A$$

Express the constant as $\ln A$.

$$\ln y = \ln A\frac{(2x-3)^2}{(x-1)}$$

Combine the ln terms using the law for combining logs.

$$\therefore y = \frac{A(2x-3)^2}{(x-1)}$$

Make y the subject of the formula.

c Given $y = 10$ when $x = 2$

$\therefore 10 = A$

Use the given coordinates to find the value of the constant.

$\therefore$ Particular solution is

$$y = \frac{10(2x-3)^2}{(x-1)}.$$

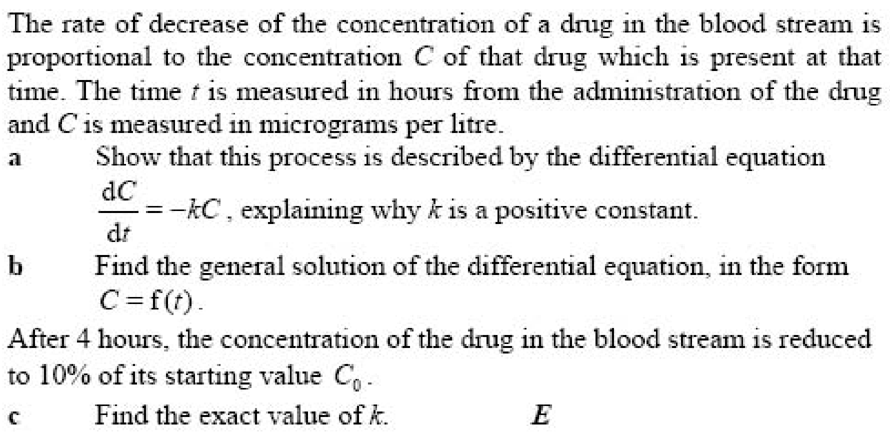
Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 45

Question:



The rate of decrease of the concentration of a drug in the blood stream is proportional to the concentration C of that drug which is present at that time. The time t is measured in hours from the administration of the drug and C is measured in micrograms per litre.

a Show that this process is described by the differential equation $\frac{dC}{dt} = -kC$, explaining why k is a positive constant.

b Find the general solution of the differential equation, in the form $C = f(t)$.

After 4 hours, the concentration of the drug in the blood stream is reduced to 10% of its starting value C_0.

c Find the exact value of k. ***E***

Solution:

a $\frac{dC}{dt}$ is the rate of change of concentration.

Explain the term $\frac{dC}{dt}$

$\frac{dC}{dt} = -kC$, because k is the constant of proportionality.

Explain the nature of k.

The negative sign and $k > 0$ indicates rate of decrease.

Explain the negative sign.

b Separate the variables.

$$\int \frac{dC}{C} = -\int k \, dt$$

$$\therefore \ln C = -kt + \ln A,$$

where $\ln A$ is a constant.

Give the constant of integration as $\ln A$.

$$\therefore \ln \frac{C}{A} = -kt$$

Combine $\ln C - \ln A = \ln \frac{C}{A}$.

$$\therefore \frac{C}{A} = e^{-kt}$$

Take exponentials of each side of the equation.

$$\therefore C = Ae^{-kt}$$

c When $t = 0, C = C_0$

The starting concentration is C_0.

$$\Rightarrow C_0 = A$$

$$\therefore C = C_0 e^{-kt}$$

when $t = 4, C = \frac{1}{10} C_0$

The concentration is 10% of its starting value after 4 hours.

$$\therefore \frac{1}{10} C_0 = C_0 e^{-4k}$$

$$\therefore e^{4k} = 10$$

$$\text{i.e. } 4k = \ln 10$$

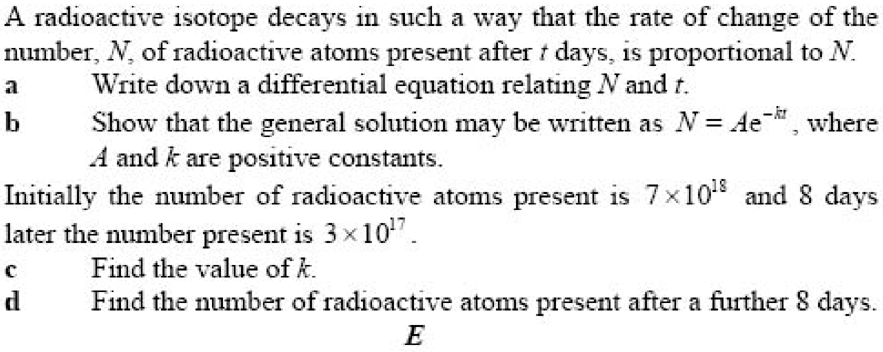
$$\therefore k = \frac{1}{4} \ln 10$$

As an exact value of k is required – give this answer.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 46

Question:



A radioactive isotope decays in such a way that the rate of change of the number, N, of radioactive atoms present after t days, is proportional to N.

a Write down a differential equation relating N and t.

b Show that the general solution may be written as $N = Ae^{-kt}$, where A and k are positive constants.

Initially the number of radioactive atoms present is 7×10^{18} and 8 days later the number present is 3×10^{17}.

c Find the value of k.

d Find the number of radioactive atoms present after a further 8 days.

E

Solution:

a $\frac{dN}{dt} = -kN$, where k is a positive constant.

$\frac{dN}{dt}$ is rate of change of N.

k is the constant of proportionality.

$-$ sign because 'decays' implies that N is decreasing.

b Separate the variables

$$\therefore \int \frac{dN}{N} = -\int k\,dt$$

$$\therefore \ln N = -kt + \ln A$$

$$\therefore \ln \frac{N}{A} = -kt$$

$$\therefore \frac{N}{A} = e^{-kt}$$

$$\therefore N = Ae^{-kt} \quad *$$

Put your arbitrary constant as $\ln A$.

Collect the ln terms, as $\ln N - \ln A = \ln\frac{N}{A}$.

Take exponentials to give the required answer.

c When $t = 8, N = 7\times10^{18}$

$\therefore 7\times10^{18} = A$

when $t = 8, N = 3\times10^{17}$

$$\therefore 3\times10^{17} = 7\times10^{18}e^{-8k}$$

$$\therefore e^{8k} = \frac{7\times10^{18}}{3\times10^{17}}$$

$$= \frac{70}{3}$$

$$\therefore 8k = \ln\left[\frac{70}{3}\right]$$

$$\therefore k = \frac{1}{8}\ln\left[\frac{70}{3}\right] = 0.394 \text{ (3 s.f.)}$$

Initially $-$ means when $t = 0$ substitute into equation *

Substitute $t = 8, N = 3\times10^{17}$ and $A = 7\times10^{18}$ into equation *.

Take lns of both sides of the equation.

d When $t = 16$

$$N = 7\times10^{18}e^{-2\ln\frac{70}{3}}$$

$$= 7\times10^{18} \div e^{\ln\left(\frac{70}{3}\right)^2}$$

$$= 7\times10^{18}\times\frac{9}{4900}$$

$$= 1.286\times10^{16}\text{(4 s.f.)}$$

After a further 8 days means that $t = 16$.

Substitute $A = 7\times10^{18}, kt = 2\ln\frac{70}{3}$ into equation *.

Give your answer to an appropriate accuracy.

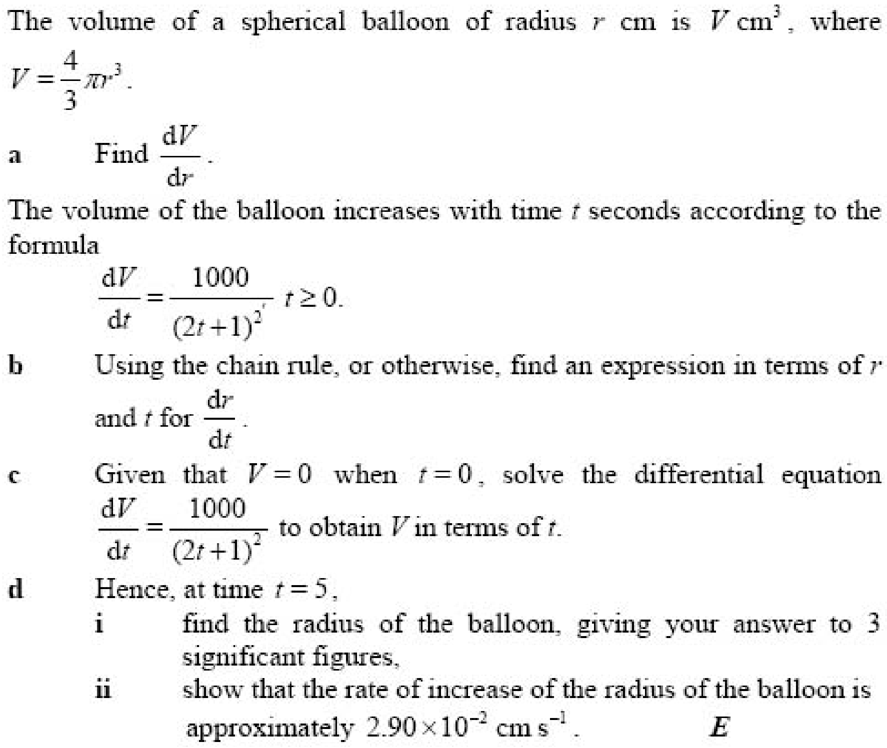
Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 47

Question:



The volume of a spherical balloon of radius r cm is $V\text{ cm}^3$, where $V = \frac{4}{3}\pi r^3$.

a Find $\frac{dV}{dr}$.

The volume of the balloon increases with time t seconds according to the formula

$$\frac{dV}{dt} = \frac{1000}{(2t+1)^2},\ t \geq 0.$$

b Using the chain rule, or otherwise, find an expression in terms of r and t for $\frac{dr}{dt}$.

c Given that $V = 0$ when $t = 0$, solve the differential equation $\frac{dV}{dt} = \frac{1000}{(2t+1)^2}$ to obtain V in terms of t.

d Hence, at time $t = 5$,

- **i** find the radius of the balloon, giving your answer to 3 significant figures,
- **ii** show that the rate of increase of the radius of the balloon is approximately $2.90 \times 10^{-2}\text{ cm s}^{-1}$. ***E***

Solution:

a

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

b From the chain rule:-

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

Use the chain rule to connect $\frac{dV}{dt}, \frac{dV}{dr}$ and $\frac{dr}{dt}$.

As

$$\frac{dV}{dt} = \frac{1000}{(2t+1)^2}$$

$$\therefore \frac{1000}{(2t+1)^2} = 4\pi r^2 \times \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{250}{\pi(2t+1)^2 r^2} \quad *$$

Make $\frac{dr}{dt}$ the subject of the formula.

c $\frac{dV}{dt} = \frac{1000}{(2t+1)^2}$

Separating the variables.

$$\int dV = \int \frac{1000}{(2t+1)^2} dt$$

This integration is reverse of the chain rule.

$$\therefore V = -500(2t+1)^{-1} + c$$

Do not forget the constant of integration, c.

But $V = 0$ when $t = 0$

$$\therefore 0 = -500 + c$$

$$\text{i.e. } c = 500$$

Use initial conditions to find c.

$$\therefore V = 500 - \frac{500}{(2t+1)}$$

d **(i)** When $t = 5$,

$$V = 500 - \frac{500}{11}$$

$$= 454.5\cdots$$

Find volume and then use $V = \frac{4}{3}\pi r^3$ to find the radius $r = \sqrt[3]{\frac{V}{\frac{4}{3}\pi}}$.

Using

$$V = \frac{4}{3}\pi r^3 = 454.5\ldots$$

$$\therefore r = 4.77 \text{ (3 s.f.)}$$

(ii) Substitute $r = 4.77,\ t = 5$ into $*$

Use the answer to part **b**.

$$\therefore \frac{dr}{dt} = 0.0289\ldots \approx 2.90 \times 10^{-2}$$

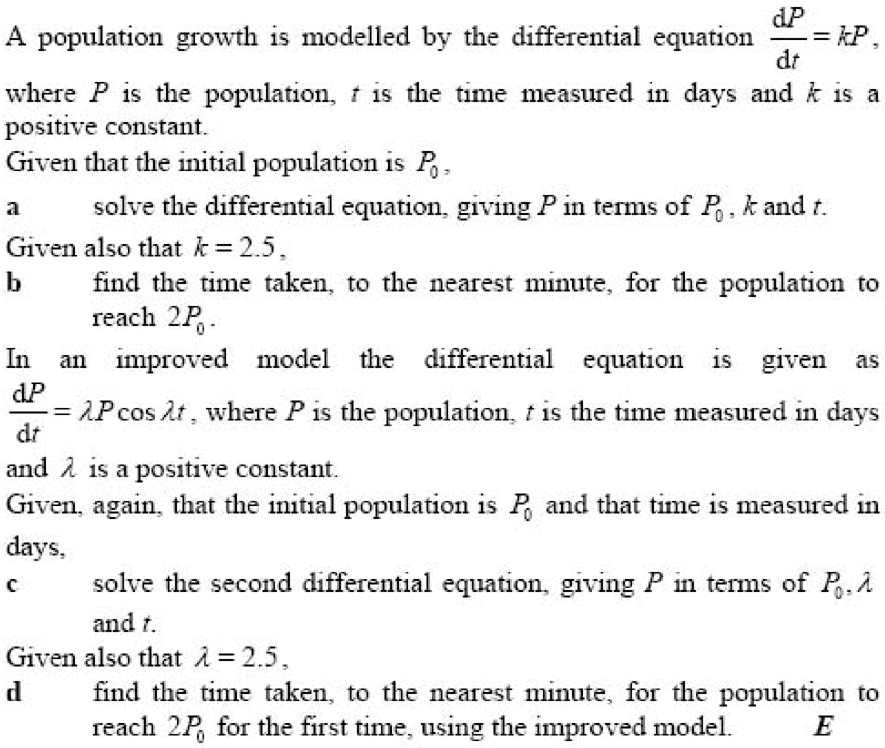
Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 48

Question:



A population growth is modelled by the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$, where P is the population, t is the time measured in days and k is a positive constant.

Given that the initial population is P_0,

a solve the differential equation, giving P in terms of P_0, k and t.

Given also that $k = 2.5$,

b find the time taken, to the nearest minute, for the population to reach $2P_0$.

In an improved model the differential equation is given as $\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda P \cos \lambda t$, where P is the population, t is the time measured in days and λ is a positive constant.

Given, again, that the initial population is P_0 and that time is measured in days,

c solve the second differential equation, giving P in terms of P_0, λ and t.

Given also that $\lambda = 2.5$,

d find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model. ***E***

Solution:

a

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

Separate the variables.

$$\int \frac{\mathrm{d}P}{P} = \int k\,\mathrm{d}t$$

$$\therefore \ln P = kt + \ln P_0$$

(as $P = P_0$ when $t = 0$)

$\ln P_0$ is the arbitrary constant which is found from the initial condition.

$$\therefore \ln P - \ln P_0 = kt$$

$$\ln \frac{P}{P_0} = kt$$

i.e. $$\therefore \frac{P}{P_0} = e^{kt}$$

Collect the two ln terms and use the law that $\ln P - \ln P_0 = \ln \frac{P}{P_0}$.

$$\therefore P = P_0\,e^{kt}$$

Take exponentials and make P the subject of the formula.

b Substitute $k = 2.5$ and $P = 2P_0$

$$\therefore 2P_0 = P_0 e^{2.5t}$$

$$\therefore e^{2.5t} = 2$$

$$\therefore 2.5t = \ln 2$$

Take lns and make t the subject of the formula.

$$t = \frac{1}{2.5}\ln 2$$

$$= 0.277\ldots \text{days}$$

$$= 6.65\text{ h}$$

$$= 6\text{ h}39\text{ minutes}$$

The units are days and need to be converted to minutes, so multiply by 24 then by 60.

c $$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda P \cos \lambda t$$

Separate the variables.

$$\int \frac{\mathrm{d}P}{P} = \int \lambda \cos \lambda t\,\mathrm{d}t$$

$$\therefore \ln P = \sin \lambda t + \ln P_0$$

$$\therefore \ln \frac{P}{P_0} = \sin \lambda t$$

$$\therefore P = P_0 e^{\sin \lambda t}$$

The method is similar to that used in part **a**.

d Substitute $P = 2P_0$ and $\lambda = 2.5$

$$\therefore e^{\sin 2.5t} = 2$$
$$\therefore \sin 2.5t = \ln 2$$
$$\therefore 2.5t = \sin^{-1}(\ln 2)$$

$\therefore t = 0.306$ days

$= 7.35$ h

$= 441$ mins or

7 h 21 min

Use radians to calculate $\sin^{-1}(\ln 2)$.

Again change the time from days to minutes.

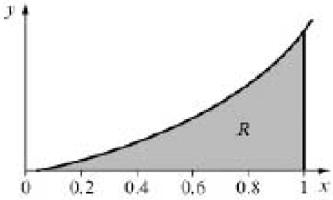
Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 49

Question:



The diagram shows the graph of the curve with equation

$y = xe^{2x}, x \geq 0.$

The finite region R bounded by the lines $x = 1$, the x-axis and the curve is shown shaded in the diagram.

a Use integration to find the exact value of the area for R.

b Complete the table with the values of y corresponding to $x = 0.4$ and 0.8.

x	0	0.2	0.4	0.6	0.8	1
$y = xe^{2x}$	0	0.29836		1.99207		7.38906

c Use the trapezium rule with all the values in the table to find an approximate value of this area, giving your answer to 4 significant figures. ***E***

Solution:

a Let $I = \int_0^1 x\,e^{2x}\,dx$

Let $u = x$ and $\frac{dv}{dx} = e^{2x}$.

$u = x \Rightarrow \frac{du}{dx} = 1$

$v = \frac{1}{2}e^{2x} \Leftarrow \frac{dv}{dx} = e^{2x}$

Complete the table for u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$.
Take care to differentiate u but integrate $\frac{dv}{dx}$.

Use the integration by parts formula

$$I = \left[\frac{1}{2}x\,e^{2x}\right]_0^1 - \int_0^1 1 \times \frac{1}{2}e^{2x}\,dx$$

$$= \left[\frac{1}{2}xe^{2x}\right]_0^1 - \left[\frac{1}{4}e^{2x}\right]_0^1$$

$$= \frac{1}{2}e^2 - \left[\frac{1}{4}e^2 - \frac{1}{4}\right]$$

$$= \frac{1}{4}e^2 + \frac{1}{4}$$

Notice that $\int \not\equiv \frac{du}{dx}\,dx$ is a simpler integral than $\int u\frac{dv}{dx}\,dx$.

Apply the limits on the uv term and to the integral term.

b

x	0	0.2	0.4	0.6	0.8	1
y	0	0.29836	0.89022	1.99207	3.96243	7.38906

Complete the table to find the values of y.

c

$$I = \frac{1}{2} \times 0.2\left[0 + 2(0.29836 + 0.89022 + 1.99207 + 3.96243) + 7.38906\right]$$

$$= 0.1[21.67522]$$

$$= 2.168 \text{ (4 s.f.)}$$

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 50

Question:

a Given that $y = \sec x$, complete the table with the values of y corresponding to $x = \frac{\pi}{16}, \frac{\pi}{8}$ and $\frac{\pi}{4}$.

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	1			1.20269	

b Use the trapezium rule, with all the values for y in the completed table, to obtain an estimate for $\int_0^{\frac{\pi}{4}} \sec x \, dx$.

Show all the steps of your working and give your answer to 4 decimal places.

The exact value of $\int_0^{\frac{\pi}{4}} \sec x \, dx$ is $\ln(1+\sqrt{2})$.

c Calculate the % error in using the estimate you obtained in part **b**.

E

Solution:

a

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	1	1.01959	1.08239	1.20269	1.41421

Complete the table. Ensure that your calculator is set to use radians and use $\left(\sec\frac{\pi}{16}\right) = \left[\cos\left(\frac{\pi}{16}\right)\right]^{-1}$ or $\frac{1}{\cos\frac{\pi}{16}}$.

b

$$I = \frac{1}{2}\cdot\frac{\pi}{16}\left[1+2(1.01959+1.08239+1.20269)+1.41421\right]$$

$$= \frac{\pi}{32}\times 9.02355$$

$$= 0.88588\ldots = 0.8859 \text{ (4 d.p.)}$$

c Percentage error is

$$\frac{\left(0.8859-\ln(1+\sqrt{2})\right)}{\ln\left(1+\sqrt{2}\right)}\times 100 =$$

0.5136% (4 d.p.)

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 51

Question:

$I = \int_0^5 e^{\sqrt{(3x+1)}}\,dx$.

a Given that $y = e^{\sqrt{(3x+1)}}$, complete the table with the values of y corresponding to $x = 2$, 3 and 4.

x	0	1	2	3	4	5
y	e^1	e^2				e^4

b Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the original integral I, giving your answer to 4 significant figures.

c Use the substitution $t = \sqrt{(3x+1)}$ to show that I may be expressed as $\int_a^b kte^t\,dt$, giving the values of a, b and k.

d Use integration by parts to evaluate this integral, and hence find the value of I correct to 4 significant figures, showing all the steps in your working. ***E***

Solution:

a

x	0	1	2	3	4	5
y	e^1	e^2	14.094	23.624	36.802	e^4

You could complete the table with $e^{\sqrt{7}}$, $e^{\sqrt{10}}$ and $e^{\sqrt{13}}$.

b

$$I = \frac{1}{2}\times 1\left[e^1 + 2\left(e^2 + 14.094 + 23.624 + 36.802\right) + e^4\right]$$

$$= \frac{1}{2}\times 221.1\ldots$$

$$= 110.6 \text{ (4 s.f.)}$$

c $I = \int_0^5 e^{\sqrt{(3x+1)}}\,dx$

You need to replace each x term with a corresponding t term. First replace dx with a term in dt.

Let

$$t = \sqrt{(3x+1)}$$

$$\frac{dt}{dx} = \frac{3}{2}(3x+1)^{-\frac{1}{2}} = \frac{3}{2t}$$

Replace dx with $\frac{2t}{3}dt$

Use $t = \sqrt{(3x+1)}$ to change the limits. When $x = 0, t = 1$ and when $x = 5, t = 4$.

x	t
0	1
5	4

So $I = \int_1^4 e^t \cdot \frac{2t}{3}dt = \int_1^4 \frac{2}{3}te^t\,dt$

i.e. $a = 1, b = 4$ and $k = \frac{2}{3}$.

d

$$u = \frac{2}{3}t \Rightarrow \frac{du}{dt} = \frac{2}{3}$$

Let $u = \frac{2}{3}t$ and $\frac{dv}{dt} = e^t$

$$v = e^t \Leftarrow \frac{dv}{dt} = e^t$$

Complete the table for $u, v, \frac{du}{dt}$ and $\frac{dv}{dt}$.

$$\therefore I = \left[\frac{2}{3}t\,e^t\right]_1^4 - \int_1^4 \frac{2}{3}e^t\,dt$$

$$= \frac{8}{3}e^4 - \frac{2}{3}e - \left[\frac{2}{3}e^t\right]_1^4$$

$$= \frac{8}{3}e^4 - \frac{2}{3}e - \frac{2}{3}e^4 + \frac{2}{3}e$$

Apply the limits to both terms.

$$= 2e^4$$

$$= 109.2 \text{ (4 s.f.)}$$

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 52

Question:

The following is a table of values for $y = \sqrt{(1 + \sin x)}$, where x is in radians.

x	0	0.5	1	1.5	2
y	1	1.216	p	1.413	q

a Find the value of p and the value of q.

b Use the trapezium rule and all the values of y in the completed table to obtain an estimate of I, where

$$I = \int_0^2 \sqrt{(1 + \sin x)}\, dx.$$ ***E***

Solution:

a

$p = 1.357 \text{ (3 d.p.)}$

$q = 1.382 \text{ (3 d.p.)}$

Your calculator should be in radian mode.

b Using the trapezium rule

$$I = \frac{1}{2} \times 0.5[1 + 2(1.216 + 1.357 + 1.413) + 1.382]$$

$$= 0.25 \times 10.354$$

$$= 2.5885$$

$$= 2.589 \text{ (4 s.f.)}$$

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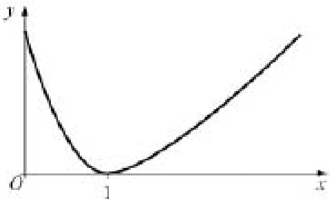
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 53

Question:



The figure shows a sketch of the curve with equation $y = (x-1)\ln x, x > 0$.

a Copy and complete the table with the values of y corresponding to $x = 1.5$ and $x = 2.5$.

x	1	1.5	2	2.5	3
y	0		ln 2		2 ln 3

Given that $I = \int_1^3 (x-1)\ln x \, dx$,

b Use the trapezium rule

- **i** with values of y at $x = 1, 2$ and 3 to find an approximate value for I to 4 significant figures,
- **ii** with values of y at $x = 1, 1.5, 2, 2.5$ and 3 to find another approximate value for I to 4 significant figures.

c Explain, with reference to the figure, why an increase in the number of values improves the accuracy of the approximation.

d Show, by integration, that the exact value of $\int_1^3 (x-1)\ln x \, dx$ is $\frac{3}{2}\ln 3$. ***E***

Solution:

a

x	1	1.5	2	2.5	3
y	0	0.5 ln 1.5	ln 2	1.5 ln 2.5	2 ln 3

You may leave your answers in terms of ln at this stage.

b **i** Trapezium rule with 2 strips

$$I = \frac{1}{2}\times 1\left[0+2\times\ln 2+2\ln 3\right]$$

$$= \frac{1}{2}\times 3.5835\ldots$$

$$= 1.792 \text{ (4 s.f.)}$$

ii Trapezium rule with 4 strips:

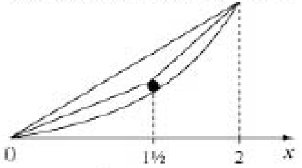
$$I = \frac{1}{2}\times 0.5\left[0+2\left(0.5\ln 1.5+\ln 2+1.5\ln 2.5\right)+2\ln 3\right]$$

Show all your working.

$$= 0.25\times 6.737856\cdots$$

$$= 1.684 \text{ (4 s.f.)}$$

c The trapezia are closer to the required area when there are more strips.



A diagram can help you to explain.

d Let $I = \int_1^3 (x-1)\ln x \, dx$.

Let $u = \ln x$ and $\frac{dv}{dx} = x-1$.

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$v = \frac{x^2}{2} - x \Leftarrow \frac{dv}{dx} = x - 1$$

Complete the table for $u, v, \frac{du}{dx}$ and $\frac{dv}{dx}$.

$$\therefore I = \left[\left(\frac{x^2}{2}-x\right)\ln x\right]_1^3 - \int_1^3 \frac{1}{x}\cdot\left(\frac{x^2}{2}-x\right)dx$$

Apply the limits to the uv term and to the integral term.

$$= \frac{3}{2}\ln 3 - \int_1^3\left(\frac{x}{2}-1\right)dx$$

$$= \frac{3}{2}\ln 3 - \left[\frac{x^2}{4}-x\right]_1^3$$

$$= \frac{3}{2}\ln 3 - \left[\left(\frac{9}{4}-3\right)-\left(\frac{1}{4}-1\right)\right]$$

$$= \frac{3}{2}\ln 3$$

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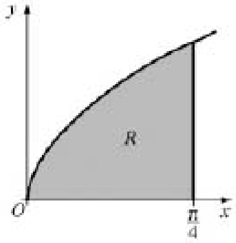
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 54

Question:



The figure shows part of the curve with equation $\sqrt{(\tan x)}$. The finite region R, which is bounded by the curve, the x-axis and the line $x = \frac{\pi}{4}$, is shown shaded in the figure.

a Given that $y = \sqrt{(\tan x)}$, copy and complete the table with the values of y corresponding to $x = \frac{\pi}{16}, \frac{\pi}{8}$ and $\frac{3\pi}{16}$, giving your answers to 5 decimal places.

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	0				1

b Use the trapezium rule with all the values of y in the completed table to obtain an estimate for the area of the shaded region R, giving your answer to 4 decimal places.

The region R is rotated through 2π radians around the x-axis to generate a solid of revolution.

c Use integration to find an exact value for the volume of the solid generated. ***E***

Solution:

a

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	0	0.44600	0.64360	0.81742	1

Ensure that your calculator is in radian mode.

b From the trapezium rule

$$\text{Area} \approx \frac{1}{2}\times\frac{\pi}{16}[0+2(0.44600+0.64360+0.81742)+1]$$

$$\approx \frac{\pi}{32}\times 4.81404$$

$$= 0.4726$$

c Volume

Use the formula $v = \pi\int y^2 \, dx$.

$$= \pi\int_0^{\frac{\pi}{4}} (\sqrt{\tan x})^2 \, dx$$

$$= \pi\int_0^{\frac{\pi}{4}} \tan x \, dx$$

$\int \tan x \, dx = \ln|\cos x|$ - is given in your formula book.

$$= \pi\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$$

or $\pi[\ln\sec x]_0^{\frac{\pi}{4}}$

$$= \pi[-\ln\cos x]_0^{\frac{\pi}{4}}$$

$$= \pi\left[-\ln\frac{1}{\sqrt{2}}\right]$$

$$= \pi\ln\sqrt{2} \text{ or } \frac{1}{2}\pi\ln 2$$

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 55

Question:

Using the substitution $u^2 = 2x - 1$, or otherwise, find the exact value of

$$\int_1^5 \frac{3x}{\sqrt{(2x-1)}}\,dx.$$ ***E***

Solution:

Let $I = \int_1^5 \frac{3x}{\sqrt{(2x-1)}}\,dx$

Replace each term in x with a term in u.

Let $u^2 = 2x - 1$

$2u\frac{du}{dx} = 2$

So replace dx with u du and $x = \frac{u^2+1}{2}$

Also $\frac{1}{\sqrt{(2x-1)}} = \frac{1}{u}$

x	u
1	1
5	3

Change the limits. When $x = 1$, $u^2 = 1$ and when $x = 5, u^2 = 9$.

So

$$I = \int_1^3 \frac{\frac{3(u^2+1)}{2}}{u} \times u\,du$$

$$= \int_1^3 \left(\frac{3}{2}u^2 + \frac{3}{2}\right) du.$$

Simplify and integrate.

$$= \left[\frac{1}{2}u^3 + \frac{3}{2}u\right]_1^3$$

$$= \left(\frac{27}{2} + \frac{9}{2}\right) - \left(\frac{1}{2} + \frac{3}{2}\right)$$

Evaluate the integral using the new u limits.

$$= 18 - 2$$

$$= 16$$

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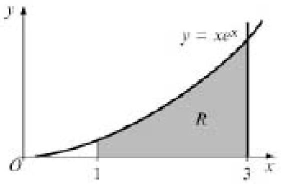
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 56

Question:



The figure shows the finite region R, which is bounded by the curve $y = xe^x$, the line $x = 1$, the line $x = 3$ and the x-axis.
The region R is rotated through 360 degrees about the x-axis.
Use integration by parts to find an exact value for the volume of the solid generated. ***E***

Solution:

Use

$$V = \pi \int y^2 \mathrm{d}x$$

$$= \pi \int_1^3 x^2 e^{2x} \mathrm{d}x$$

Let $u = x^2$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^{2x}$

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

$$v = \frac{1}{2}e^{2x} \Leftarrow \frac{\mathrm{d}v}{\mathrm{d}x} = e^{2x}$$

Complete the table for u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$. Take care to differentiate u but integrate $\frac{\mathrm{d}v}{\mathrm{d}x}$.

$$\therefore V = \pi\left[x^2 \cdot \frac{1}{2}e^{2x}\right]_1^3 - \pi\int_1^3 \frac{1}{2}e^{2x} \cdot 2x \,\mathrm{d}x.$$

$$\text{i.e. } V = \pi\left[\frac{9}{2}e^6 - \frac{1}{2}e^2\right] - \pi\int_1^3 xe^{2x} \,\mathrm{d}x.$$

This integral is simpler than V but still not one you can write down. Use integration by parts again with $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^{2x}$.

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$v = \frac{1}{2}e^{2x} \Leftarrow \frac{\mathrm{d}v}{\mathrm{d}x} = e^{2x}$$

Complete a new table for the new u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.

$$\therefore V = \pi\left[\frac{9}{2}e^6 - \frac{1}{2}e^2\right] - \pi\left[x \cdot \frac{1}{2}e^{2x}\right]_1^3 + \pi\int_1^3 \frac{1}{2}e^{2x} \cdot 1 \,\mathrm{d}x$$

$$= \pi\left[\frac{9}{2}e^6 - \frac{1}{2}e^2\right] - \pi\left[\frac{3}{2}e^6 - \frac{1}{2}e^2\right] + \pi\left[\frac{1}{4}e^{2x}\right]_1^3$$

$$= \frac{13}{4}\pi e^6 - \frac{\pi}{4}e^2$$

Apply the integration by parts formula a second time.

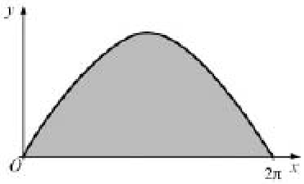
Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 57

Question:



The curve with equation $y = 3\sin\frac{x}{2}, 0 \le x \le 2\pi$, is shown in the figure.

The finite region enclosed by the curve and the x-axis is shaded.

a Find, by integration, the area of the shaded region.

This region is rotated through 2π radians about the x-axis.

b Find the volume of the solid generated.

Solution:

a

$$\text{Area} = \int_0^{2\pi} 3\sin\frac{x}{2}\,dx$$

Recall (5) in the introduction to integration. Integrating a sin function gives a change of sign and a cos function.

$$= \left[3 \times -2\cos\frac{x}{2}\right]_0^{2\pi}$$

The 2 here is obtained from dividing by $\frac{1}{2}$ which arises from the chain rule.

$$= \left[-6\cos\frac{x}{2}\right]_0^{2\pi}$$

$$= [6 - (-6)]$$

$$\text{Area} = 12$$

b

$$\text{Volume} = \pi\int_0^{2\pi} 9\sin^2\frac{x}{2}\,dx$$

You cannot integrate $\sin^2\frac{x}{2}$, but you can write this in terms of $\cos x$.

Recall $\cos 2A = 1 - 2\sin^2 A$

So $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

So $\sin^2\frac{x}{2} = \frac{1}{2}(1 - \cos x)$

$$\therefore \text{Volume} = \frac{9\pi}{2}\int_0^{2\pi}(1 - \cos x)\,dx$$

You can now integrate each term directly.

$$= \frac{9\pi}{2}[x + \sin x]_0^{2\pi}$$

$$= \frac{9\pi}{2} \times (2\pi - 0)$$

$$\text{volume} = 9\pi^2$$

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 58

Question:

Use integration by parts to find the exact value of $\int_1^3 x^2 \ln x \, dx$. ***E***

Solution:

Let $I = \int_1^3 x^2 \ln x dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$v = \frac{x^3}{3} \Leftarrow \frac{dv}{dx} = x^2$

Since there is a ln x term

Let $u = \ln x$ and $\frac{dv}{dx} = x^2$.

Complete the table for $u, v, \frac{du}{dx}$ and $\frac{dv}{dx}$.

Take care to differentiate u but integrate $\frac{dv}{dx}$.

Using the integration by parts formula.

$$I = \left[\frac{x^3}{3}\ln x\right]_1^3 - \int_1^3 \frac{x^3}{3} \cdot \frac{1}{x} dx$$

Apply the integration by parts formula.

$$= 9\ln 3 - \int_1^3 \frac{x^2}{3} dx$$

Simplify the $v\frac{du}{dx}$ term.

$$= 9\ln 3 - \left[\frac{x^3}{9}\right]_1^3$$

$$= 9\ln 3 - \left[3 - \frac{1}{9}\right]$$

$$= 9\ln 3 - \frac{26}{9}.$$

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 59

Question:

Use the substitution $u = 1 - x^2$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{x^3}{(1-x^2)^{\frac{1}{2}}}\,\mathrm{d}x.$$

Solution:

Let $u = 1 - x^2$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = -2x$

This implies that $x\mathrm{d}x = -\frac{\mathrm{d}u}{2}$.

and $x^2 = 1 - u$

so $\int \frac{x^3}{(1-x^2)^{\frac{1}{2}}}\mathrm{d}x = \int \frac{x^2}{(1-x^2)^{\frac{1}{2}}}x\mathrm{d}x = \int \frac{1-u}{u^{\frac{1}{2}}}(-\frac{\mathrm{d}u}{2})$

Use $x^3 = x^2 x$ and $x^2 = 1 - u$ with $x\mathrm{d}x = -\frac{\mathrm{d}u}{2}$.

$= -\frac{1}{2}\int \frac{1-u}{u^{\frac{1}{2}}}\mathrm{d}u = -\frac{1}{2}\int u^{-\frac{1}{2}} - u^{\frac{1}{2}}\mathrm{d}u = \left[-u^{\frac{1}{2}} + \frac{1}{3}u^{\frac{3}{2}}\right]$

Simplify and integrate

As limits for x were 0 and $\frac{1}{2}$, limits for u are 1 and $\frac{3}{4}$

As the variable has changed, so must the limits. So use $u = 1 - x^2$ to find the new limits.

So evaluate $\left[-u^{\frac{1}{2}} + \frac{1}{3}u^{\frac{3}{2}}\right]_1^{\frac{3}{4}} = (-\frac{\sqrt{3}}{2} + \times\frac{3\sqrt{3}}{3\times 4\sqrt{4}}) - (-1 + \frac{1}{3})$

$= (-\frac{3\sqrt{3}}{8}) - (-\frac{2}{3})$

Use the new limits to evaluate the answer.

$= \frac{2}{3} - \frac{3\sqrt{3}}{8}$

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Review Exercise
Exercise A, Question 60

Question:

a Express $\frac{5x+3}{(2x-3)(x+2)}$ in partial fractions.

b Hence find the exact value of $\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,dx$, giving your answer as a single logarithm. ***E***

Solution:

a

$$\frac{5x+3}{(2x-3)(x+2)} \equiv \frac{A}{(2x-3)}+\frac{B}{(x+2)}$$

Use denominators $(2x-3)$ and $(x+2)$.

$$\equiv \frac{A(x+2)+B(2x-3)}{(2x-3)(x+2)}$$

$\therefore 5x+3 \equiv A(x+2)+B(2x-3)$

Equate numerators.

Put $x=-2$, then $-7=0-7B \Rightarrow B=1$

Put $x=\frac{3}{2}$, then $\frac{21}{2}=\frac{7}{2}A \Rightarrow A=3$

$\therefore \frac{5x+3}{(2x-3)(x+2)} \equiv \frac{3}{2x-3}+\frac{1}{x+2}$

b

$$\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,dx = \int_2^6 \frac{3}{2x-3}\,dx+\int_2^6 \frac{1}{x+2}\,dx$$

Rewrite the integral using partial fractions.

$$= \left[\frac{3}{2}\ln(2x-3)+\ln(x+2)\right]_2^6$$

Integrate and do not forget to divide by 2.

$$= \frac{3}{2}\ln 9+\ln 8-\ln 4$$

Substitute the limits noting $\ln 1 = 0$.

$$= \ln 9^{\frac{3}{2}}+\ln\frac{8}{4}$$

$$= \ln 27+\ln 2$$

$$= \ln 54$$

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Review Exercise
Exercise A, Question 61

Question:

a Use integration by parts to find

$$\int x\cos 2x\,\mathrm{d}x.$$

b Prove that the answer to part a may be expressed as

$$\frac{1}{2}\sin x(2x\cos x-\sin x)+C,$$

where C is an arbitrary constant. ***E***

Solution:

a Let $I=\int x\cos 2x\mathrm{d}x$

Let $u=x$ and $\frac{\mathrm{d}v}{\mathrm{d}x}=\cos 2x$.

$$u = x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=1$$

$$v = \frac{1}{2}\sin 2x\Leftarrow\frac{\mathrm{d}v}{\mathrm{d}x}=\cos 2x$$

Complete the table for $u, v, \frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.

$$\therefore I = \frac{1}{2}x\sin 2x-\int\frac{1}{2}\sin 2x\cdot 1\,\mathrm{d}x$$

This integral can now be integrated directly.

$$= \frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$$

b

$$\therefore I = \frac{1}{2}x\cdot 2\sin x\cos x+\frac{1}{4}\left(1-2\sin^2 x\right)+c$$

Use double angle formulae: $\sin 2x=2\sin x\cos x$ and $\cos 2x=1-2\sin^2 x$.

$$= \frac{1}{2}\sin x\left(2x\cos x-\sin x\right)+\frac{1}{4}+c$$

$$= \frac{1}{2}\sin x\left(2x\cos x-\sin x\right)+c'$$

Where $c'=\frac{1}{4}+c$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 62

Question:

Use the substitution $u = 2^x$ to find the exact value of

$$\int_0^1 \frac{2^x}{(2^x+1)}\,\mathrm{d}x. \qquad \textbf{\textit{E}}$$

Solution:

Let $I = \int_0^1 \frac{2^x}{2^x+1}\,\mathrm{d}x$.

Let $u = 2^x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 2^x \cdot \ln 2$

You need to replace each `x' term with a corresponding `u' term.

Replace $2^x\,\mathrm{d}x$ by $\frac{1}{\ln 2}\mathrm{d}u$.

x	u
0	1
1	2

Change the limits:
when $x = 0, u = 2^0 = 1$
$x = 1, u = 2^1 = 2$.

Then $I = \int_1^2 \frac{1}{u+1} \cdot \frac{1}{\ln 2}\,\mathrm{d}u$.

$$= \frac{1}{\ln 2}\left[\ln(u+1)\right]_1^2$$

$$= \frac{1}{\ln 2}\left[\ln 3 - \ln 2\right]$$

$$= \frac{1}{\ln 2}\ln\frac{3}{2}.$$

Use the limits for u to evaluate the integral.

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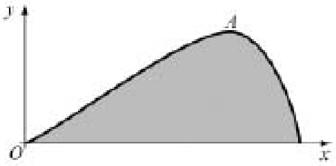
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 63

Question:



The figure shows a graph of $y = x\sqrt{\sin x}, 0 < x < \pi$.

The finite region enclosed by the curve and the x-axis is shaded as shown in the figure. A solid body S is generated by rotating this region through 2π radians about the x-axis. Find the exact value of the volume of S.

***E*(adapted)**

Solution:

$$\text{Volume} = \pi\int_0^{\pi}(x\sqrt{\sin x})^2\,dx$$

Use $v = \pi\int y^2 dx$.

$$= \pi\int_0^{\pi} x^2 \sin x\,dx$$

Use integration by parts.

Let $u = x^2$ and $\frac{dv}{dx} = \sin x$.

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x$$

$$v = -\cos x \Leftarrow \frac{dv}{dx} = \sin x$$

Complete the table for $u, v, \frac{du}{dx}$ and $\frac{dv}{dx}$.

$$\therefore \text{Volume} = \pi\left[[-x^2\cos x]_0^{\pi} - \int_0^{\pi} -2x\cos x\,dx\right]$$

$$= \pi\left(\pi^2 + \int_0^{\pi} 2x\cos x\,dx\right)$$

This integral is simpler than the original one but you will need to use integration by parts again, with $u = 2x$ and $\frac{dv}{dx} = \cos x$.

$$u = 2x \Rightarrow \frac{du}{dx} = 2$$

$$v = \sin x \Leftarrow \frac{dv}{dx} = \cos x$$

$$\therefore \text{Volume} = \pi\left[\pi^2 + [2x\sin x]_0^{\pi} - \int_0^{\pi} 2\sin x\,dx\right]$$

This term becomes zero as $2\pi\sin\pi - 0 = 0$.

$$= \pi\left(\pi^2 + [2\cos x]_0^{\pi}\right)$$

$$= \pi\left(\pi^2 + [-2-2]\right)$$

$$= \pi\left(\pi^2 - 4\right)$$

$$= \pi^3 - 4\pi$$

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Review Exercise

Exercise A, Question 64

Question:

a Find $\int x \cos 2x \, \mathrm{d}x$.

b Hence, using the identity $\cos 2x = 2\cos^2 x - 1$, deduce $\int x \cos^2 x \, \mathrm{d}x$. ***E***

Solution:

a Let $I = \int x \cos 2x \, \mathrm{d}x$

use integration by parts and let $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$.

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\therefore v = \frac{1}{2}\sin 2x \Leftarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$$

Complete the table for $u, v, \frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.

$$\therefore I = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x . 1 \, \mathrm{d}x$$

$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$$

Do not forget to add the constant.

b $\therefore \int x(2\cos^2 x - 1)\mathrm{d}x = I$

So $\int x \cos^2 x \, \mathrm{d}x = \frac{1}{2}I + \frac{1}{2}\int x \, \mathrm{d}x$.

$$= \left(\frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x\right) + \frac{1}{4}x^2 + c$$

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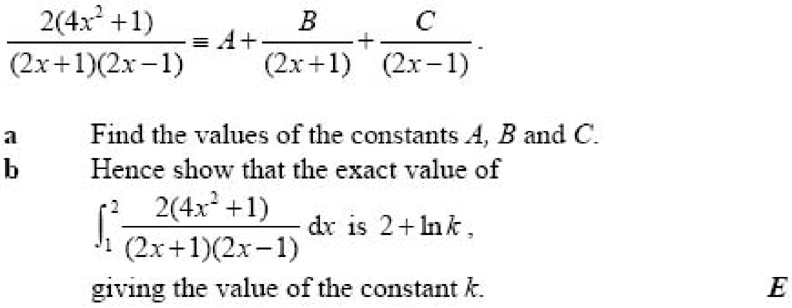
Review Exercise

Exercise A, Question 65

Question:



$$\frac{2(4x^2+1)}{(2x+1)(2x-1)} \equiv A + \frac{B}{(2x+1)} + \frac{C}{(2x-1)}.$$

a Find the values of the constants A, B and C.

b Hence show that the exact value of

$$\int_1^2 \frac{2(4x^2+1)}{(2x+1)(2x-1)}\,\mathrm{d}x \text{ is } 2+\ln k,$$

giving the value of the constant k. ***E***

Solution:

a Let

$$\begin{aligned} f(x) &= \frac{2(4x^2+1)}{(2x+1)(2x-1)} \\ &= \frac{8x^2+2}{4x^2-1} \end{aligned}$$

$$\begin{array}{r} 2\ r\ 4 \\ 4x^2-1 \;\big)\, 8x^2+2 \\ \underline{8x^2-2} \\ 4 \end{array}$$

Divide the denominator into the numerator.

$$\begin{aligned} \therefore f(x) &\equiv 2+\frac{4}{(2x+1)(2x-1)} \\ &\equiv 2+\frac{A}{2x+1}+\frac{B}{2x-1} \end{aligned}$$

Express as partial fractions, using denominators $2x+1$ and $2x-1$.

where $\frac{4}{(2x+1)(2x-1)} \equiv \frac{A(2x-1)+B(2x+1)}{(2x+1)(2x-1)}$

Equate numerators

$4 \equiv A(2x-1)+B(2x+1)$

Put

$x = \frac{1}{2};\ 4=2B \Rightarrow B=2$

$x = -\frac{1}{2};\ 4=-2A \Rightarrow A=-2$

$\therefore f(x) \equiv 2-\frac{2}{(2x+1)}+\frac{2}{(2x-1)}$

or $A=2, B=-2, C=2$

b

$$\begin{aligned} \therefore \int_1^2 f(x)\,dx &= \int_1^2 2-\frac{2}{2x+1}+\frac{2}{2x-1}\,dx \\ &= \left[2x-\ln|2x+1|+\ln|2x-1|\right]_1^2 \\ &= 4-\ln 5+\ln 3-(2-\ln 3) \\ &= 2-\ln 5+2\ln 3 \\ &= 2+\ln 9-\ln 5 \\ &= 2+\ln\frac{9}{5}. \end{aligned}$$

Use the partial fractions from part **a**.

Integrate each term using $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$.

Use the laws of lns to combine the log terms, noting that $2\ln 3 = \ln 3^2 = \ln 9$.

i.e. $k=\frac{9}{5}$ or 1.8.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 66

Question:

$\mathrm{f}(x) = (x^2+1)\ln x$.

Find the exact value of $\int_1^{\mathrm{e}} \mathrm{f}(x)\,\mathrm{d}x$. ***E***

Solution:

Let $I = \int_1^{\mathrm{e}} (x^2+1)\ln x$

Use integration by parts with $u = \ln x$ and so $\frac{\mathrm{d}v}{\mathrm{d}x} = x^2+1$.

Let

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$

$$v = \frac{x^3}{3} + x \Leftarrow \frac{\mathrm{d}v}{\mathrm{d}x} = (x^2+1)$$

Complete the table for $u, v, \frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.

Using integration by parts:

$$\therefore I = \left[\left(\frac{x^3}{3}+x\right)\ln x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \frac{1}{x}\left(\frac{x^3}{3}+x\right)\mathrm{d}x$$

Apply the limits to the uv term and to $\int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$.

$$= \left(\frac{\mathrm{e}^3}{3}+\mathrm{e}\right) - \int_1^{\mathrm{e}}\left(\frac{x^2}{3}+1\right)\mathrm{d}x$$

Evaluate the limits on uv and remember $\ln 1 = 0$.

$$= \frac{\mathrm{e}^3}{3}+\mathrm{e} - \left[\frac{x^3}{9}+x\right]_1^{\mathrm{e}}$$

$$= \frac{\mathrm{e}^3}{3}+\mathrm{e} - \left[\frac{\mathrm{e}^3}{9}+\mathrm{e}-\frac{1}{9}-1\right]$$

$$= \frac{2\mathrm{e}^3}{9}+\frac{10}{9}$$

$$= \frac{1}{9}\left(2\mathrm{e}^3+10\right)$$

This is an exact answer.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 67

Question:

The curve C is described by the parametric equations

$x = 3\cos t, y = \cos 2t, 0 \le t \le \pi.$

Find a Cartesian equation of the curve C. ***E***

Solution:

$$x = 3\cos t$$
$$\therefore \cos t = \frac{x}{3}$$

Rearrange to make cos t the subject of the formula.

$$\begin{aligned} y &= \cos 2t \\ &= 2\cos^2 t - 1 \end{aligned}$$

Use the double angle formula.

$$y = 2\left(\frac{x}{3}\right)^2 - 1$$

Eliminate t to give a Cartesian equation.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 68

Question:

The point $P(a, 4)$ lies on a curve C. C has parametric equations $x = 3t \sin t, y = 2 \sec t, 0 \le t < \frac{\pi}{2}$. Find the exact value of a. ***E***

Solution:

Put $y = 2 \sec t = 4$

As point P has y coordinate 4.

Then $\sec t = 2$

Now solve the resulting trigonometric equation.

So $\cos t = \frac{1}{2}$

$\therefore t = \frac{\pi}{3}$

Give your answer in radians as $0 \le t < \frac{\pi}{2}$.

$\therefore x = 3 \frac{\pi}{3} \sin \frac{\pi}{3}$

Substitute the value of t into $x = 3t \sin t$.

i.e. $x = \pi \frac{\sqrt{3}}{2}$

So $a = \frac{\pi\sqrt{3}}{2}$

This is the x coordinate of P and so is equal to a.

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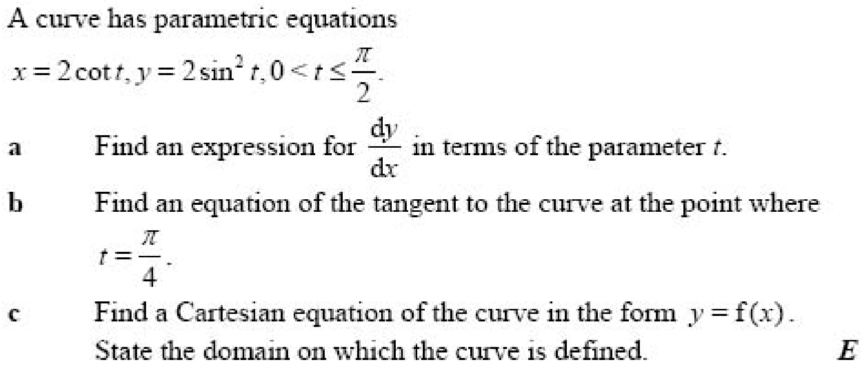
Review Exercise

Exercise A, Question 69

Question:



A curve has parametric equations

$x = 2\cot t,\ y = 2\sin^2 t,\ 0 < t \le \frac{\pi}{2}$.

a Find an expression for $\frac{dy}{dx}$ in terms of the parameter t.

b Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$.

c Find a Cartesian equation of the curve in the form $y = f(x)$.
State the domain on which the curve is defined. *E*

Solution:

a

$$x = 2\cot t, y = 2\sin^2 t$$

$$\frac{dx}{dt} = -2\text{cosec}^2 t, \frac{dy}{dt} = 4\sin t\cos t$$

Use the chain rule to differentiate $2\sin^2 t$.

$$\therefore \frac{dy}{dx} = \frac{4\sin t\cos t}{-2\text{cosec}^2 t}$$

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.

$$= -2\sin^3 t\cos t$$

Simplify using $\text{cosec}\, t = \frac{1}{\sin t}$.

b At $t = \frac{\pi}{4}$, gradient $= -2 \times \left(\frac{1}{\sqrt{2}}\right)^3 \times \left(\frac{1}{\sqrt{2}}\right)$

$$= -\frac{1}{2}.$$

Find the value of the gradient of the curve at $t = \frac{\pi}{4}$.

The coordinates of the point where $t = \frac{\pi}{4}$ are:

$$x = 2, y = 2 \times \left(\frac{1}{\sqrt{2}}\right)^2 = 1$$

∴ The equation of the tangent is

$$(y-1) = -\frac{1}{2}(x-2)$$

The tangent has the same gradient as the curve.

$$\therefore y = \frac{-1}{2}x + 2.$$

c As $x = 2\cot t, \cot t = \frac{x}{2}$.

Also as $y = 2\sin^2 t, \sin^2 t = \frac{y}{2}$ and

$$\text{cosec}^2 t = \frac{2}{y}$$

Rearrange to make $\cot t$ and $\text{cosec}^2 t$ the subjects of the formulae.

use $1 + \cot^2 t = \text{cosec}^2 t$

Write down the relationship between $\cot^2 t$ and $\text{cosec}^2 t$.

then $1 + \left(\frac{x}{2}\right)^2 = \left(\frac{2}{y}\right)$

Eliminate t to give a Cartesian equation.

$$\therefore \left(\frac{2}{y}\right) = \frac{4+x^2}{4}$$

Take the reciprocal of each side of the equation.

$$\left(\frac{y}{2}\right) = \frac{4}{4+x^2}$$

$$y = \frac{8}{4+x^2}$$

As $0 < t \leq \frac{\pi}{2}, \cot t \geq 0$

As $x = 2\cot t, x \geq 0$

This is the domain of the function.

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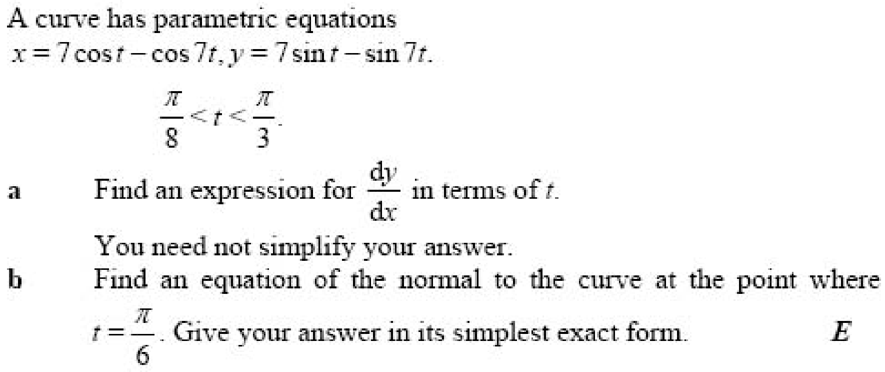
Review Exercise

Exercise A, Question 70

Question:



A curve has parametric equations

$x = 7\cos t - \cos 7t,\ y = 7\sin t - \sin 7t.$

$$\frac{\pi}{8} < t < \frac{\pi}{3}.$$

a Find an expression for $\frac{dy}{dx}$ in terms of t.

You need not simplify your answer.

b Find an equation of the normal to the curve at the point where $t = \frac{\pi}{6}$. Give your answer in its simplest exact form. ***E***

Solution:

a

$$x = 7\cos t - \cos 7t; y = 7\sin t - \sin 7t$$

$$\frac{dx}{dt} = -7\sin t + 7\sin 7t; \frac{dy}{dt} = 7\cos t - 7\cos 7t$$

using the chain rule :

$$\frac{dy}{dx} = \frac{7\cos t - 7\cos 7t}{-7\sin t + 7\sin 7t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

b When $t = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{7 \times \frac{\sqrt{3}}{2} + 7 \times \frac{\sqrt{3}}{2}}{-7 \times \frac{1}{2} - 7 \times \frac{1}{2}}$$

$$= \frac{7\sqrt{3}}{-7}$$

$$= -\sqrt{3}$$

Substitute $t = \frac{\pi}{6}$ to find the gradient of the curve.

$\therefore$ Gradient of the normal at the point where $t = \frac{\pi}{6}$ is $\frac{1}{\sqrt{3}}$.

Use $mm^1 = -1$, the condition for perpendicular lines to find the gradient of the normal.

When $t = \frac{\pi}{6}$,

$$x = 7\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 4\sqrt{3}$$

$$y = 7 \times \frac{1}{2} + \frac{1}{2} = 4$$

Find the co-ordinates of the point on the curve when $t = \frac{\pi}{6}$.

$\therefore$ Equation of the normal is

$$y - 4 = \frac{1}{\sqrt{3}}(x - 4\sqrt{3})$$

$$\therefore y\sqrt{3} = x$$

Use $y - y_1 = m(x - x_1)$ for the equation of a straight line.

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Review Exercise

Exercise A, Question 71

Question:

A curve has parametric equations

$x = \tan^2 t, y = \sin t, 0 < t < \frac{\pi}{2}.$

a Find an expression for $\frac{dy}{dx}$ in terms of t.

You need not simplify your answer.

b Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$.

Give your answer in the form $y = ax + b$, where a and b are constants to be determined.

c Find a Cartesian equation of the curve in the form $y^2 = f(x)$. *E*

Solution:

a

$$x = \tan^2 t,\ y = \sin t$$

Use the chain rule to differentiate $\tan^2 t$.

$$\frac{dx}{dt} = 2\tan t\sec^2 t,\ \frac{dy}{dt} = \cos t$$

using the chain rule:

$$\frac{dy}{dx} = \frac{\cos t}{2\tan t\sec^2 t}$$

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.

b When $t = \frac{\pi}{4}$

$$\frac{dy}{dx} = \frac{\frac{1}{\sqrt{2}}}{2\times 1\times(\sqrt{2})^2} = \frac{1}{4\sqrt{2}}$$

Substitute $t = \frac{\pi}{4}$ to find the value of the gradient of the curve.

$\therefore$ Gradient of the tangent where $t = \frac{\pi}{4}$ is $\frac{1}{4\sqrt{2}}$.

At $t = \frac{\pi}{4}, x = 1, y = \frac{1}{\sqrt{2}}$.

Find the co-ordinates of the point on the curve where $t = \pi/4$.

The equation of the tangent is:

$$y - \frac{1}{\sqrt{2}} = \frac{1}{4\sqrt{2}}(x-1)$$

$$\therefore y = \frac{1}{4\sqrt{2}}x + \frac{3}{4\sqrt{2}} \text{ or } y = \frac{\sqrt{2}}{8}x + \frac{3\sqrt{2}}{8}$$

The tangent has the same gradient as the curve at the point $\left(1, \frac{1}{\sqrt{2}}\right)$.

Use $y - y_1 = m(x - x_1)$.

c

$$y^2 = \sin^2 t$$

$$= 1 - \cos^2 t$$

Use $\cos^2 t + \sin^2 t = 1$.

$$= 1 - \frac{1}{\sec^2 t}$$

Use $\sec t = \frac{1}{\cos t}$.

$$= 1 - \frac{1}{1+\tan^2 t}$$

Use $1 + \tan^2 t = \sec^2 t$.

$$= 1 - \frac{1}{1+x} \text{ or } y^2 = \frac{x}{1+x}$$

Eliminate t by using $\tan^2 t = x$.

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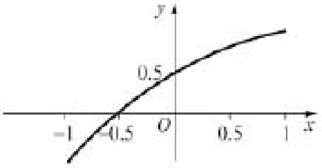
Review Exercise
Exercise A, Question 72

Question:



The curve shown in the figure has parametric equations

$$x=\sin t, y=\sin\left(t+\frac{\pi}{6}\right), -\frac{\pi}{2}<t<\frac{\pi}{2}.$$

a Find an equation of the tangent to the curve at the point where $t=\frac{\pi}{6}$.

b Show that a Cartesian equation of the curve is

$$y=\frac{\sqrt{3}}{2}x+\frac{1}{2}\sqrt{(1-x^2)}, -1<x<1.$$ ***E***

Solution:

a

$$x = \sin t, y = \sin\left(t+\frac{\pi}{6}\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos t, \frac{\mathrm{d}y}{\mathrm{d}t} = \cos\left(t+\frac{\pi}{6}\right)$$

using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos(t+\frac{\pi}{6})}{\cos t}$$

Use $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$.

At the point where $t = \frac{\pi}{6}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

Substitute $t = \frac{\pi}{6}$ to find the gradient of the curve which is also the gradient of the tangent.

$\therefore$ Gradient of the tangent at $t = \frac{\pi}{6}$ is $\frac{1}{\sqrt{3}}$.

Find the values of x and y when $t = \frac{\pi}{6}$.

Also at $t = \frac{\pi}{6}, x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$

Equation of the tangent is:

$$y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}\left(x-\frac{1}{2}\right)$$

$$\therefore y = \frac{1}{\sqrt{3}}x - \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2}$$

Use $y - y_1 = m(x - x_1)$ for the equation of a straight line.

i.e. $y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$

b

$$y = \sin(t+\frac{\pi}{6})$$

$$= \sin t\cos\frac{\pi}{6} + \cos t\sin\frac{\pi}{6}.$$

$$= \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\cos t$$

Expand, using addition formula.

Replace $\cos\frac{\pi}{6}$ by $\frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6}$ by $\frac{1}{2}$.

As $x = \sin t$, using $\cos^2 t = 1 - \sin^2 t$

means that $\cos t = \sqrt{1-x^2}$

$$\therefore y = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{(1-x^2)}$$

Eliminate t by using $\sin t = x$ and $\cos t = \sqrt{(1-x^2)}$.

As $-1 < \sin t < 1 \Rightarrow -1 < x < 1$.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 73

Question:

The curve C has parametric equations

$$x = \frac{1}{1+t}, y = \frac{1}{1-t}, -1 < t < 1.$$

The line l is a tangent to C at the point where $t = \frac{1}{2}$.

a Find an equation for the line l.

b Show that a Cartesian equation for the curve C is $y = \frac{x}{2x-1}$. ***E***

Solution:

a

Differentiate $(1+t)^{-1}$ and $(1-t)^{-1}$ using the chain rule.

$$x = \frac{1}{1+t}, y = \frac{1}{1-t}$$

$$\frac{dx}{dt} = \frac{-1}{(1+t)^2}, \frac{dy}{dt} = \frac{1}{(1-t)^2}$$

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.

$$\therefore \frac{dy}{dx} = -\frac{(1+t)^2}{(1-t)^2}$$

At the point where $t = \frac{1}{2}$,

$$\frac{dy}{dx} = -\frac{\frac{9}{4}}{\frac{1}{4}}$$

$$= -9$$

Substitute $t = \frac{1}{2}$ to find the gradient of the curve and thus the tangent.

$\therefore$ Gradient of tangent, where $t = \frac{1}{2}$, is -9.

Also $x = \frac{2}{3}$ and $y = 2$ where $t = \frac{1}{2}$.

Find the values of x and y when $t = \frac{1}{2}$.

$\therefore$ Equation of tangent is

$$y - 2 = -9\left(x - \frac{2}{3}\right)$$

Use $y - y_1 = m(x - x_1)$.

i.e. $y = -9x + 8$.

b As $x = \frac{1}{1+t}$

$$1 + t = \frac{1}{x}$$

$$\therefore t = \frac{1}{x} - 1$$

Rearrange to make t the subject of the formula.

Substitute into $y = \frac{1}{1-t}$

$$\therefore y = \frac{1}{1 - \left(\frac{1}{x} - 1\right)}$$

$$= \frac{1}{2 - \frac{1}{x}}$$

$$= \frac{x}{2x - 1}$$

Eliminate t and simplify the fraction multiplying numerator and denominator by x.

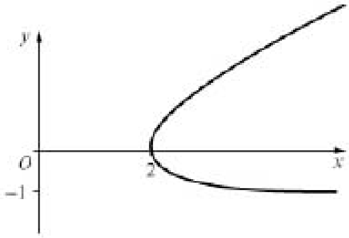
Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 74

Question:



The curve shown has parametric equations

$x = t + \frac{1}{t}, y = t - 1$ for $t > 0$.

a Find the value of the parameter t at each of the points where $x = 2\frac{1}{2}$.

b Find the gradient of the curve at each of these points.

c Find the area of the finite region enclosed between the curve and the line $x = 2\frac{1}{2}$. ***E***

Solution:

a $x=t+\frac{1}{t},\ y=t-1$ for $t>0$

As $x=2\frac{1}{2},\ t+\frac{1}{t}=2\frac{1}{2}$

Multiply both sides of this equation by t and collect the terms to give a quadratic equation.

$$\therefore t^2-2\frac{1}{2}t+1 = 0$$

$$\text{i.e. } 2t^2-5t+2 = 0$$

$$\therefore (2t-1)(t-2) = 0$$

$$\Rightarrow t=\frac{1}{2} \text{ or } 2$$

b

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1-\frac{1}{t^2},\ \frac{\mathrm{d}y}{\mathrm{d}t}=1$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = 1-\frac{1}{t^3}=\frac{t^2}{t^2-1}$$

Find $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ and use the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$

When $t=\frac{1}{2}$, gradient $=\frac{\frac{1}{4}}{\frac{1}{4}-1}=\frac{-1}{3}$

Substitute the values of t found in part **a**.

$t=2$, gradient $=\frac{4}{4-1}=\frac{4}{3}$.

c

$$\text{Area} = \int y\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

$$= \int_{\frac{1}{2}}^{2}(t-1)\left(1-\frac{1}{t^2}\right)\mathrm{d}t$$

Substitute $y=t-1$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=1-\frac{1}{t^2}$.

$$= \int_{\frac{1}{2}}^{2}t-1-\frac{1}{t}+\frac{1}{t^2}\,\mathrm{d}t$$

Expand the brackets.

$$= \left[\frac{t^2}{2}-t-\ln t-\frac{1}{t}\right]_{\frac{1}{2}}^{2}$$

Integrate each term.

use of limits $t=2$ and $t=\frac{1}{2}$ to give

$$\text{area} = -\ln 2-\frac{1}{2}-\left(\frac{1}{8}-\frac{1}{2}-\ln\frac{1}{2}-2\right)$$

Substitute $t=2$ and $t=\frac{1}{2}$ then subtract.

$$= 1\frac{7}{8}-2\ln 2=0.4887$$

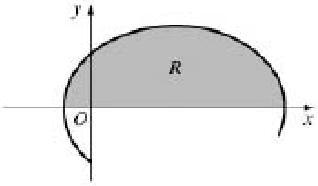
Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 75

Question:



The curve shown in the figure has parametric equations

$x = t - 2\sin t, y = 1 - 2\cos t,$

$0 \leq t \leq 2\pi.$

a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the figure

b Show that the area R is given by the integral

$$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \, dt.$$

c Use this integral to find the exact value of the shaded area. ***E***

Solution:

a The curve crosses the x-axis when $y = 0$.

As $y = 1 - 2\cos t$, when $y = 0$

$\cos t = \frac{1}{2}$

$\therefore t = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

b Area of R is given by $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} y \frac{dx}{dt}\, dt$

As $x = t - 2\sin t$,

$\frac{dx}{dt} = 1 - 2\cos t$.

$$\therefore \text{Area} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)(1 - 2\cos t)\, dt$$

Substitute $y = 1 - 2\cos t$ and $\frac{dx}{dt} = 1 - 2\cos t$ into the integral.

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2\, dt.$$

c

$$\therefore \text{Area} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 4\cos t + 4\cos^2 t)\, dt$$

Expand the bracket.

$$= [t - 4\sin t]_{\frac{\pi}{3}}^{\frac{5\pi}{3}} + 2\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (\cos 2t + 1)\, dt$$

Integrate $1 - 4\cos t$ directly.

Use double angle formula $\cos 2t = 2\cos^2 t - 1$ to replace $4\cos^2 t$ with $2(\cos 2t + 1)$.

$$= [t - 4\sin t]_{\frac{\pi}{3}}^{\frac{5\pi}{3}} + [\sin 2t + 2t]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

Now integrate $(2\cos 2t + 2)$.

$$= [3t - 4\sin t + \sin 2t]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

Collect the terms.

$$= \left(5\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) - \left(\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right)$$

$$= 4\pi + 4\sqrt{3} - \sqrt{3}$$

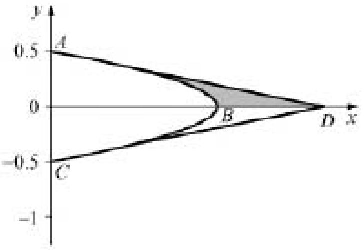
$$= 4\pi + 3\sqrt{3}$$

Use the limits to find an exact answer.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 76

Question:



The curve shown in the figure has parametric equations

$x = a\cos 3t, y = a\sin t, -\frac{\pi}{6} \le t \le \frac{\pi}{6}.$

The curve meets the axes at points A, B and C, as shown.
The straight lines shown are tangents to the curve at the points A and C and meet the x-axis at point D. Find, in terms of a

a the equation of the tangent to A,

b the area of the finite region between the curve, the tangent at A and the x-axis, shown shaded in the figure.

Given that the total area of the finite region between the two tangents and the curve is 10 cm^2

c find the value of a. *E*

Solution:

a At point A, $x = 0$

$\therefore a\cos 3t = 0 \Rightarrow 3t = \frac{\pi}{2}$

Find the co-ordinates of the point A.

$\therefore t = \frac{\pi}{6}$.

But $y = a\sin t$

At $t = \frac{\pi}{6}, y = \frac{a}{2}$.

$\therefore A$ is the point $(0, \frac{a}{2})$

$$x = a\cos 3t, y = a\sin t$$

$$\frac{dx}{dt} = -3a\sin 3t, \frac{dy}{dt} = a\cos t$$

$$\therefore \frac{dy}{dx} = -\frac{\cos t}{3\sin 3t}$$

Use $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

when $t = \frac{\pi}{6}, \frac{dy}{dx} = -\frac{\frac{\sqrt{3}}{2}}{3} = -\frac{\sqrt{3}}{6}$.

Find the gradient at the point A.

$\therefore$ Equation of the tangent at A is $y - \frac{a}{2} = -\frac{\sqrt{3}}{6}(x - 0)$

Use $y - y_1 = m(x - x_1)$.

$\therefore y = \frac{-\sqrt{3}}{6}x + \frac{a}{2}$

b This tangent meets the x-axis when $y = 0$, at the point D.

$$\therefore \frac{\sqrt{3}}{6}x = \frac{a}{2}$$

$$\therefore x = \sqrt{3}a$$

Find the point where the tangent meets the x-axis.

Area of triangle AOD is $\frac{1}{2} \times \sqrt{3}a \times \frac{a}{2}$

$= \frac{1}{4}\sqrt{3}a^2$

Use area of triangle $= \frac{1}{2}$ base $\times$ height i.e. $\frac{1}{2}OD \times OA$.

At the point B, $t = 0$

$\therefore$ Area of region required $= \frac{1}{4}\sqrt{3}a^2 - \int y\frac{dx}{dt}\,dt$

$$\therefore \text{Area} = \frac{1}{4}\sqrt{3}a^2 - \int_{\pi/6}^{0} a\sin t(-3a\sin 3t)\,\mathrm{d}t$$

area = area of triangle − area beneath the curve .

$$= \frac{1}{4}\sqrt{3}a^2 + \frac{3a^2}{2}\int_{\frac{\pi}{6}}^{0} \cos 2t - \cos 4t\ \mathrm{d}t$$

Use $2\sin t\sin 3t = \cos 2t - \cos 4t$ This is from the trigonometric `factor formulae' − see C3.

$$= \frac{1}{4}\sqrt{3}a^2 + \frac{3a^2}{2}\left[\frac{1}{2}\sin 2t - \frac{1}{4}\sin 4t\right]_{\frac{\pi}{6}}^{0}$$

The $\frac{\pi}{6}$ limit corresponds to point A and the 0 limit corresponds to point B.

$$= \frac{1}{4}\sqrt{3}a^2 + \frac{3a^2}{2}\left[0 - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8}\right]$$

Use the limits and the result $\sin\frac{\pi}{3} = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ to give an exact answer.

$$= \frac{1}{4}\sqrt{3}a^2 - \frac{3}{16}\sqrt{3}a^2$$

$$= \frac{1}{16}\sqrt{3}a^2$$

c Total area is $2 \times \frac{1}{16}\sqrt{3}a^2 = 10$

The total area is twice the area found in part **b**.

$$\therefore a^2 = \frac{80}{\sqrt{3}}$$

$$\therefore a = 6.796 \text{ (4 s.f.)}$$

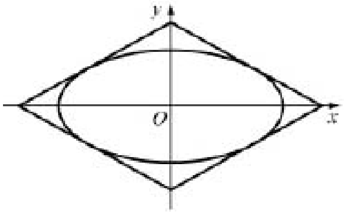
Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 77

Question:



A table top, in the shape of a parallelogram, is made from two types of wood. The design is shown in the figure. The area inside the ellipse is made from one type of wood, and the surrounding area is made from a second type of wood.

The ellipse has parametric equations,

$$x = 5\cos\theta, y = 4\sin\theta, 0 \le \theta \le 2\pi.$$

The parallelogram consists of four line segments, which are tangents to the ellipse at the points where

$\theta = \alpha,\ \theta = -\alpha,\ \theta = \pi - \alpha,\ \theta = -\pi + \alpha.$

- **a** Find an equation of the tangent to the ellipse at $(5\cos\alpha, 4\sin\alpha)$, and show that it can be written in the form $5y\sin\alpha + 4x\cos\alpha = 20.$
- **b** Find by integration the area enclosed by the ellipse.
- **c** Hence show that the area enclosed between the ellipse and the parallelogram is

$$\frac{80}{\sin 2\alpha} - 20\pi. \qquad \textbf{\textit{E}}$$

Solution:

a

$$x = 5\cos\theta, y = 4\sin\theta$$

$$\frac{dx}{d\theta} = -5\sin\theta, \frac{dy}{d\theta} = 4\cos\theta$$

From the chain rule

$$\frac{dy}{dx} = \frac{4\cos\theta}{-5\sin\theta}$$

Use $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$.

$$= -\frac{4}{5}\cot\theta$$

The gradient of the tangent at $(5\cos\alpha, 4\sin\alpha) = \frac{-4}{5}\cot\alpha$

Substitute $\theta = \alpha$ to give gradient at particular point.

$\therefore$ Equation of the tangent is

$$y - 4\sin\alpha = -\frac{4}{5}\cot\alpha(x - 5\cos\alpha)$$

Use $y - y_1 = m(x - x_1)$.

$$\text{i.e. } 5y\sin\alpha - 20\sin^2\alpha = -4\cos\alpha \times x + 20\cos^2\alpha$$

Multiply both sides of the equation by sin α.

$$\therefore 5y\sin\alpha + 4x\cos\alpha = 20(\cos^2\alpha + \sin^2\alpha)$$

$$= 20 \times 1$$

$$= 20$$

Collect terms using $\cos^2\alpha + \sin^2\alpha = 1$.

b

$$\text{Area} = \int_{2\pi}^{0} y\frac{dx}{d\theta}\,d\theta$$

$$= \int_{2\pi}^{0} 4\sin\theta(-5\sin\theta)\,d\theta$$

Substitute $y = 4\sin\theta$ and $\frac{dx}{d\theta} = -5\sin\theta$ into integral.

$$= -10\int_{2\pi}^{0} 2\sin^2\theta\,d\theta$$

$$= 10\int_{2\pi}^{0} \cos 2\theta - 1\,d\theta$$

Use double angle formula $\cos 2\theta = 1 - 2\sin^2\theta$.

$$= 10\left[\frac{1}{2}\sin 2\theta - \theta\right]_{2\pi}^{\theta}$$

Integrate and use appropriate limits.

Use limits 0 and 2π to obtain area $= 20\pi$

c Area of triangle formed by tangent at $(5\cos\alpha, 4\sin\alpha)$ and the coordinate axes:

Tangent meets x-axis at $x = \dfrac{5}{\cos\alpha}$

Tangent meets y-axis at $y = \dfrac{4}{\sin\alpha}$

Find the points where the tangent crosses the x- and y-axis.

$$\therefore \text{Area of triangle} = \frac{1}{2} \times \frac{5}{\cos\alpha} \times \frac{4}{\sin\alpha}$$

$$= \frac{10}{\sin\alpha\cos\alpha}$$

$$= \frac{20}{\sin 2\alpha}.$$

Use area of triangle $= \dfrac{1}{2}$ base $\times$ height.

Parallelogram is made up of four such triangles.

$\therefore$ Area of parallelogram $= \dfrac{80}{\sin 2\alpha}$

From symmetry the area of the parallelogram is $4 \times$ area of triangle.

$\therefore$ Enclosed area $= \dfrac{80}{\sin 2\alpha} - 20\pi$.

Area required $=$ area of parallelogram $-$ Area enclosed by ellipse.