Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise A, Question 1

Question:

A particle P is moving in a straight line. At time t, the displacement of P from a fixed point on the line is x. The motion of the particle is modelled by the differential equation $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x = 0$

When $t = 0$ P is at rest at the point where $x = 2$.

a Find x as a function of t.

b Calculate the value of x when $t = \frac{\pi}{3}$.

c State whether the motion is heavily, critically or lightly damped.

Solution:

a $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x = 0$

Auxiliary equation: $m^2 + 4m + 8 = 0$ ← Solve the equation using the methods of book FP2 chapter 5.

$m = \frac{-4 \pm \sqrt{(16-32)}}{2}$

$m = -2 \pm 2\text{i}$

General solution:

$x = e^{-2t}(A\cos 2t + B\sin 2t)$

$t = 0, x = 2 \Rightarrow 2 = A$ ← Use the initial conditions given in the question to obtain values for A and B.

$\dot{x} = -2e^{-2t}(A\cos 2t + B\sin 2t) + e^{-2t}(-2A\sin 2t + 2B\cos 2t)$

$t = 0, \dot{x} = 0 \Rightarrow 0 = -2A + 2B$

$B = A$

$\therefore x = 2e^{-2t}(\cos 2t + \sin 2t)$

b $t = \frac{\pi}{3}$ $x = 2e^{-\frac{2\pi}{3}}\left(\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}\right)$

$x = 0.09014\ldots$

$\therefore x = 0.0901$ (3 s.f.)

c Lightly damped

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Damped and forced harmonic motion
Exercise A, Question 2

Question:

A particle P is moving in a straight line. At time t, the displacement of P from a fixed point on the line is x. The motion of the particle is modelled by the differential equation $\frac{d^2x}{dt^2}+8\frac{dx}{dt}+12x=0$

When $t=0$ P is at rest at the point where $x=4$.
Find x as a function of t.

Solution:

$$\frac{d^2x}{dt^2}+8\frac{dx}{dt}+12x=0$$

Auxiliary equation: $m^2+8m+12=0$ ← Solve the equation using the methods of book FP2 chapter 5.

$(m+6)(m+2)=0$

$m=-6$ or -2

General solution:

$x=Ae^{-6t}+Be^{-2t}$

$t=0,\ x=4 \Rightarrow 4=A+B$ ① ← Use the information given in the question to obtain values for A and B.

$\dot{x}=-6Ae^{-6t}-2Be^{-2t}$

$t=0,\ \dot{x}=0 \quad 0=-6A-2B$

$0=3A+B$ ②

$\therefore 2A=-4$ ← Solve equations ① and ② simultaneously.

$A=-2,\ B=6$

$\therefore x=6e^{-2t}-2e^{-6t}$

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Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise A, Question 3

Question:

A particle P is moving in a straight line. At time t, the displacement of P from a fixed point on the line is x. The motion of the particle is modelled by the differential equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+6x=0$

When $t=0$ P is at rest at the point where $x=1$.

a Find x as a function of t.

The smallest value of $t, t>0$, for which P is instantaneously at rest is T.

b Find the value of T.

Solution:

a $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+6x=0$

Auxiliary equation: $m^2+2m+6=0$ ← Solve the equation using the methods of book FP2 Chapter 5.

$m=\frac{-2\pm\sqrt{(4-24)}}{2}$

$m=-1\pm i\sqrt{5}$

General solution:

$x=e^{-t}(A\cos\sqrt{5}t+B\sin\sqrt{5}t)$

$t=0 \quad x=1\Rightarrow 1=A$ ← Use the initial conditions given in the question to obtain values for A and B.

$\dot{x}=-e^{-t}(A\cos\sqrt{5}t+B\sin\sqrt{5}t)$
$\quad +e^{-t}(-A\sqrt{5}\sin\sqrt{5}t+B\sqrt{5}\cos\sqrt{5}t)$

$t=0 \quad \dot{x}=0\Rightarrow 0=-A+B\sqrt{5}$

$B=\frac{1}{\sqrt{5}}$

$\therefore x=e^{-t}\left(\cos\sqrt{5}t+\frac{1}{\sqrt{5}}\sin\sqrt{5}t\right)$

b $\dot{x}=0 \quad t=T$

$\Rightarrow 0=-e^{-T}\left(\cos\sqrt{5}T+\frac{1}{\sqrt{5}}\sin\sqrt{5}T\right)+e^{-T}\left(-\sqrt{5}\sin\sqrt{5}T+\frac{1}{\sqrt{5}}\sqrt{5}\cos\sqrt{5}T\right)$

$e^{-T}\neq 0$

$\therefore -\cos\sqrt{5}T-\frac{1}{\sqrt{5}}\sin\sqrt{5}T-\sqrt{5}\sin\sqrt{5}T+\cos\sqrt{5}T=0$

$\sin\sqrt{5}T=0$

$\sqrt{5}T=0,\pi,\ldots$

$T=\frac{\pi}{\sqrt{5}},\ldots$ ← $T>0$

$\therefore$ Smallest value of T is $\frac{\pi}{\sqrt{5}}$ or 1.40^c (3 s.f.)

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Damped and forced harmonic motion
Exercise A, Question 4

Question:

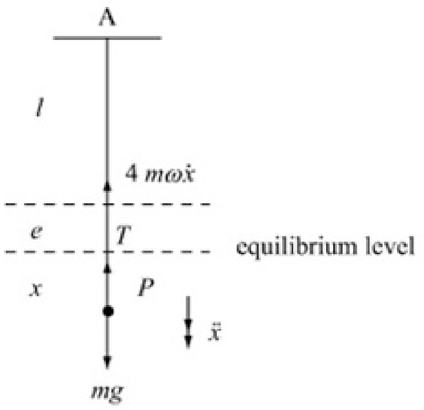
A particle P of mass m is attached to one end of a light elastic spring of natural length l and modulus of elasticity $4m\omega^2 l$, where ω is a positive constant. The other end of the spring is attached to a fixed point A and P hangs in equilibrium vertically below A. At time $t = 0$, P is projected vertically downwards with speed u. A resistance of magnitude $4m\omega v$, where v is the speed of P, acts on P. The displacement of P downwards from its equilibrium position at time t is x.

a Show that $\frac{d^2x}{dt^2} + 4\omega\frac{dx}{dt} + 4\omega^2 x = 0$

b Find an expression for x in terms of u, t and ω.

c Find the time at which P comes to instantaneous rest.

Solution:



$\lambda = 4m\omega^2 l$

a In equilibrium: $R(\uparrow)\ T = mg$

Hooke's Law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{4m\omega^2 e}{l}$$

$\therefore 4m\omega^2 e = mg$ ①

When extension is $(e + x)$

$$T = \frac{\lambda(e+x)}{l} = \frac{4m\omega^2 l(e+x)}{l}$$

$F = ma$:

$$mg - T - 4m\omega\dot{x} = m\ddot{x}$$

$$mg - 4m\omega^2(e + x) - 4m\omega\dot{x} = m\ddot{x}$$

$$mg - mg - 4m\omega^2 x - 4m\omega\dot{x} = m\ddot{x}$$ ← Use 1.

$$\ddot{x} + 4\omega\dot{x} + 4\omega^2 x = 0$$

or $\dfrac{d^2x}{dt^2} + 4\omega\dfrac{dx}{dt} + 4\omega^2 x = 0$

b Auxiliary equation: $m^2 + 4\omega m + 4\omega^2 = 0$ ← Now solve the differential equation using the methods of book FP2 chapter 5

$(m + 2\omega)^2 = 0$

$m = -2\omega$ (twice)

General solution: $x = (A + Bt)e^{-2\omega t}$

$t = 0,\ x = 0 \Rightarrow 0 = A$

$\dot{x} = Be^{-2\omega t} - 2\omega Bte^{-2\omega t}$

$t = 0,\ \dot{x} = u \Rightarrow u = B$

$\therefore x = ute^{-2\omega t}$

c $\dot{x} = ue^{-2\omega t} - 2\omega ut\, e^{-2\omega t}$

$= u\, e^{-2\omega t}(1 - 2\omega t)$

$\dot{x} = 0 \quad 1 - 2\omega t = 0$ ← $e^{-2\omega t} \neq 0$

$$t = \frac{1}{2\omega}$$

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Damped and forced harmonic motion
Exercise A, Question 5

Question:

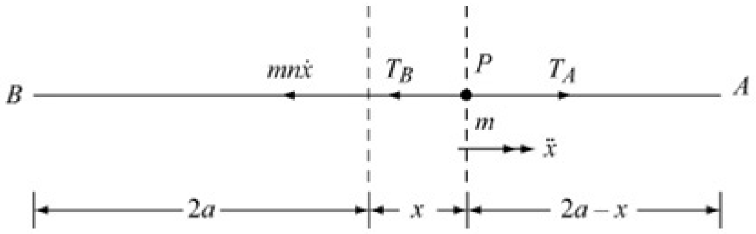
A particle P of mass m is attached to the mid-point of a light elastic string AB of natural length $2a$ and modulus of elasticity mg. The ends A and B of the string are attached to fixed points on a smooth horizontal table with $AB = 4a$. The particle is released from rest at the point C where A, C and B lie in a straight line and $AC = \frac{3}{2}a$. At time t the displacement of P from its equilibrium position is x. The particle is subject to a resisting force of magnitude mnv where v is the speed of P and $n = \sqrt{\frac{2g}{a}}$.

a Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + n\frac{\mathrm{d}x}{\mathrm{d}t} + n^2x = 0$.

b Find an expression for x in terms of a, n and t.

Solution:

a



$n = \sqrt{\frac{2g}{a}}$ $\qquad \lambda = mg$

$l = a$

Consider each half string separately.

$T = \frac{\lambda x}{l}$

Use Hooke's Law.

$T_A = \frac{mg(a-x)}{a} \quad T_B = \frac{mg(a+x)}{a}$

$F = ma$

$$T_A - T_B - mn\dot{x} = m\ddot{x}$$

$$\frac{mg(a-x)}{a} - \frac{mg(a+x)}{a} - mn\dot{x} = m\ddot{x}$$

$$-\frac{2gx}{a} - n\dot{x} = \ddot{x}$$

$$\ddot{x} + n\dot{x} + \frac{2g}{a}x = 0$$

or $\frac{d^2x}{dt^2} + n\frac{dx}{dt} + n^2x = 0$

From the question, $n^2 = \frac{2g}{a}$

b Auxiliary equation: $m^2 + nm + n^2 = 0$

Now solve the differential equation using the methods of book FP2 chapter 5

$$m = \frac{-n \pm \sqrt{(n^2 - 4n^2)}}{2}$$

$$m = \frac{-n \pm in\sqrt{3}}{2}$$

General solution:

$$x = e^{-\frac{1}{2}nt}\left(A\cos\frac{n\sqrt{3}}{2}t + B\sin\frac{n\sqrt{3}}{2}t\right)$$

$t = 0 \quad x = \frac{1}{2}a \quad \Rightarrow \quad \frac{1}{2}a = A$

Use the initial conditions given in the question to obtain values for A and B.

$$\dot{x} = -\frac{1}{2}ne^{-\frac{1}{2}nt}\left(A\cos\frac{n\sqrt{3}}{2}t + B\sin\frac{n\sqrt{3}}{2}t\right)$$

$$+e^{-\frac{1}{2}nt}\left(\frac{-n\sqrt{3}}{2}A\sin\frac{n\sqrt{3}}{2}t + \frac{n\sqrt{3}}{2}B\cos\frac{n\sqrt{3}}{2}t\right)$$

$t = 0 \quad \dot{x} = 0 \quad \Rightarrow 0 = -\frac{1}{2}nA + \frac{n\sqrt{3}}{2}B$

$B = \frac{A}{\sqrt{3}} = \frac{a}{2\sqrt{3}}$

$\therefore x = e^{-\frac{1}{2}nt}\left(\frac{1}{2}a\cos\frac{n\sqrt{3}}{2}t + \frac{a}{2\sqrt{3}}\sin\frac{n\sqrt{3}}{2}t\right)$

or $x = \frac{a}{2}e^{-\frac{1}{2}nt}\left(\cos\frac{n\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{n\sqrt{3}}{2}t\right)$

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Damped and forced harmonic motion
Exercise B, Question 1

Question:

A particle P is attached to end A of a light elastic spring AB. The end B of the spring is oscillating. At time t the displacement of P from a fixed point is x. When $t = 0, x = 0$ and $\frac{dx}{dt} = \frac{k}{5}$ where k is a constant. Given that x satisfies the differential equation $\frac{d^2x}{dt^2} + 9x = k\cos t$, find x as a function of t.

Solution:

$$\frac{d^2x}{dt^2} + 9x = k\cos t$$

Solve the equation using the methods of book FP2 chapter 5.

Auxiliary equation:

$$m^2 + 9 = 0$$
$$m = \pm 3i$$

Complementary function:

$$x = A\cos 3t + B\sin 3t$$

Particular integral:

try $x = p\cos t + q\sin t$

$$\dot{x} = -p\sin t + q\cos t$$
$$\ddot{x} = -p\cos t - q\sin t$$

$$-p\cos t - q\sin t + 9(p\cos t + q\sin t) = k\cos t$$

Substitute the above results in the differential equation.

$$-p + 9p = k$$

Equating coefficients of $\cos t$ …

$$p = \frac{k}{8}$$

$$-q + 9q = 0 \quad q = 0$$

… and of $\sin t$.

Complete solution:

$$x = A\cos 3t + B\sin 3t + \frac{k}{8}\cos t$$

$$t = 0 \; x = 0 \Rightarrow 0 = A + \frac{k}{8} \quad A = -\frac{k}{8}$$

$$\dot{x} = -3A\sin 3t + 3B\cos 3t - \frac{k}{8}\sin t$$

Use the initial conditions given in the question to obtain values of A and B.

$$t = 0, \; \dot{x} = \frac{k}{5} \Rightarrow \frac{k}{5} = 3B \quad B = \frac{k}{15}$$

$$\therefore x = -\frac{k}{8}\cos 3t + \frac{k}{15}\sin 3t + \frac{k}{8}\cos t$$

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Damped and forced harmonic motion
Exercise B, Question 2

Question:

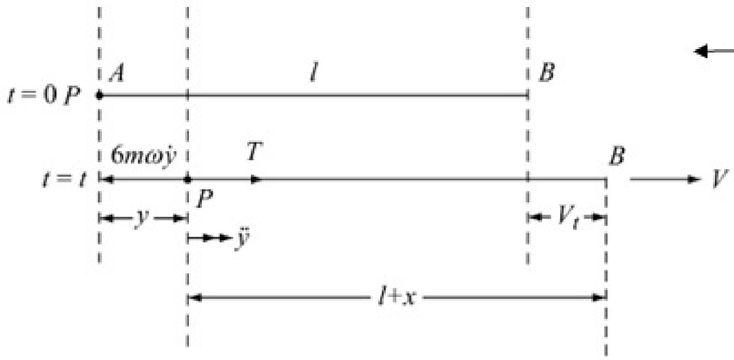
A particle P of mass m lies at rest on a horizontal table attached to end A of a light elastic spring AB of natural length l and modulus of elasticity $9m\omega^2 l$. At time $t = 0, AB = l$. The end B of the spring is now moved along the table in the direction AB with constant speed V. The resistance to motion of P has magnitude $6m\omega v$, where v is the speed of P and ω is a constant. At time t the extension of the spring is x and the displacement of P from its initial position is y.
Show that

a $x + y = Vt$,

b $\frac{d^2x}{dt^2} + 6\omega\frac{dx}{dt} + 9\omega^2 x = 6\omega V$.

c Find an expression for x in terms of t, ω and V.

Solution:



Draw diagrams to show the initial situation and the situation at time t. Mark all distances clearly.

$\lambda = 9m\,w^2 l$

a $y + (l + x) = l + Vt$

$x + y = Vt$ ①

Use the diagrams to form this equation.

b Hooke's law: $T = \dfrac{\lambda x}{l} = \dfrac{9m\omega^2 l}{l}x$

$T = 9m\omega^2 x$

$F = ma$: $\quad T - 6m\omega\dot{y} = m\ddot{y}$

$9m\omega^2 x - 6m\omega\dot{y} = m\ddot{y}$

The displacement of P from its initial position is y, not x.

From ①

$\dot{x} + \dot{y} = V$

$\ddot{x} + \ddot{y} = 0$

Use ① to obtain $\dot{y}$ and $\ddot{y}$ in terms of $\dot{x}$ and $\ddot{x}$

$\therefore 9m\omega^2 x - 6m\omega(V - \dot{x}) = m(-\ddot{x})$

$\ddot{x} + 6\omega\dot{x} + 9\omega^2 x = 6\omega V$

or $\dfrac{d^2x}{dt^2} + 6\omega\dfrac{dx}{dt} + 9\omega^2 x = 6\omega V$

c Auxiliary equation: $m^2 + 6m\omega + 9\omega^2 = 0$

Solve the equation using the methods of book FP2 Chapter 5.

$(m + 3\omega)^2 = 0$

$m = -3\omega$ (twice)

Complementary function:

$x = (A + Bt)e^{-3\omega t}$

Particular integral: try $x = k$

$\dot{x} = \ddot{x} = 0$

$9\omega^2 k = 6\omega V$

$k = \dfrac{2V}{3\omega}$

$\therefore$ Complete solution:

$x = (A + Bt)e^{-3\omega t} + \dfrac{2V}{3\omega}$

$t = 0,\ x = 0 \Rightarrow 0 = A + \dfrac{2V}{3\omega}$

Use the initial conditions given in the question to obtain expressions for A and B.

$A = -\dfrac{2V}{3\omega}$

$\dot{x} = Be^{-3\omega t} - 3\omega(A + Bt)e^{-3\omega t}$

$t = 0,\ \dot{x} = 0 \Rightarrow 0 = B - 3\omega A$

$B = 3\omega A = -2V$

$\therefore x = \left(-\dfrac{2V}{3\omega} - 2Vt\right)e^{-3\omega t} + \dfrac{2V}{3\omega}$

or $x = \dfrac{2V}{3\omega}(1 - e^{-3\omega t} - 3\omega t\, e^{-3\omega t})$

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Damped and forced harmonic motion
Exercise B, Question 3

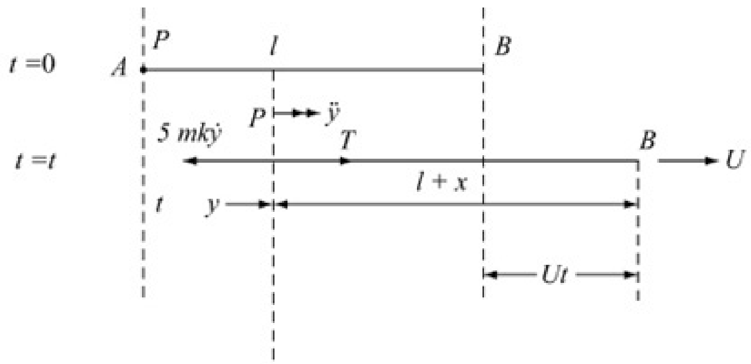
Question:

A particle P of mass m is attached to end A of a light elastic spring AB of natural length l and modulus of elasticity $6mk^2l$. Initially the spring and the particle lie at rest on a horizontal surface with $AB = l$. The end B of the spring is then moved in a straight line in the direction AB with constant speed U. As P moves on the surface it is subject to a resistance of magnitude $5mkv$ where v is the speed of P. At time $t, t > 0$, the extension of the spring is x.

a Show that $\frac{d^2x}{dt^2} + 5k\frac{dx}{dt} + 6k^2x = 5kU$.

b Find an expression for x in terms of t.

Solution:



Draw diagrams to show the initial situation and the situation at time t.
Let the distance moved by P from its initial position be y.
Mark all distances clearly.

a $y+(l+x)=l+Ut$

$y+x=Ut$ ①

Obtain a connection between y and the extension x.

Hooke's law: $T=\frac{\lambda x}{l}=\frac{6mk^2lx}{l}$

$T=6mk^2x$

$F=ma \quad T-5mk\dot{y}=m\ddot{y}$

Using ① $\dot{y}+\dot{x}=U$

$\ddot{y}+\ddot{x}=0$

Use ① to obtain $\dot{y}$ and $\ddot{y}$ in terms of $\dot{x}$ and $\ddot{x}$.

$\therefore 6mk^2x-5mk(U-\dot{x})=m(-\ddot{x})$

$\ddot{x}+5k\dot{x}+6k^2x=5kU$

or $\frac{d^2x}{dt^2}+5k\frac{dx}{dt}+6k^2x=5kU$

b Auxiliary equation: $m^2+5km+6k^2=0$

Now solve the equation using the methods of book FP2 Chapter 5.

$(m+3k)(m+2k)=0$

$m=-3k$ or $-2k$

Complementary function:

$x=Ae^{-3kt}+Be^{-2kt}$

Particular integral: try $x=a$

$\dot{x}=\ddot{x}=0$

$\therefore 6k^2a=5kU$

$a=\frac{5U}{6k}$

Complete Solution: $x=Ae^{-3kt}+Be^{-2kt}+\frac{5U}{6k}$

$t=0, x=0 \Rightarrow 0=A+B+\frac{5U}{6k}$ ①

$\dot{x}=-3kAe^{-3kt}-2kBe^{-2kt}$

$t=0, \dot{x}=0 \Rightarrow 0=-3kA-2kB$

$3A+2B=0$ ②

$\therefore 2A-3A+\frac{5U}{3k}=0$

Solve ① and ② simultaneously.

$A=\frac{5U}{3k}, B=-\frac{5U}{2k}$

$\therefore x=\frac{5U}{3k}e^{-3kt}-\frac{5U}{2k}e^{-2kt}+\frac{5U}{6k}$

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Damped and forced harmonic motion
Exercise B, Question 4

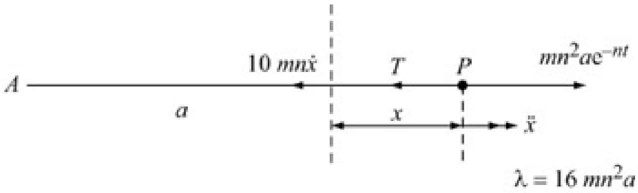
Question:

A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity $16mn^2a$. The other end of the string is attached to a fixed point A on the horizontal table on which P lies. At time $t = 0$, P is at rest on the table with $AP = a$. A force of magnitude mn^2ae^{-nt}, $t \geq 0$, acting in the direction AP is applied to P. The motion of P is opposed by a resistance of magnitude $10mnv$, where v is the speed of P. At time t, $t > 0$, the extension of the string is x.

a Show that $\frac{d^2x}{dt^2} + 10n\frac{dx}{dt} + 16n^2x = n^2ae^{-nt}$.

b Find an expression for x in terms of t.

Solution:



a Hooke's law: $T = \frac{\lambda x}{l}$

$T = \frac{16\,mn^2a}{a}x = 16mn^2x$

$F = ma$

$mn^2ae^{-nt} - T - 10mn\dot{x} = m\ddot{x}$

$\ddot{x} + 10n\dot{x} + 16n^2x = n^2ae^{-nt}$

or $\frac{d^2x}{dt^2} + 10n\frac{dx}{dt} + 16n^2x = n^2ae^{-nt}$

b Auxiliary equation: $m^2 + 10mn + 16n^2 = 0$

$(m+2n)(m+8n) = 0$

$m = -8n, m = -2n$

∴ Complementary function:

$x = Ae^{-8nt} + Be^{-2nt}$

Particular integral: try $x = ke^{-nt}$

$\dot{x} = -nke^{-nt}$

$\ddot{x} = n^2ke^{-nt}$

$\therefore n^2ke^{-nt} - 10n^2ke^{-nt} + 16n^2ke^{-nt} = n^2ae^{-nt}$

$7n^2ke^{-nt} = n^2ae^{-nt}$

$k = \frac{a}{7}$

Solve the equation using the methods of book FP2 Chapter 5.

Complete solution:

$x = Ae^{-8nt} + Be^{-2nt} + \frac{a}{7}e^{-nt}$

$t = 0, x = 0 \Rightarrow 0 = A + B + \frac{a}{7}$ ①

$\dot{x} = -8nAe^{-8nt} - 2nBe^{-2nt} - \frac{an}{7}e^{-nt}$

$t = 0, \dot{x} = 0 \quad 0 = -8nA - 2nB - \frac{an}{7}$

Use the initial conditions given in the question to obtain values for A and B.

$8A + 2B + \frac{a}{7} = 0$ ②

$6A - \frac{a}{7} = 0$

$A = \frac{a}{42}$

$B = -\frac{a}{42} - \frac{a}{7} = -\frac{a}{6}$

$\therefore x = \frac{a}{42}e^{-8nt} - \frac{a}{6}e^{-2nt} + \frac{a}{7}e^{-nt}$

Solve equations ① and ② simultaneously.

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Damped and forced harmonic motion
Exercise B, Question 5

Question:

A particle P of mass 0.5 kg is attached to end A of a light elastic string AB of natural length 0.8 m and modulus of elasticity 5 N. The particle and string lie on a smooth horizontal plane with $AB = 0.8$ m. At time $t = 0$ a variable force F N is applied to the end B of the string which then moves with a constant speed 5 m s^{-1} in the direction AB. The particle moves along the plane and is subject to air resistance of magnitude $0.5v$ newtons, where $v \text{ m s}^{-1}$ is the speed of P. At time t seconds the displacement of P from its initial position is y metres and the extension of the string is x metres.

Show that, while the string is taut,

a $x + y = 5t$,

b $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 12.5x = 5.$

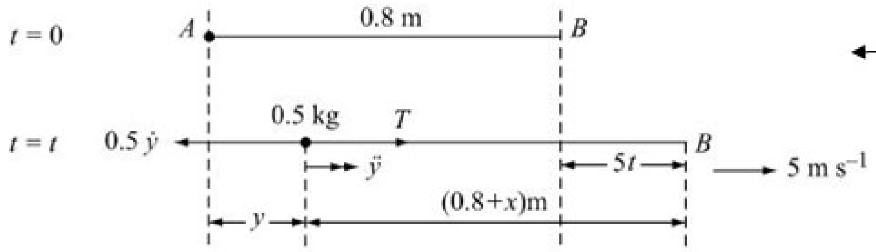
Find

c an expression for x in terms of t,

d the exact distance travelled by P in the first π seconds,

e the exact value of F when $t = \pi$.

Solution:



Draw diagrams to show all the distances at $t = 0$ and at time t clearly.

a $y + (0.8 + x) = 0.8 + 5t$

$x + y = 5t$ ①

Use the diagrams to form this equation

b Hooke's law: $T = \frac{\lambda x}{l} = \frac{5x}{0.8} = 6.25x$

$F = ma \quad T - 0.5\dot{y} = 0.5\ddot{y}$

$6.25x - 0.5\dot{y} = 0.5\ddot{y}$

$\dot{x} + \dot{y} = 5$

Use equation ①.

$\ddot{x} + \ddot{y} = 0$

$\therefore 6.25x - 0.5(5 - \dot{x}) = 0.5(-\ddot{x})$

$\ddot{x} + \dot{x} + 12.5x = 5$

or $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 12.5x = 5$

c Auxiliary equation: $m^2 + m + 12.5 = 0$

Now solve the equation using the methods of book FP2 Chapter 6.

$m = \frac{-1 \pm \sqrt{(1 - 50)}}{2}$

$m = \frac{-1 \pm 7i}{2}$

Complementary function is

$x = e^{-\frac{1}{2}t}\left(A\cos\frac{7}{2}t + B\sin\frac{7}{2}t\right)$

Particular integral: try $x = k$

$\dot{x} = \ddot{x} = 0$

$12.5k = 5$

$k = \frac{5}{12.5} = \frac{2}{5}$

General solution:

$x = e^{-\frac{1}{2}t}\left(A\cos\frac{7}{2}t + B\sin\frac{7}{2}t\right) + \frac{2}{5}$

$t = 0,\ x = 0 \Rightarrow 0 = A + \frac{2}{5} \Rightarrow A = -\frac{2}{5}$

Use the initial conditions given in the question to obtain values for A and B.

$\dot{x} = -\frac{1}{2}e^{-\frac{1}{2}t}\left(A\cos\frac{7}{2}t + B\sin\frac{7}{2}t\right)$

$$+\,\mathrm{e}^{-\frac{1}{2}t}\left(-\frac{7}{2}A\sin\frac{7}{2}t+\frac{7}{2}B\cos\frac{7}{2}t\right)$$

$$t=0,\ \dot{x}=0\Rightarrow 0=-\frac{1}{2}A+\frac{7}{2}B$$

$$B=\frac{A}{7}=-\frac{2}{35}$$

$$\therefore x=\mathrm{e}^{-\frac{1}{2}t}\left(-\frac{2}{5}\cos\frac{7}{2}t-\frac{2}{35}\sin\frac{7}{2}t\right)+\frac{2}{5}$$

d $t=\pi,\ x=\frac{2}{5}-\mathrm{e}^{-\pi/2}\times\left(\frac{-2}{35}\right)(-1)$

x is the extension of the string, not the distance travelled by P.

$$y=5t-x$$

$$=5\pi-\frac{2}{5}+\frac{2}{35}\mathrm{e}^{-\frac{\pi}{2}}$$

e $F=T=\frac{25x}{4}$

End B is moving at a constant speed, so it is in equilibrium.

$$t=\pi\ \ F=\frac{25}{4}\left(\frac{2}{5}-\frac{2}{35}\mathrm{e}^{-\frac{\pi}{2}}\right)$$

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Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise C, Question 1

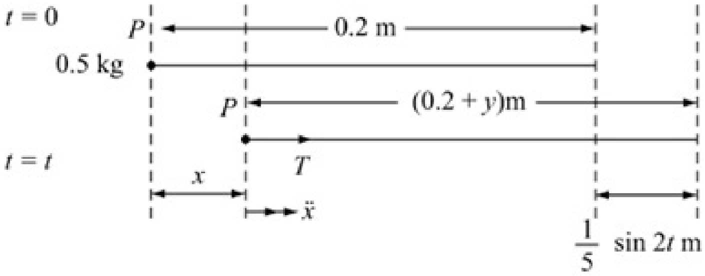
Question:

A particle P of mass 0.5 kg is free to move horizontally inside a smooth cylindrical tube. The particle is attached to one end of a light elastic spring of natural length 0.2 m and modulus of elasticity 5 N. At time $t = 0$ the system is at rest with the spring at its natural length. The other end of the spring is then forced to oscillate with simple harmonic motion so that at time t seconds, $t > 0$, its displacement from its initial position is $\frac{1}{5}\sin 2t$ metres and the displacement of P from its initial position is x metres.

a Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 50x = 10\sin 2t$.

b Find an expression for x in terms of t.

Solution:



Let the extension of the spring be y m and draw diagrams showing the situation when $t = 0$ and at time t seconds.

a Hooke's law: $T = \frac{\lambda x}{l} = \frac{5y}{0.2} = 25y$

$F = ma$:

$T = 0.5\ddot{x}$

$25y = 0.5\ddot{x}$

From the diagrams:

$$0.2 + \frac{1}{5}\sin 2t = (0.2 + y) + x$$

Use the lengths shown in the diagrams to form this equation.

$$x + y = \frac{1}{5}\sin 2t$$

$$\therefore 25\left(\frac{1}{5}\sin 2t - x\right) = 0.5\ddot{x}$$

$$\ddot{x} + 50x = 10\sin 2t$$

$$\text{or } \frac{d^2x}{dt^2} + 50x = 10\sin 2t$$

b Auxiliary equation: $m^2 + 50 = 0$

$$m = \pm 5i\sqrt{2}$$

Now solve the differential equation using the methods of book FP2 Chapter 5.

Complementary function:

$x = A\cos 5\sqrt{2}t + B\sin 5\sqrt{2}t$

Particular integral:

Try : $x = P\cos 2t + Q\sin 2t$

$\dot{x} = -2P\sin 2t + 2Q\cos 2t$

$\ddot{x} = -4P\cos 2t - 4Q\sin 2t$

$\therefore -4P\cos 2t - 4Q\sin 2t + 50(P\cos 2t + Q\sin 2t)$

$= 10\sin 2t$

$\Rightarrow 46Q = 10 \quad Q = \frac{10}{46} = \frac{5}{23}$

$P = 0$

Equate coefficients of sin $2t$ and cos $2t$.

$\therefore$ Complete solution is

$$x = A\cos 5\sqrt{2}t + B\sin 5\sqrt{2}t + \frac{5}{23}\sin 2t$$

Use the initial conditions given in the question to obtain values for A and B.

$t = 0, x = 0 \Rightarrow 0 = A$

$\dot{x} = 5\sqrt{2}B\cos 5\sqrt{2}t + \frac{10}{23}\cos 2t$

$t = 0, \dot{x} = 0 \quad 0 = 5\sqrt{2}B + \frac{10}{23}$

$B = -\frac{\sqrt{2}}{23}$

$$\therefore x = \frac{5}{23}\sin 2t - \frac{\sqrt{2}}{23}\sin 5\sqrt{2}t$$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise C, Question 2

Question:

A particle P of mass m is moving in a straight line. At time t the displacement of P from a fixed point O of the line is x. Given that x satisfies the differential equation $\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+n^2x=0$ where k and n are positive constants with $k<n$,

a find an expression for x in terms of k, n and t.

b Write down the period of the motion.

Solution:

a $\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+n^2x=0$

Solve the equation using the methods of book FP2 Chapter 5.

Auxiliary equation: $m^2+2km+n^2=0$

$$m=\frac{-2k\pm\sqrt{(4k^2-4n^2)}}{2}$$

$$m=-k\pm\sqrt{(k^2-n^2)}$$

$0<k<n \Rightarrow k^2-n^2<0$

$$\therefore m=-k\pm i\sqrt{(n^2-k^2)}$$

General solution:

$$x=e^{-kt}(A\cos\sqrt{(n^2-k^2)}t+B\sin\sqrt{(n^2-k^2)}t)$$

b Period $=\frac{2\pi}{\sqrt{(n^2-k^2)}}$

[You can write the general solution in its alternative form $x=A'e^{-kt}\cos(\omega t+\varepsilon)$ where $\omega=\sqrt{(n^2-k^2)}$ if you prefer.]

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Damped and forced harmonic motion
Exercise C, Question 3

Question:

A particle P of mass m is attached to one end of light elastic spring of natural length l and modulus of elasticity $2mk^2l$. The other end of the spring is attached to a fixed point A and P is hanging in equilibrium with AP vertical.

a Find the length of the spring.

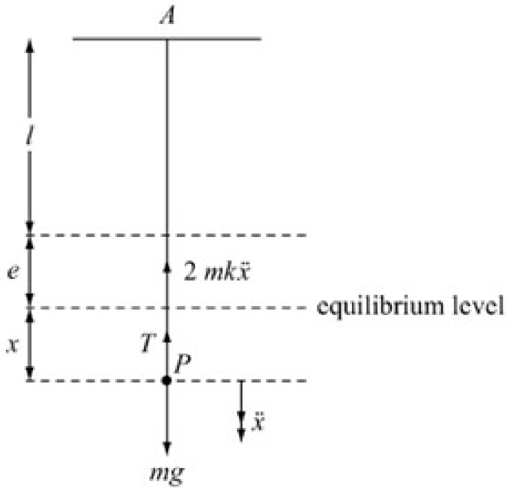
The particle is now projected vertically downwards from its equilibrium position with speed U. A resistance of magnitude $2mkv$, where v is the speed of P, acts on P. At time $t, t > 0$, the displacement of P from its equilibrium position is x.

b Show that $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0$.

c Show that P is instantaneously at rest when $kt = (n + \frac{1}{4})\pi$, where $n \in \mathbb{N}$

d Sketch the graph of x against t.

Solution:



$\lambda = 2mk^2 l$

a In equilibrium: $R(\uparrow) T = mg$

Hooke's law $T = \frac{\lambda x}{l} = 2mk^2 e$

$\therefore mg = 2mk^2 e$ ①

$e = \frac{g}{2k^2}$

The length of the spring is $l + \frac{g}{2k^2}$

b Hooke's law: $T = \frac{2mk^2 l}{l}(x + e)$

$F = ma : mg - T - 2mk\dot{x} = m\ddot{x}$

$g - 2k^2(x + e) - 2k\dot{x} = m\ddot{x}$

$g - 2k^2 x - g - 2k\dot{x} = \ddot{x}$

$\ddot{x} + 2k\dot{x} + 2k^2 x = 0$

or $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2 x = 0$

From ① $2k^2 e = g$

c Auxiliary equation: $m^2 + 2km + 2k^2 = 0$

$m = \frac{-2k \pm \sqrt{(4k^2 - 8k^2)}}{2}$

$m = \frac{-2k \pm \sqrt{(-4k^2)}}{2}$

$m = -k \pm k\mathrm{i}$

An expression for x must be found in order to answer parts **c** and **d**. Use the methods of book FP2 chapter 5 to solve the differential equation.

General solution:

$$x = e^{-kt}(A\cos kt + B\sin kt)$$

$$t = 0,\ x = 0 \Rightarrow 0 = A$$

$$\dot{x} = -ke^{-kt}B\sin kt + Be^{-kt}k\cos kt$$

$$t = 0,\ \dot{x} = U \Rightarrow U = Bk$$

$$B = \frac{U}{k}$$

$$\therefore x = e^{-kt}\frac{U}{k}\sin kt$$

$$\dot{x} = -ke^{-kt}\frac{U}{k}\sin kt + \frac{U}{k}e^{-kt}k\cos kt$$

$$\dot{x} = 0 \quad Ue^{-kt}(\sin kt - \cos kt) = 0$$

$$\sin kt = \cos kt$$

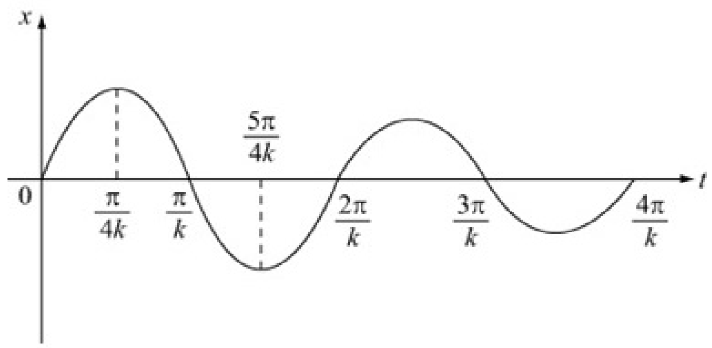
$$\tan kt = 1$$

$$kt = \frac{\pi}{4} + n\pi$$

$$kt = \left(n + \frac{1}{4}\right)\pi, n \in \mathbb{N}$$

Use the initial conditions given in the question to obtain expressions for A and B.

d



From **c**, the maxima and minima occur when $kt = \left(n + \frac{1}{4}\right)\pi$.

Multiplying $\sin kt$ by e^{-kt} causes the amplitude to decrease as t increases.

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Damped and forced harmonic motion
Exercise C, Question 4

Question:

A particle P of mass m is attached to one end of light elastic spring of natural length l and modulus of elasticity mn^2l. The other end of the spring is attached to the roof of a stationary lift. The particle is hanging in equilibrium with the spring vertical. At time $t = 0$ the lift starts to move vertically upwards with constant speed U. At time $t, t > 0$, the displacement of P from its initial position is x.
By considering the extension in the spring,

a show that $\frac{d^2x}{dt^2} + n^2x = n^2Ut$,

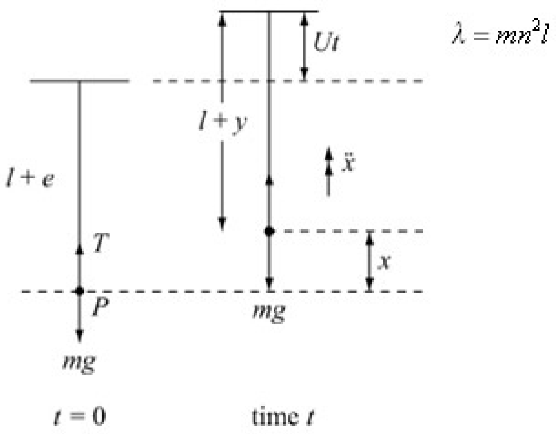
b find an expression for x in terms of t and n.

At time $t = T$, the particle is instantaneously at rest. Find

c the smallest value of T,

d the displacement of P from its initial position at this time.

Solution:



a Let the extension in the spring at time t be y.

$l + y + x = l + e + Ut$

$y + x = e + Ut$ ①

Use the distances on the diagrams to form this equation.

When $t = 0$, P is in equilibrium

R(↑) $T = Mg$

Hooke's Law: $T = \frac{\lambda x}{l} = \frac{mn^2 l}{l} \times e$

$\therefore mn^2 e = mg$ ②

At time t:

Hooke's law: $T = \frac{mn^2 l}{l} y$

$F = ma$

$T - mg = m\ddot{x}$

$mn^2 y - mg = m\ddot{x}$

Using ①:

$mn^2(e + Ut - x) - mg = m\ddot{x}$

From ①
$y = e + Ut - x$

Using ②:

$mg + mn^2 Ut - mn^2 x - mg = m\ddot{x}$

$\ddot{x} + n^2 x = n^2 Ut$

or $\frac{d^2 x}{dt^2} + n^2 x = n^2 Ut$

From ②
$mn^2 e = mg$

b Auxiliary equation:

$m^2 + n^2 = 0$

$m = \pm in$

Solve the differential equation using the methods of book FP2 Chapter 5.

Complementary function:

$x = A\cos nt + B\sin nt$

Particular integral:

try $x = Ct + D$

$\dot{x} = C$

$\ddot{x} = 0$

$\therefore n^2(Ct + D) = n^2Ut$

$C = U \quad D = 0$

Complete solution:

$x = A\cos nt + B\sin nt + Ut$

$t = 0,\ x = 0 \Rightarrow A = 0$

$\dot{x} = Bn\cos nt + U$

Use the initial conditions given in the question to obtain expressions for A and B.

$t = 0,\ \dot{x} = 0 \Rightarrow 0 = Bn + U$

$B = -\frac{U}{n}$

$\therefore x = Ut - \frac{U}{n}\sin nt$

c $\dot{x} = Bn\cos ut + U$

$\dot{x} = U - U\cos nt$

From **b**.

$\dot{x} = 0 \quad 0 = 1 - \cos nt$

$\cos nt = 1$

$nt = 0, 2\pi, \ldots$

$\therefore$ Smallest T is $\frac{2\pi}{n}$

d $t = \frac{2\pi}{n} \Rightarrow x = U \times \frac{2\pi}{n} - \frac{U}{n}\sin 2\pi$

$x = \frac{2U\pi}{n} - 0$

P has moved a distance $\frac{2U\pi}{n}$ when it first comes to rest.

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Damped and forced harmonic motion
Exercise C, Question 5

Question:

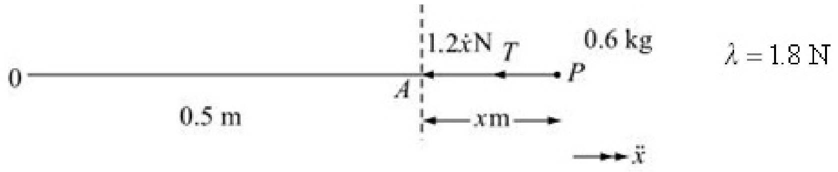
A particle P of mass 0.6 kg is attached to one end of light elastic spring of natural length 0.5 m and modulus of elasticity 1.8 N. The other end of the spring is attached to a fixed point O of the horizontal table on which P lies. At time $t = 0, P$ is at the point A, where $OA = 0.5$ m. The particle is then projected in the direction OA with speed 6 m s^{-1}. The particle is subject to a resistance of magnitude $1.2v$ N, where v m s^{-1} is the speed of P. At time t seconds the extension in the spring is x metres.

a Show that $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 6x = 0$.

b Find x in terms of t.

c Find the value of t the first time P comes to instantaneous rest.

Solution:



a Hooke's Law:

$T = \frac{\lambda x}{l}$

$T = \frac{1.8x}{0.5} = 3.6x$

$F = ma$:

$$T + 1.2\dot{x} = -0.6\ddot{x}$$
$$3.6x + 1.2\dot{x} = -0.6\ddot{x}$$
$$\ddot{x} + 2\dot{x} + 6x = 0$$

or $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 6x = 0$

b Auxiliary equation: $m^2 + 2m + 6 = 0$

$m = \frac{-2 \pm \sqrt{(4 - 24)}}{2}$

$m = -1 \pm i\sqrt{5}$

Solve the differential equation using the methods of book FP2 Chapter 5.

General solution:

$$x = e^{-t}\left(A\cos\sqrt{5}t + B\sin\sqrt{5}t\right)$$

$t = 0,\ x = 0 \Rightarrow 0 = A$

$$\dot{x} = -e^{-t}B\sin\sqrt{5}t + e^{-t}\sqrt{5}B\cos\sqrt{5}t$$

$t = 0,\ \dot{x} = 6 \Rightarrow 6 = \sqrt{5}B$

$$B = \frac{6}{\sqrt{5}}$$

$$\therefore x = \frac{6}{\sqrt{5}}e^{-t}\sin\sqrt{5}t$$

Use the initial conditions given in the question to obtain values for A and B.

c $\dot{x} = -\frac{6}{\sqrt{5}}e^{-t}\sin\sqrt{5}t + 6e^{-t}\cos\sqrt{5}t$

$6e^{-t} \neq 0$

$$\dot{x} = 0 \Rightarrow \frac{1}{\sqrt{5}}\sin\sqrt{5}t = \cos\sqrt{5}t$$

$\tan\sqrt{5}t = \sqrt{5}$

$$t = \frac{1}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

$$t = 0.5144\ldots$$

P first comes to instantaneous rest when $t = 0.514$ s (3 s.f.)

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Edexcel AS and A Level Modular Mathematics

Damped and forced harmonic motion
Exercise C, Question 6

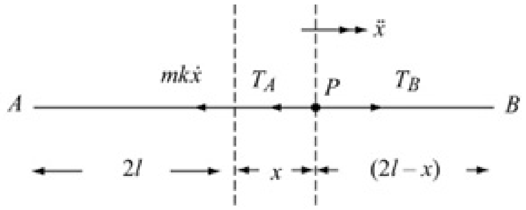
Question:

A particle P of mass m is attached to one end of each of two identical elastic strings of natural length l and modulus of elasticity $2mg$. The free ends of the strings are fixed at points A and B on a smooth horizontal plane where $AB = 4l$. At time $t = 0, P$ is at rest at its equilibrium position. The particle is then projected along the line AB with speed U and moves in a straight line. At time t the displacement of P from its equilibrium position is x. A resistance of magnitude mkv, where v is the speed of P and $k = \sqrt{\frac{g}{l}}$, acts on P. Both strings remain taut throughout the motion.

a Show that $\frac{d^2x}{dt^2} + k\frac{dx}{dt} + 4k^2x = 0$.

b Find an expression for x in terms of U, k, and t.

Solution:



$\lambda = 2mg$

a Hooke's law: $T = \dfrac{\lambda x}{l}$

$T_A = \dfrac{2mg\,(l + x)}{l}, T_B = \dfrac{2mg\,(l - x)}{l}$

$F = ma$:

$$T_B - T_A - mk\dot{x} = m\ddot{x}$$

$$\frac{2mg\,(l - x)}{l} - \frac{2mg\,(l + x)}{l} - mk\dot{x} = m\ddot{x}$$

$$-4\frac{mgx}{l} - mk\dot{x} = m\ddot{x}$$

$$\ddot{x} + k\dot{x} + \frac{4gx}{l} = 0$$

$$\text{or} \quad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + 4k^2 x = 0$$

$$\text{where } k = \sqrt{\frac{g}{l}}$$

b Auxiliary equation: $m^2 + km + 4k^2 = 0$

$$m = \frac{-k \pm \sqrt{\left(k^2 - 16k^2\right)}}{2}$$

$$m = \frac{-k \pm \mathrm{i}k\sqrt{15}}{2}$$

General solution:

$$x = \mathrm{e}^{-\frac{kt}{2}}(A\cos k\frac{\sqrt{15}}{2}t + B\sin k\frac{\sqrt{15}}{2}t)$$

$$t = 0, x = 0 \Rightarrow A = 0$$

$$\dot{x} = -\frac{k}{2}\mathrm{e}^{-\frac{kt}{2}}B\sin\frac{k\sqrt{15}}{2}t + \mathrm{e}^{-\frac{kt}{2}}\frac{k\sqrt{15}}{2}B\cos\frac{k\sqrt{15}}{2}t$$

$$t = 0, \dot{x} = U \Rightarrow U = \frac{Bk\sqrt{15}}{2}$$

$$B = \frac{2U}{k\sqrt{15}}$$

$$\therefore x = \frac{U}{k\sqrt{15}}\mathrm{e}^{\frac{-kt}{2}}\sin\frac{k\sqrt{15}}{2}t$$