# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 1**

**Question:**

Find the magnitude of the vector $(-\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k})$. **[E]**

**Solution:**

$$(-\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -1 & 1\\ -1 & 1 & -1\end{vmatrix}$$

Formulae for finding the vector product are given in the Edexcel formulae booklet which is provided for the examination. One form it gives is $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$ and that has been used here.

$$=\begin{vmatrix}-1 & 1\\ 1 & -1\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1 & 1\\ -1 & -1\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & -1\\ -1 & 1\end{vmatrix}\mathbf{k}$$

$$=\left((-1\times-1)-(1\times1)\right)\mathbf{i}-\left((-1\times-1)-(-1\times1)\right)\mathbf{j}+\left((-1\times1)-(-1\times-1)\right)\mathbf{k}$$

$$=(1-1)\mathbf{i}-(1-(-1))\mathbf{j}+(-1-1)\mathbf{k}=-2\mathbf{j}-2\mathbf{k}$$

Hence

$$\left|(-\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k})\right|=\sqrt{\left((-2)^2+(-2)^2\right)}$$

$$=\sqrt{8}=2\sqrt{2}$$

You use the formula for the magnitude of a vector $|x\mathbf{i}+y\mathbf{j}+z\mathbf{k}|=\sqrt{(x^2+y^2+z^2)}$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 2**

**Question:**

Given that $\mathbf{p} = 3\mathbf{i} + \mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 3\mathbf{j} + c\mathbf{k}$, find the value of the constant $c$ for which the vector $(\mathbf{p} \times \mathbf{q}) + \mathbf{p}$ is parallel to the vector $\mathbf{k}$. **[E]**

**Solution:**

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 1 \\ 1 & 3 & c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 \\ 3 & c \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & c \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 0 \\ 1 & 3 \end{vmatrix}\mathbf{k}$$

$$= -3\mathbf{i} - (3c - 1)\mathbf{j} + 9\mathbf{k}$$

$$\mathbf{p} \times \mathbf{q} + \mathbf{p} = (-3\mathbf{i} - (3c-1)\mathbf{j} + 9\mathbf{k}) + (3\mathbf{i} + \mathbf{k})$$

$$= (1 - 3c)\mathbf{j} + 10\mathbf{k}$$

If $\mathbf{p} \times \mathbf{q} + \mathbf{p}$ is parallel to $\mathbf{k}$ then

$$1 - 3c = 0 \Rightarrow c = \frac{1}{3}$$

You work out $2 \times 2$ determinants using the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, which is in the FP1 specification. Knowledge of the FP1 specification is a prerequisite of the FP3 specification.

If a vector is parallel to $\mathbf{k}$ then both its $\mathbf{i}$ and $\mathbf{j}$ components must be 0. The $\mathbf{i}$ component of $\mathbf{p} \times \mathbf{q} + \mathbf{p}$ is 0 and the $\mathbf{j}$ component, $1 - 3c$ must equal 0, which gives you a simple equation to find $c$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 3**

**Question:**

Referred to a fixed origin $O$, the position vectors of three non-linear points $A$, $B$ and $C$ are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. By considering $\overrightarrow{AB} \times \overrightarrow{AC}$, prove that the area of $\Delta ABC$ can be expressed in the form $\frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$. **[E]**

**Solution:**

$\overrightarrow{AB} = \mathbf{b}-\mathbf{a}, \quad \overrightarrow{AC} = \mathbf{c}-\mathbf{a}$

$\overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$

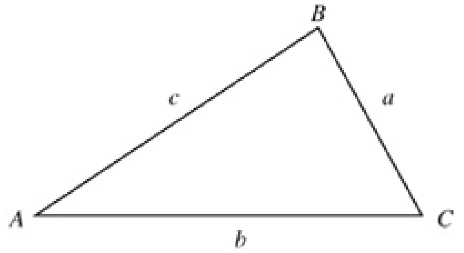
$= \mathbf{b}\times\mathbf{c} - \mathbf{b}\times\mathbf{a} - \mathbf{a}\times\mathbf{c} + \mathbf{a}\times\mathbf{a}$

As $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$, $\mathbf{c}\times\mathbf{a} = -\mathbf{a}\times\mathbf{c}$ and $\mathbf{a}\times\mathbf{a} = \mathbf{0}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{b}\times\mathbf{c} + \mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{a}$

$= \mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}$

You multiply out the brackets using the usual rules of algebra. You must take care with the order in which the vectors are multiplied as the vector product is not commutative. For a vector product $\mathbf{x}\times\mathbf{y} = -\mathbf{y}\times\mathbf{x}$.




The area of the triangle, $\Delta$, say, is given by

$$\Delta = \frac{1}{2}bc\sin A$$

$$= \frac{1}{2}AC \times AB\sin A$$

$$= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|, \text{ as required.}$$

.The magnitude of the vector product $\mathbf{a}\times\mathbf{b}$ is $|\mathbf{a}||\mathbf{b}|\sin\theta$, where $\theta$ is the angle between the vectors. The vector product can be interpreted as a vector with magnitude twice the area of the triangle which has the vectors as two of its sides.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 4**

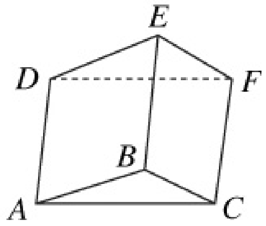
**Question:**

The figure shows a right prism with triangular ends $ABC$ and $DEF$, and parallel edges $AD$, $BE$, $CF$.
Given that $A$ is $(2,7,-1)$, $B$ is $(5, 8, 2)$, $C$ is $(6, 7, 4)$ and $D$ is $(12,1,-9)$,

a find $\overrightarrow{AB}\times\overrightarrow{AC}$,

b find $\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})$.

c Calculate the volume of the prism.




**Solution:**

a
$$\overrightarrow{AB}=(5\mathbf{i}+8\mathbf{j}+2\mathbf{k})-(2\mathbf{i}+7\mathbf{j}-\mathbf{k})$$
$$=3\mathbf{i}+\mathbf{j}+3\mathbf{k}$$
$$\overrightarrow{AC}=(6\mathbf{i}+7\mathbf{j}+4\mathbf{k})-(2\mathbf{i}+7\mathbf{j}-\mathbf{k})$$
$$=4\mathbf{i}+5\mathbf{k}$$
$$\overrightarrow{AB}\times\overrightarrow{AC}=(3\mathbf{i}+\mathbf{j}+3\mathbf{k})\times(4\mathbf{i}+5\mathbf{k})$$
$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 3\\ 4 & 0 & 5\end{vmatrix}$$
$$=\begin{vmatrix}1 & 3\\ 0 & 5\end{vmatrix}\mathbf{i}-\begin{vmatrix}3 & 3\\ 4 & 5\end{vmatrix}\mathbf{j}+\begin{vmatrix}3 & 1\\ 4 & 0\end{vmatrix}\mathbf{k}$$
$$=5\mathbf{i}-3\mathbf{j}-4\mathbf{k}$$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$. It is important to get the vectors the right way round. It is a common error to use $\overrightarrow{AB}=\overrightarrow{OA}-\overrightarrow{OB}$ and obtain the negative of the correct answer.

b
$$\overrightarrow{AD}=(12\mathbf{i}+\mathbf{j}-9\mathbf{k})-(2\mathbf{i}+7\mathbf{j}-\mathbf{k})$$
$$=10\mathbf{i}-6\mathbf{j}-8\mathbf{k}$$
$$\overrightarrow{AD}.(\overrightarrow{AB}\times\overrightarrow{AC})=(10\mathbf{i}-6\mathbf{j}-8\mathbf{k})\cdot(5\mathbf{i}-3\mathbf{j}-4\mathbf{k})$$
$$=10\times5+(-6)\times(-3)+(-8)\times(-4)$$
$$=50+18+32=100$$

$10\mathbf{i}-6\mathbf{j}-9\mathbf{k}=2(5\mathbf{i}-3\mathbf{j}-4\mathbf{k})$ so $\overrightarrow{AD}$ and $\overrightarrow{AB}\times\overrightarrow{AC}$ are parallel. As the vector product is perpendicular to $AB$ and $AC$, it follows that the line $AD$ is perpendicular to the plane of the triangle $ABC$.

c The volume of the prism, $P$ say, is given by
$$P=\frac{1}{2}\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=\frac{1}{2}\times100=50$$

In this case, the volume of the prism is the area of the triangle $ABC$, which is half the magnitude of $\overrightarrow{AB}\times\overrightarrow{AC}$, multiplied by the distance $AD$. (Even if the line $AD$ is not perpendicular to the plane of the triangle $ABC$, the triple scalar product is still twice the volume of the prism.)

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 5**

**Question:**

The plane $\Pi_1$, has vector equation $\mathbf{r} = (5\mathbf{i} + \mathbf{j}) + u(-4\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + v(\mathbf{j} + 2\mathbf{k})$, where $u$ and $v$ are parameters.

**a** Find a vector $\mathbf{n}_1$ normal to $\Pi_1$.

The plane $\Pi_2$ has equation $3x + y - z = 3$.

**b** Write down a vector $\mathbf{n}_2$ normal to $\Pi_2$.

**c** Show that $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Given that the point (1, 1, 1) lies on both $\Pi_1$ and $\Pi_2$,

**d** write down an equation of the line of intersection of $\Pi_1$ and $\Pi_2$ in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $t$ is a parameter. **[E]**

**Solution:**

**a** $\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & 3 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= -\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$

If the equation of a plane is given to you in the form $\mathbf{r} = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$, then you can find a normal to the plane by finding $\mathbf{b} \times \mathbf{c}$.

**b** $\mathbf{n}_2 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

The Cartesian equation $3x + y - z = 3$ can be written in the vector form $\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 3$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Comparison with the standard form, $\mathbf{r} \cdot \mathbf{n} = p$, gives you that $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to $\Pi_2$.

**c** $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 8 & -4 \\ 3 & 1 & -1 \end{vmatrix}$

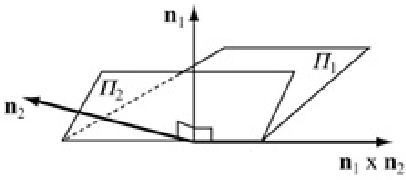
$= \begin{vmatrix} 8 & -4 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -4 \\ 3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 8 \\ 3 & 1 \end{vmatrix}\mathbf{k}$

$= -4\mathbf{i} - 13\mathbf{j} - 25\mathbf{k} = -1(4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k})$

The scalar product $\mathbf{n}_1 \times \mathbf{n}_2$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. So to show that a vector, $\mathbf{r}$ say, is perpendicular to two other vectors, you can show that $\mathbf{r}$ is parallel to the vector product of the two other vectors. An alternative method is to show that the scalar product of $\mathbf{r}$ with each of the other two vectors is zero.

$\mathbf{n}_1 \times \mathbf{n}_2$ is perpendicular to the plane containing $\mathbf{n}_1$ and $\mathbf{n}_2$, and $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$ is a multiple of $\mathbf{n}_1 \times \mathbf{n}_2$. Hence $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$.

**d** $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k})$




This diagram illustrates that the line of intersection of the planes $\Pi_1$ and $\Pi_2$ lies in the direction of $\mathbf{n}_1 \times \mathbf{n}_2$. In this case, $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k} = -\mathbf{n}_1 \times \mathbf{n}_2$ and can be used as the direction of the line, as can any other multiple of this vector.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 6**

**Question:**

The points $A$, $B$ and $C$ lie on the plane $\Pi$ and, relative to a fixed origin $O$, they have position vectors

$\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \mathbf{b} = -\mathbf{i} + 2\mathbf{j},$

$\mathbf{c} = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$

respectively.

**a** Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

**b** Obtain the equation of $\Pi$ in the form $\mathbf{r} \cdot \mathbf{n} = p$.

The point $D$ has position vector $5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

**c** Calculate the volume of the tetrahedron $ABCD$. **[E]**

**Solution:**

**a**

$$\overrightarrow{AB} = -\mathbf{i}+2\mathbf{j}-(3\mathbf{i}-\mathbf{j}+4\mathbf{k})$$
$$= -4\mathbf{i}+3\mathbf{j}-4\mathbf{k}$$
$$\overrightarrow{AC} = 5\mathbf{i}-3\mathbf{j}+7\mathbf{k}-(3\mathbf{i}-\mathbf{j}+4\mathbf{k})$$
$$= 2\mathbf{i}-2\mathbf{j}+3\mathbf{k}$$
$$\overrightarrow{AB}\times\overrightarrow{AC} = (-4\mathbf{i}+3\mathbf{j}-4\mathbf{k})\times(2\mathbf{i}-2\mathbf{j}+3\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -4 \\ 2 & -2 & 3 \end{vmatrix}$$
$$= \begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & -4 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 3 \\ 2 & -2 \end{vmatrix}\mathbf{k}$$
$$= \mathbf{i}+4\mathbf{j}+2\mathbf{k}$$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. It is important to get the vectors the right way round. It is a common error to use $\overrightarrow{AB} = \overrightarrow{OA} - \overrightarrow{OB}$ and obtain the negative of the correct answer.

**b** An equation of $\Pi$ is

$$\mathbf{r}\cdot(\mathbf{i}+4\mathbf{j}+2\mathbf{k}) = (3\mathbf{i}-\mathbf{j}+4\mathbf{k})\cdot(\mathbf{i}+4\mathbf{j}+2\mathbf{k})$$
$$= 3\times 1+(-1)\times 4+4\times 2$$
$$= 3-4+8 = 7$$

Once you have a vector $\mathbf{n}$ perpendicular to the plane, you can find a vector equation of the plane using $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$, where $\mathbf{a}$ is the position vector of any point on the plane. Here the position vector of $A$ has been used but the position vectors of $B$ and $C$ would do just as well. As the scalar product is quite quickly worked out, it is a useful check to recalculate, using one of the other points. All should give the same answer, here 7.

**c**

$$\overrightarrow{AD} = 5\mathbf{i}+2\mathbf{j}+3\mathbf{k}-(3\mathbf{i}-\mathbf{j}+4\mathbf{k})$$
$$= 2\mathbf{i}+3\mathbf{j}-\mathbf{k}$$
$$\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) = (2\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+4\mathbf{j}+2\mathbf{k})$$
$$= 2\times 1+3\times 4+(-1)\times 2$$
$$= 2+12-2 = 12$$

The volume, $V$ say, of the tetrahedron is given by

$$V = \frac{1}{6}\left|\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})\right| = \frac{1}{6}\times 12 = 2$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 7**

**Question:**

The points $A$ and $B$ have position vectors $4\mathbf{i}+\mathbf{j}-7\mathbf{k}$ and $2\mathbf{i}+6\mathbf{j}+2\mathbf{k}$ respectively relative to a fixed origin $O$.

a Show that angle $AOB$ is a right angle.

b Find a vector equation for the median $AM$ of the triangle $OAB$.

c Find a vector equation of the plane $OAB$, giving your answer in the form $\mathbf{r}\cdot\mathbf{n}=p$.

**[E]**

**Solution:**

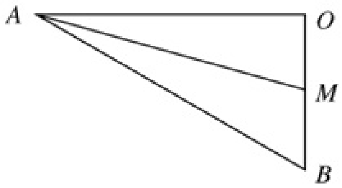
a $\overrightarrow{OA}\cdot\overrightarrow{OB}=(4\mathbf{i}+\mathbf{j}-7\mathbf{k})\cdot(2\mathbf{i}+6\mathbf{j}+2\mathbf{k})$

$=4\times 2+1\times 6+(-7)\times 2$

$=8+6-14=0$

Hence $\angle AOB=90°$, as required.

As the scalar product of two vectors $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$, where $\theta$ is the angle between the vectors, if the scalar product of two non-zero vectors is zero, then $\cos\theta=0$ and the angle between the vectors is a right angle.

b 


The median $AM$ of a triangle is the line joining the vertex $A$ of the triangle to the mid-point $M$ of the side of the triangle which is opposite to $A$.

The coordinates of $M$, the mid-point of $O$ (0, 0, 0) and $B$ (2, 6, 2) are

$\left(\frac{0+2}{2},\frac{0+6}{2},\frac{0+2}{2}\right)=(1,3,1)$

$\overrightarrow{AM}=\mathbf{i}+3\mathbf{j}+\mathbf{k}-(4\mathbf{i}+\mathbf{j}-7\mathbf{k})$

$=-3\mathbf{i}+2\mathbf{j}+8\mathbf{k}$

There are many possible alternative forms for this answer. For example, you could use $M$ as the 'starting point' of the line and obtain an answer such as $\mathbf{r}=\mathbf{i}+3\mathbf{j}+\mathbf{k}+\lambda(3\mathbf{i}-2\mathbf{j}+8\mathbf{k})$.

An equation of $AM$ is $\mathbf{r}=4\mathbf{i}+\mathbf{j}-7\mathbf{k}+\lambda(-3\mathbf{i}+2\mathbf{j}+8\mathbf{k})$

c $\overrightarrow{OA}\times\overrightarrow{OB}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 1 & -7\\ 2 & 6 & 2\end{vmatrix}$

$=\begin{vmatrix}1 & -7\\ 6 & 2\end{vmatrix}\mathbf{i}-\begin{vmatrix}4 & -7\\ 2 & 2\end{vmatrix}\mathbf{j}+\begin{vmatrix}4 & 1\\ 2 & 6\end{vmatrix}\mathbf{k}$

$=44\mathbf{i}-22\mathbf{j}+22\mathbf{k}$

You can use $44\mathbf{i}-22\mathbf{j}+22\mathbf{k}$ or any multiple of this vector as $\mathbf{n}$ in $\mathbf{r}\cdot\mathbf{n}=p$. $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ is used here as it gives a simpler answer.

A vector parallel to $44\mathbf{i}-22\mathbf{j}+22\mathbf{k}$ is $2\mathbf{i}-\mathbf{j}+\mathbf{k}$

An equation of $\Pi$ is $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})=0$

As the plane goes through the origin, the $p$ in $\mathbf{r}.\mathbf{n}=p$ must be zero.

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 8**

**Question:**

Referred to a fixed origin $O$, the point $A$ has position vector $a(4\mathbf{i}+\mathbf{j}+2\mathbf{k})$ and the plane $\Pi$ has equation $\mathbf{r}\cdot(\mathbf{i}-5\mathbf{j}+3\mathbf{k})=5a$, where $a$ is a scalar constant.

**a** Show that $A$ lies in the plane $\Pi$.

The point $B$ has position vector $a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k})$.

**b** Show that $\overrightarrow{BA}$ is perpendicular to the plane $\Pi$.

**c** Calculate, to the nearest one tenth of a degree, $\angle OBA$. **[E]**

**Solution:**

**a** $a(4\mathbf{i}+\mathbf{j}+2\mathbf{k}).(\mathbf{i}-5\mathbf{j}+3\mathbf{k})=a(4\times1+1\times(-5)+2\times3)$

$=a(4-5+6)=5a$

Hence $A$ lies in the plane $\Pi$, as required.

For $A$ to lie on the plane with equation $\mathbf{r}.\mathbf{n}=5a$, when $\mathbf{r}$ is replaced by the position vector of $A$, $\mathbf{r}.\mathbf{n}$ must give $5a$.

**b** $\overrightarrow{BA}=a(4\mathbf{i}+\mathbf{j}+2\mathbf{k})-a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k})$

$=a(2\mathbf{i}-10\mathbf{j}+6\mathbf{k})$

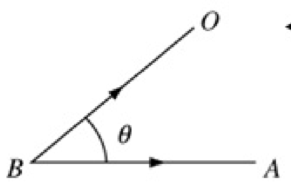
$\overrightarrow{BA}=2a(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$

$\overrightarrow{BA}$ is parallel to the vector $\mathbf{i}-5\mathbf{j}+3\mathbf{k}$, which is perpendicular to the plane $\Pi$.

Hence $\overrightarrow{BA}$ is perpendicular to the plane $\Pi$, as required.

When a plane has an equation of the form $\mathbf{r}.\mathbf{n}=p$, the vector $\mathbf{n}$ is perpendicular to the plane.

**c**




The angle $OBA$ is the angle between $BO$ and $BA$. Both these line segments have a definite sense and so you must use the scalar product $\overrightarrow{BO}.\overrightarrow{BA}$ to find $\theta$. If you used $\overrightarrow{OB}.\overrightarrow{BA}$, you would obtain the supplementary angle $(180°-\theta)$, which is not the correct answer.

Let $\angle OAB=\theta$

$|\overrightarrow{BO}|=a\sqrt{((-2^2)+(-11)^2+4^2)}=a\sqrt{(141)}$

$|\overrightarrow{BA}|=a\sqrt{(2^2+(-10)^2+6^2)}=a\sqrt{(140)}$

$\overrightarrow{BO}.\overrightarrow{BA}=a(-2\mathbf{i}-11\mathbf{j}+4\mathbf{k}).a(2\mathbf{i}-10\mathbf{j}+6\mathbf{k})$

$|\overrightarrow{BO}|.|\overrightarrow{BA}|\cos\theta=a^2((-2)\times2+(-11)\times(-10)+4\times6)$

$a\sqrt{(141)}\times a\sqrt{(140)}\cos\theta=a^2(-4+110+24)$

$\cos\theta=\dfrac{130}{\sqrt{(141)}\sqrt{(140)}}=0.925272\ldots$

$\theta=22.3°$ (to the nearest one tenth of a degree)

Finding the angle between two vectors using the scalar product is part of the C4 specification. Knowledge of the C4 specification is a pre-requisite of the FP3 specification.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 9**

**Question:**

The points $A$, $B$, $C$ and $D$ have coordinates (3, 1, 2), (5, 2, −1), (6, 4, 5) and (−7, 6, −3) respectively.

**a** Find $\overrightarrow{AC} \times \overrightarrow{AD}$.

**b** Find a vector equation of the line through $A$ which is perpendicular to $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

**c** Verify that $B$ lies on this line.

**d** Find the volume of the tetrahedron $ABCD$. **[E]**

**Solution:**

**a** $\overrightarrow{AC} = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

$\overrightarrow{AD} = \begin{pmatrix} -7 \\ 6 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix}$

$\overrightarrow{AC} \times \overrightarrow{AD} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \times \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 3\times(-5) - 3\times 5 \\ 3\times(-10) - 3\times(-5) \\ 3\times 5 - 3\times(-10) \end{pmatrix}$

$= \begin{pmatrix} -30 \\ -15 \\ 45 \end{pmatrix}$

For writing vectors, you can use either the form with **i**s, **j**s and **k**s, or column vectors, which are used in this solution. Sometimes it may even be appropriate to use a mixture of the two. The form using **i**, **j** and **k** usually gives a more compact solution but many find column vectors quicker to write. The choice is entirely up to you and you may choose to vary it from question to question.

**b** $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$

The vector $\begin{pmatrix} -30 \\ -15 \\ 45 \end{pmatrix}$ is perpendicular to both $\overrightarrow{AC}$ and $\overrightarrow{AD}$. This vector or any multiple of it may be used for the equation of the line.

**c** For $B$ to lie on the line there must be a value of $\lambda$ for which

$\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$

Equating the $x$ components of the vectors

$5 = 3 - 2\lambda \Rightarrow \lambda = -1$

Checking this value of $\lambda$ for the other components

$y$ component:

$1 + \lambda \times (-1) = 1 + (-1)\times(-1) = 2$, as required

$z$ component:

$2 + \lambda \times 3 = 2 + (-1)\times 3 = -1$, as required

Hence, $B$ lies on the line.

**d** $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -30 \\ -15 \\ 45 \end{pmatrix} = 2\times(-30) + 1\times(-15) + (-3)\times 45$

$= -60 - 15 - 135 = -210$

The volume of the tetrahedron, $V$ say, is given by

$V = \frac{1}{6}\left|\overrightarrow{AB}.(\overrightarrow{AC} \times \overrightarrow{AD})\right| = \frac{1}{6}|-210| = \frac{1}{6} \times 210 = 35$

The volume of the tetrahedron is one sixth of the triple scalar product.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 10**

**Question:**

The line $l_1$ has equation $\mathbf{r} = \mathbf{i} + 6\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{k})$ and the line $l_2$ has equation $\mathbf{r} = 3\mathbf{i} + p\mathbf{j} + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, where $p$ is a constant.

The plane $\Pi_1$ contains $l_1$ and $l_2$.

**a** Find a vector which is normal to $\Pi_1$.

**b** Show that an equation for $\Pi_1$ is $6x + y - 4z = 16$.

**c** Find the value of $p$.

The plane $\Pi_2$ has equation $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2$.

**d** Find an equation for the line of intersection of $\Pi_1$ and $\Pi_2$, giving your answer in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$. **[E]**

**Solution:**

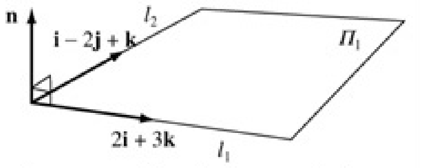
**a** A vector $\mathbf{n}$ perpendicular to $l_1$ and $l_2$ is given by

$$\mathbf{n} = (2\mathbf{i} + 3\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 0 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + \mathbf{j} - 4\mathbf{k}$$




The vector $2\mathbf{i} + 3\mathbf{k}$ is parallel to $l_1$ and $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ is parallel to $l_2$. The vector product of two vectors is perpendicular to both vectors and so is perpendicular to the plane containing both lines

**b** An equation for $\Pi_1$ has the form

$$\mathbf{r}.(6\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = p$$

$$p = (\mathbf{i} + 6\mathbf{j} - \mathbf{k}).(6\mathbf{i} + \mathbf{j} - 4\mathbf{k})$$

$$= 6 + 6 + 4 = 16$$

A vector equation of $\Pi_1$ is

$$\mathbf{r}.(6\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 16$$

A Cartesian equation of $\Pi_1$ is given by

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).(6\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 16$$

$$6x + y - 4z = 16, \text{ as required.}$$

To obtain a Cartesian equation of a plane when you have a vector equation in the form $\mathbf{r}.\mathbf{n} = p$, replace $\mathbf{r}$ by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and work out the scalar product.

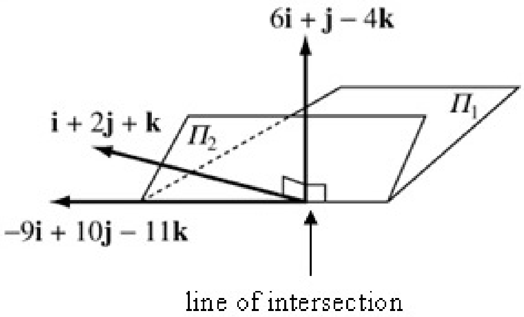
**c** The point with coordinates $(3, p, 0)$ lies on $l_1$ and, hence, must lie on $\Pi_1$.
Substituting $(3, p, 0)$ into the result of part **b**
$18 + p = 16 \Rightarrow p = -2$

**d** The line of intersection lies in the direction given by

$$(\mathbf{i}+2\mathbf{j}+\mathbf{k})\times(6\mathbf{i}+\mathbf{j}-4\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 6 & 1 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 6 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 6 & 1 \end{vmatrix}\mathbf{k}$$

$$= -9\mathbf{i}+10\mathbf{j}-11\mathbf{k}$$




To find one point that lies on both $\Pi_1$ and $\Pi_2$

$\Pi_1$: $6x+y-4z=16$ ①

$\Pi_2$: $x+2y+z=2$ ②

① $+4\times$ ② gives $10x+9y=24$

Choose $x=-3, y=6$

Substitute into ②

$-3+12+z=2 \Rightarrow z=-7$

One point on the line is $(-3, 6, -7)$

An equation of the line is

$(\mathbf{r}-(-3\mathbf{i}+6\mathbf{j}-7\mathbf{k}))\times(-9\mathbf{i}+10\mathbf{j}-11\mathbf{k})=0$

> You need to find just one point that is on both planes and there are infinitely many possibilities. Here you can choose any pair of values of $x$ and $y$ which fit this equation. A choice here has been made which gives whole numbers but you may find, for example, $y=0$, $x=2.4$ easier to see.

> The form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$, for the equation of a straight line, represents a line that passes through the point with position vector $\mathbf{a}$ and is parallel to the vector $\mathbf{b}$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 11**

**Question:**

The plane $\Pi$ passes through the points $A(-2,3,5), B(1,-3,1)$ and $C(4,-6,-7)$.

**a** Find $\overrightarrow{AC} \times \overrightarrow{BC}$.

**b** Hence, or otherwise, find the equation of the plane $\Pi$ in the form $\mathbf{r} \cdot \mathbf{n} = \mathbf{p}$.

The perpendicular from the point (25, 5, 7) to $\Pi$ meets the plane at the point $F$.

**c** Find the coordinates of $F$. **[E]**

**Solution:**

**a**

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ -6 \\ -7 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ -12 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} 4 \\ -6 \\ -7 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -8 \end{pmatrix}$$

$$\overrightarrow{AC} \times \overrightarrow{BC} = \begin{pmatrix} 6 \\ -9 \\ -12 \end{pmatrix} \times \begin{pmatrix} 3 \\ -3 \\ -8 \end{pmatrix} = \begin{pmatrix} 72-36 \\ -36+48 \\ -18+27 \end{pmatrix}$$

$$= \begin{pmatrix} 36 \\ 12 \\ 9 \end{pmatrix}$$

This vector, or any multiple of this vector, can be used for the vector perpendicular to $\Pi$ in part **b**. The working in later parts of the question will usually be simplest if you take the multiple which gives the smallest possible integers. In this case one third of the vector has been used in part **b**.

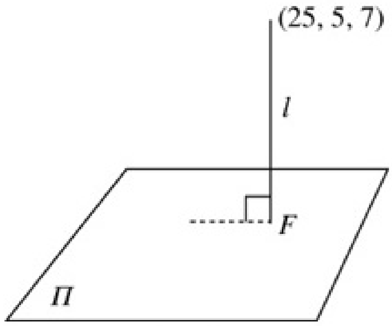
**b** An equation of $\Pi$ is

$$\mathbf{r}.\begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}.\begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} = -24+12+15$$

$$\mathbf{r}.\begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} = 3$$

The position vector of the point $A$ has been used to evaluate $p$ in $\mathbf{r}.\mathbf{n} = p$. You could use the position vector of any of the points, $A$, $B$ and $C$.

c




An equation of the line, $l$ say, which passes through (25, 5, 7) and is perpendicular to $\Pi$ is

$$\mathbf{r} = \begin{pmatrix}25\\5\\7\end{pmatrix} + t\begin{pmatrix}12\\4\\3\end{pmatrix}$$

The equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ represents a line passing through the point with position vector $\mathbf{a}$, in this case $\begin{pmatrix}25\\5\\7\end{pmatrix}$, which is parallel to the vector $\mathbf{b}$. In this case, $l$ is parallel to the normal to the plane, $\begin{pmatrix}12\\4\\7\end{pmatrix}$.

Parametric equations of $l$ are

$x = 25 + 12t, y = 5 + 4t, z = 7 + 3t$

A Cartesian equation of $\Pi$ is

$12x + 4y + 3z = 3$

Writing $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$, the vector equation $\mathbf{r}.\begin{pmatrix}12\\4\\3\end{pmatrix} = 3$ becomes the Cartesian equation $12x + 4y + 3z = 3$.

Substituting $(25 + 12t, 5 + 4t, 7 + 3t)$ into the Cartesian equation of $\Pi$

$$12(25 + 12t) + 4(5 + 4t) + 3(7 + 3t) = 3$$
$$300 + 144t + 20 + 16t + 21 + 9t = 3$$
$$169t = -338$$
$$t = -2$$

The coordinates of $F$ are given by

$(25 + 12t, 5 + 4t, 7 + 3t)$

$= (25 - 24, 5 - 8, 7 - 6)$

$= (1, -3, 1)$

$t = -2$ is the value of the parameter $t$ at the point where the line intersects the plane. Substituting $t = -2$ into the parametric form of the line then gives you the coordinates of $F$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 12**

**Question:**

The plane $\Pi$ passes through the points $P(-1,3,-2)$, $Q(4,-1,-1)$ and $R(3, 0, c)$, where $c$ is a constant.

**a** Find, in terms of $c$, $\overrightarrow{RP} \times \overrightarrow{RQ}$.

Given that $\overrightarrow{RP} \times \overrightarrow{RQ} = 3\mathbf{i} + d\mathbf{j} + \mathbf{k}$, where $d$ is a constant,

**b** find the value of $c$ and show that $d = 4$.

**c** Find an equation of $\Pi$ in the form $\mathbf{r} \cdot \mathbf{n} = p$, where $p$ is a constant.

The point $S$ has position vector $\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$. The point $S'$ is the image of $S$ under reflection in $\Pi$.

**d** Find the position vector of $S'$. **[E]**

**Solution:**

**a** $\overrightarrow{RP} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ -2-c \end{pmatrix}$

$\overrightarrow{RQ} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1-c \end{pmatrix}$

$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{pmatrix} -4 \\ 3 \\ -2-c \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -1-c \end{pmatrix}$

$= \begin{pmatrix} 3(-1-c)-(2+c) \\ -2-c-4(1+c) \\ 4-3 \end{pmatrix} = \begin{pmatrix} -5-4c \\ -6-5c \\ 1 \end{pmatrix}$

In this solution, the vector product has be found using the formula $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$. This formula can be found in the Edexcel formulae booklet which is provided for the examination.

**b** $\begin{pmatrix} -5-4c \\ -6-5c \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ d \\ 1 \end{pmatrix}$

Equating the $x$ components

$-5-4c = 3 \Rightarrow 4c = -8 \Rightarrow c = -2$

Equating the $y$ components

$d = -6-5c = -6-5\times(-2) = -6+10$

$= 4$, as required.

**c** When $c = -2$

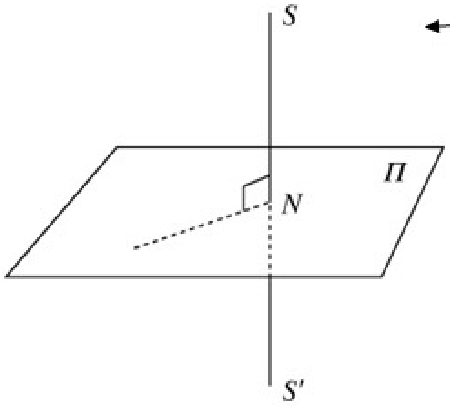
$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{pmatrix} -5-4c \\ -6-5c \\ 1 \end{pmatrix} = \begin{pmatrix} -5+8 \\ -6+10 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

An equation of $\Pi$ is

$\mathbf{r}.\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}.\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = -3+12-2$

$\mathbf{r}.\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 7$

**d**




In this diagram, the point $N$ is the intersection of $SS'$ and the plane. As $S'$ is the reflection of $S$ in $\Pi$, $SS'$ is perpendicular to $\Pi$ and $N$ is the mid-point of $SS'$. Hence the translation (or displacement) from $S$ to $N$ is the same as the translation (or displacement) from $N$ to $S'$.
The method used in this solution is to find the position vector of $N$ and, then, to find the translation which maps $S$ to $N$. This translation can then be used to find the position vector of $S'$ from the position vector of $N$.

A vector equation of $SS'$ is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + t\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

Parametric equations for $SS'$ are

$x = 1 + 3t, y = 5 + 4t, z = 10 + t$ ①

A Cartesian equation of $\Pi$ is

$3x + 4y + z = 7$ ②

To find the position vector of $N$, the point of intersection of $SS'$ and $\Pi$, substitute ① into ②

$$3(1+3t) + 4(5+4t) + 10 + t = 7$$
$$3 + 9t + 20 + 16t + 10 + t = 7$$
$$26t = -26 \Rightarrow t = -1$$

The position vector of $N$ is $\begin{pmatrix} 1+3t \\ 5+4t \\ 10+t \end{pmatrix} = \begin{pmatrix} 1-3 \\ 5-4 \\ 10-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix}$

The translation which maps $S$ to $N$ is represented by the vector

$$\overrightarrow{SN} = \begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix}$$

The translation which maps $S$ to $N$ will also map $N$ to $S'$.

The position vector of $S'$ is given by

$$\begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \\ 8 \end{pmatrix}$$

The position vector of $S'$ is the position vector of $N$ added to the vector representing the translation.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 13**

**Question:**

The points $A$, $B$ and $C$ lie on the plane $\Pi_1$ and, relative to a fixed origin $O$, they have position vectors
$\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}, \mathbf{b} = 3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$,
$\mathbf{c} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$
respectively.

**a** Find $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ ..

**b** Find an equation of $\Pi_1$, giving your answer in the form $\mathbf{r} \cdot \mathbf{n} = p$ .

The plane $\Pi_2$ has Cartesian equation $x + z = 3$ and $\Pi_1$ and $\Pi_2$ intersect in the line $l$.

**c** Find an equation of $l$ in the form $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = 0$ .

The point $P$ is the point on $l$ that is nearest to the origin $O$.

**d** Find the coordinates of $P$. **[E]**

**Solution:**

**a** $\mathbf{b}-\mathbf{a}=3\mathbf{i}+3\mathbf{j}-4\mathbf{k}-(\mathbf{i}+3\mathbf{j}-\mathbf{k})=2\mathbf{i}-3\mathbf{k}$

$\mathbf{c}-\mathbf{a}=5\mathbf{i}-2\mathbf{j}-2\mathbf{k}-(\mathbf{i}+3\mathbf{j}-\mathbf{k})=4\mathbf{i}-5\mathbf{j}-\mathbf{k}$

$$(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & -3\\ 4 & -5 & -1\end{vmatrix}$$

$$=\begin{vmatrix}0 & -3\\ -5 & -1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2 & -3\\ 4 & -1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2 & 0\\ 4 & -5\end{vmatrix}\mathbf{k}$$

$$=-15\mathbf{i}-10\mathbf{j}-10\mathbf{k}$$

The vector $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$ is perpendicular to both $AB$ and $AC$ and, so, is perpendicular to the plane $\Pi_1$. You can use this vector, or any parallel vector, as the $\mathbf{n}$ in the equation $\mathbf{r}.\mathbf{n}=p$ in part **b**. Here each coefficient has been divided by $-5$. This eases later working and avoids negative signs

**b** A vector perpendicular to $\Pi_1$ is $3\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

A vector equation of $\Pi_1$ is

$$\mathbf{r}.(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})=(\mathbf{i}+3\mathbf{j}-\mathbf{k}).(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})$$

$$=3+6-2=7$$

**c** The line $l$ is parallel to the vector

$$(\mathbf{i}+\mathbf{k})\times(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 1\\ 3 & 2 & 2\end{vmatrix}$$

$$=-2\mathbf{i}+\mathbf{j}+2\mathbf{k}$$

To find one point on both $\Pi_1$ and $\Pi_2$

For $\Pi_1$ $\quad x+z=3$

Let $z=0$, then $x=3$

The form $(\mathbf{r}-\mathbf{p})\times\mathbf{q}=0$ is that of a line passing through a point with position vector $\mathbf{p}$, parallel to the vector $\mathbf{q}$. So you need to find one point on the line; that is any point which is on both $\Pi_1$ and $\Pi_2$. As there are infinitely many such points, there are many possible answers to this question. The choice of $z=0$ here is an arbitrary one.

Substituting $z=0, x=3$ into a Cartesian equation of $\Pi_2$

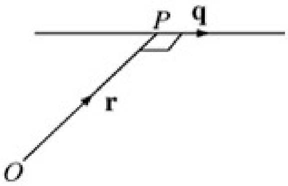
$3x+2y+z=7$

$9+2y+0 \;=7\Rightarrow y=-1$

One point on $\Pi_1$ and $\Pi_2$ and, hence on $l$ is $(3,-1,0)$

Hence, a vector equation of $l$ is $(\mathbf{r}-(3\mathbf{i}-\mathbf{j}))\times(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})=0$

**d** A vector equation of $l$ is




$\mathbf{r}=(3\mathbf{i}-\mathbf{j})+t(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})$

$\quad=(3-2t)\mathbf{i}+(-1+t)\mathbf{j}+2t\mathbf{k}$

At $P$, $\mathbf{r}$ is perpendicular to $l$

$((3-2t)\mathbf{i}+(-1+t)\mathbf{j}+2t\mathbf{k}).(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})=0$

$$-6+4t-1+t+4t=0\Rightarrow 9t=7\Rightarrow t=\frac{7}{9}$$

The coordinates of $P$ are

$$(3-2t,-1+t,2t)=\left(\frac{13}{9},-\frac{2}{9},\frac{14}{9}\right)$$

At the point $P$ which is nearest to the origin $O$, the position vector of $P$, $\mathbf{r}$, is perpendicular to the direction of the line, $\mathbf{q}$. Forming the scalar product $\mathbf{r}.\mathbf{q}$ and equating to zero gives you an equation in $t$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 14**

**Question:**

The points $A(2, 0, -1)$ and $B(4, 3, 1)$ have position vectors **a** and **b** respectively with respect to a fixed origin $O$.

**a** Find $\mathbf{a} \times \mathbf{b}$ .

The plane $\Pi_1$ contains the points $O$, $A$ and $B$.

**b** Verify that an equation of $\Pi_1$ is $x - 2y + 2z = 0$ .

The plane $\Pi_2$ has equation $\mathbf{r} \cdot \mathbf{n} = d$ where $n = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $d$ is a constant. Given that $B$ lies on $\Pi_2$ ,

**c** find the value of $d$.

The planes $\Pi_1$ and $\Pi_2$ intersect in the line $L$.

**d** Find an equation of $L$ in the form $\mathbf{r} = \mathbf{p} + t\mathbf{q}$ , where $t$ is a parameter.

**e** Find the position vector of the point $X$ on $L$ where $OX$ is perpendicular to $L$. **[E]**

**Solution:**

**a** $\mathbf{a}\times\mathbf{b} = (2\mathbf{i}-\mathbf{k})\times(4\mathbf{i}+3\mathbf{j}+\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 4 & 3 & 1 \end{vmatrix} = 3\mathbf{i}-6\mathbf{j}+6\mathbf{k}$$

**b** Substituting (0, 0, 0) into $x-2y+2z$

$0-2\times 0+2\times 0=0$

So the plane with equation $x-2y+2z=0$ contains $O$.

Similarly as

$2-2\times 0+2\times(-1)=2-2=0$

and $4-2\times 3+2\times 1=4-6+2=0$,

the plane with equation $x-2y+2z=0$

contains $A(2,0,-1)$ and $B(4,3,1)$.

'Verify' means check that the equation is satisfied by the data of this particular question. To do this you can just show that the coordinates of $O$, $A$ and $B$ satisfy $x-2y+2z=0$. You do not have to show any general methods.

**c** For $B$ to lie on the plane with equation

$$\mathbf{r}.\mathbf{n}=d$$

$(4\mathbf{i}+3\mathbf{j}+\mathbf{k}).(3\mathbf{i}+\mathbf{j}-\mathbf{k})=d$

$$d=4\times 3+3\times 1+1\times(-1)=12+3-1=14$$

**d** The line of intersection $L$ lies in the direction given by

$$(\mathbf{i}-2\mathbf{j}+2\mathbf{k})\times(3\mathbf{i}+\mathbf{j}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ 3 & 1 & -1 \end{vmatrix}$$

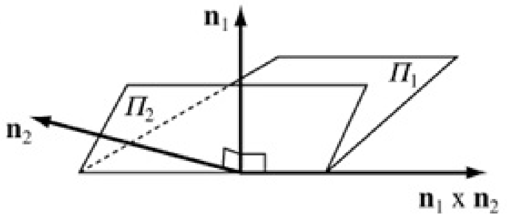
$$= 0\mathbf{i}+7\mathbf{j}+7\mathbf{k}$$

A vector parallel to $7\mathbf{j}+7\mathbf{k}$ is $\mathbf{j}+\mathbf{k}$ and this is parallel to the line $L$.

The point $B$, which has position vector $4\mathbf{i}+3\mathbf{j}+\mathbf{k}$, lies on both $\Pi_1$ and $\Pi_2$ and, hence, on $L$.

A vector equation of $L$ is

$\mathbf{r}=4\mathbf{i}+3\mathbf{j}+\mathbf{k}+t(\mathbf{j}+\mathbf{k})$




The vector $\mathbf{n}_1=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ is perpendicular to $\Pi_1$ and the vector $\mathbf{n}_2=3\mathbf{i}+\mathbf{j}-\mathbf{k}$ is perpendicular to $\Pi_2$. This diagram illustrates the line of intersection of the planes is parallel to $\mathbf{n}_1\times\mathbf{n}_2$. This gives you the direction of $L$. To find the equation of $L$, you also need one point on $L$. In this case, the information given in the question shows you that you already have such a point, $B$.

**e** Rearranging the answer to part **d**

$\mathbf{r}=4\mathbf{i}+(3+t)\mathbf{j}+(1+t)\mathbf{k}$

At the point $X$ on $L$ where $OX$ is perpendicular to $L$

$$\mathbf{r}.(\mathbf{j}+\mathbf{k})=0$$

$(4\mathbf{i}+(3+t)\mathbf{j}+(1+t)\mathbf{k}).(\mathbf{j}+\mathbf{k})=3+t+1+t=0$

$$2t=-4\Rightarrow t=-2$$

The position vector of $X$ is

$4\mathbf{i}+(3-2)\mathbf{j}+(1-2)\mathbf{k}=4\mathbf{i}+\mathbf{j}-\mathbf{k}$

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 15**

**Question:**

The points $A$, $B$ and $C$ have position vectors, relative to a fixed origin $O$,
$\mathbf{a} = 2\mathbf{i} - \mathbf{j}, \mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$,
$\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$,
respectively. The plane $\Pi$ passes through $A$, $B$ and $C$.

**a** Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

**b** Show that a Cartesian equation of $\Pi$ is $3x - y + 2z = 7$.

The line $l$ has equation $(\mathbf{r} - 5\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 0$.
The line $l$ and the plane $\Pi$ intersect at the point $T$.

**c** Find the coordinates of $T$.

**d** Show that $A$, $B$ and $T$ lie on the same straight line. **[E]**

**Solution:**

**a** $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}$

**b** A vector equation of $\Pi$ is $\mathbf{r}.\begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.\begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix} = -12 - 2 = -14$

Once you have a vector $\mathbf{n}$ perpendicular to the plane, you can find a vector equation of the plane using $\mathbf{r.n} = \mathbf{a.n}$, where $\mathbf{a}$ is the position vector of any point on the plane. Here the position vector of $A$ has been used but the position vectors of $B$ and $C$ would do just as well. As the scalar product is quite quickly worked out, it is a useful check to recalculate, using one of the other points. All should give the same answer, here $-14$.

Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\mathbf{r}.\begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix} = -6x + 2y - 4z = -14$

A Cartesian equation of $\Pi$ is

$-6x + 2y - 4z = -14$

Dividing throughout by $-2$

$3x - y + 2z = 7$, as required

**c** A vector equation of the line $l$ is

$\mathbf{r} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

The two vector forms of a straight line $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$ and $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ are equivalent and you can always interchange one with the other, Here

$\mathbf{a} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

Parametric equations of $l$ are

$x = 5 + 2t, y = 5 - t, z = 3 - 2t$

Substituting the parametric equations into

$$3x - y + 2z = 7$$

$$3(5 + 2t) - (5 - t) + 2(3 - 2t) = 7$$

$$15 + 6t - 5 + t + 6 - 4t = 7$$

$$3t = -9 \Rightarrow t = -3$$

The coordinates of $T$ are

$(5 + 2t, 5 - t, 3 - 2t) = (5 - 6, 5 + 3, 3 + 6)$

$= (-1, 8, 9)$

**d** $\overrightarrow{BT} = \overrightarrow{OT} - \overrightarrow{OB} = \begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 6 \end{pmatrix}$

When $A$, $B$ and $T$ lie on the same straight line, $AB$ and $BT$ are in the same direction. So you show that the vectors $\overrightarrow{AB}$ and $\overrightarrow{BT}$ are parallel.

From part a

$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$

Hence

$\overrightarrow{AB} = \frac{1}{2}\overrightarrow{BT}$ and $AB$ is parallel to $BT$.

Hence $A$, $B$ and $T$ lie in the same straight line.

Points which lie on the same straight line are called **collinear** points.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 16**

**Question:**

The plane $\Pi$ passes through the points $A(-1,-1,1)$, $B(4,2,1)$ and $C(2, 1, 0)$.

**a** Find a vector equation of the line perpendicular to $\Pi$ which passes through the point $D(1, 2, 3)$.

**b** Find the volume of the tetrahedron $ABCD$.

**c** Obtain the equation of $\Pi$ in the form $\mathbf{r}\cdot\mathbf{n}=p$.

The perpendicular from $D$ to the plane $\Pi$ meets $\Pi$ at the point $E$.

**d** Find the coordinates of $E$.

**e** Show that $DE=\frac{11\sqrt{35}}{35}$.

The point $D'$ is the reflection of $D$ in $\Pi$.

**f** Find the coordinates of $D'$. **[E]**

**Solution:**

**a** $\overrightarrow{AB} = \begin{pmatrix}4\\2\\1\end{pmatrix} - \begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}5\\3\\0\end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix}2\\1\\0\end{pmatrix} - \begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}5\\3\\0\end{pmatrix} \times \begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}-3\\5\\1\end{pmatrix}$

> The vector product $\overrightarrow{AB} \times \overrightarrow{AC}$ is, by definition, perpendicular to both $AB$ and $AC$. So it will also be perpendicular to the plane containing $AB$ and $AC$.

An equation of the line, $l$ say, which passes through $D$ and is perpendicular to $\Pi$ is $\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}-3\\5\\1\end{pmatrix}$

**b** $\overrightarrow{AD} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\3\\2\end{pmatrix}$

$\overrightarrow{AD}.(\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{pmatrix}2\\3\\2\end{pmatrix}.\begin{pmatrix}-3\\5\\1\end{pmatrix} = -6+15+2 = 11$

The volume of the tetrahedron, $V$ say, is given by

$V = \frac{1}{6}\left|\overrightarrow{AD}.(\overrightarrow{AB} \times \overrightarrow{AC})\right| = \frac{1}{6}|11| = \frac{11}{6}$

**c** An equation for $\Pi$ is

$\mathbf{r}.\begin{pmatrix}-3\\5\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}.\begin{pmatrix}-3\\5\\1\end{pmatrix} = 3-5+1$

$\mathbf{r}.\begin{pmatrix}-3\\5\\1\end{pmatrix} = -1$

> The vector equation $\mathbf{r}.\begin{pmatrix}a\\b\\c\end{pmatrix} = p$ and the Cartesian equation $ax+by+cz=p$ are equivalents and one can always be replaced by the other.

**d** A Cartesian equation for $\Pi$ is

$-3x+5y+z=-1$

Parametric equations corresponding to the equation of $l$ found in part **a** are

$x = 1-3t, y = 2+5t, z = 3+t$

Substituting these parametric equations into the Cartesian equation for $\Pi$

$-3(1-3t)+5(2+5t)+3+t = -1$

$-3+9t+10+25t+3+t = -1$

$35t = -11 \Rightarrow t = -\frac{11}{35}$

The coordinates of $E$ are given by

$(1-3t, 2+5t, 3+t)$

$= \left(1+3\times\frac{11}{35}, 2-5\times\frac{11}{35}, 3-\frac{11}{35}\right)$

$= \left(\frac{68}{35}, \frac{15}{35}, \frac{94}{35}\right)$

Use your calculator to help you work out these awkward fractions.

Of course, $\frac{15}{35} = \frac{5}{7}$ and this is acceptable as part of the answer. However, the subsequent working is easier if all the coordinates have the same denominator.

**e** $DE^2 = \left(1-\frac{68}{35}\right)^2 + \left(2-\frac{15}{35}\right)^2 + \left(3-\frac{94}{35}\right)^2$

$= \left(\frac{33}{35}\right)^2 + \left(\frac{55}{35}\right)^2 + \left(\frac{11}{35}\right)^2$

$= \frac{33^2+55^2+11^2}{35^2} = \frac{4235}{1225} = \frac{121}{35}$

Hence $DE = \sqrt{\left(\frac{121}{35}\right)} = \frac{11}{\sqrt{35}} = \frac{11\sqrt{35}}{35}$, as required.

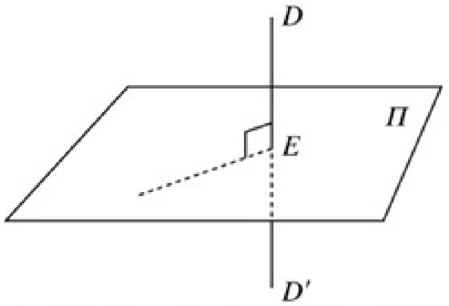
The distance $d$ between points with coordinates $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ is given by $d^2 = (x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2$.

**f** The translation mapping $D$ to $E$ is represented by the vector

$$\overrightarrow{DE} = \begin{pmatrix} \frac{68}{35} \\ \frac{15}{35} \\ \frac{94}{35} \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{33}{35} \\ -\frac{55}{35} \\ -\frac{11}{35} \end{pmatrix}$$

The position vector of $D'$ is given by

$$\overrightarrow{OD'} = \overrightarrow{OE} + \overrightarrow{DE} = \begin{pmatrix} \frac{68}{35} \\ \frac{15}{35} \\ \frac{94}{35} \end{pmatrix} + \begin{pmatrix} \frac{33}{35} \\ -\frac{55}{35} \\ -\frac{11}{35} \end{pmatrix} = \begin{pmatrix} \frac{101}{35} \\ -\frac{40}{35} \\ \frac{83}{35} \end{pmatrix}$$




As $D'$ is the reflection of $D$ in $\Pi$, $E$ is the mid-point of $DD'$ and the translation which maps $D$ to $E$ also maps $E$ to $D'$. So you can find the position vector of $D'$ by adding $\begin{pmatrix} \frac{33}{35} \\ -\frac{55}{35} \\ -\frac{11}{35} \end{pmatrix}$ to the position vector of $E$.

The coordinates of $D'$ are $\left(\frac{101}{35}, -\frac{40}{35}, \frac{83}{35}\right)$

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 17**

**Question:**

The points $A$, $B$ and $C$ have position vectors $(\mathbf{j}+2\mathbf{k}), (2\mathbf{i}+3\mathbf{j}+\mathbf{k})$ and $(\mathbf{i}+\mathbf{j}+3\mathbf{k})$, respectively, relative to the origin $O$. The plane $\Pi$ contains the points $A$, $B$ and $C$.

**a** Find a vector which is perpendicular to $\Pi$.

**b** Find the area of $\Delta ABC$.

**c** Find a vector equation of $\Pi$ in the form $\mathbf{r}\cdot\mathbf{n}=p$.

**d** Hence, or otherwise, obtain a Cartesian equation of $\Pi$.

**e** Find the distance of the origin $O$ from $\Pi$.

The point $D$ has position vector $(3\mathbf{i}+4\mathbf{j}+\mathbf{k})$. The distance of $D$ from $\Pi$ is $\frac{1}{\sqrt{17}}$.

**f** Using this distance, or otherwise, calculate the acute angle between the line $AD$ and $\Pi$, giving your answer in degrees to one decimal place. **[E]**

**Solution:**

**a** Let $\mathbf{a} = \mathbf{j} + 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, and $\mathbf{c} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$\mathbf{b} - \mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - (\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\mathbf{c} - \mathbf{a} = \mathbf{i} + \mathbf{j} + 3\mathbf{k} - (\mathbf{j} + 2\mathbf{k}) = \mathbf{i} + \mathbf{k}$

A vector which is perpendicular to $\Pi$ is

$$(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$$

The vector product $(\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})$ is, by definition, perpendicular to both $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ and, so, it is perpendicular to both $AB$ and $AC$. It will also be perpendicular to the plane containing $AB$ and $AC$.

**b**

$$\Delta ABC = \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$$
$$= \frac{1}{2}|2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}|$$
$$= \frac{1}{2}\sqrt{(2^2 + (-3)^2 + (-2)^2)}$$
$$= \frac{\sqrt{17}}{2}$$

The vector product can be interpreted as a vector with magnitude twice the area of the triangle which has the vectors as two of its sides.

**c** A vector equation of $\Pi$ is

$$\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (\mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = 0 - 3 - 4$$
$$= -7$$

**d** A Cartesian equation of $\Pi$ is $2x - 3y - 2z = -7$

The vector equation $\mathbf{r} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = p$ and the Cartesian equation $ax + by + cz = p$ are equivalents.

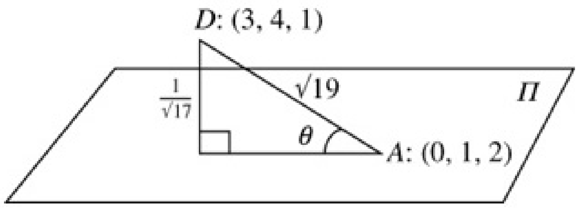
**e** The distance from a point $(\alpha, \beta, \gamma)$ to a plane $n_1x + n_2y + n_2z + d = 0$ is $\left|\dfrac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{(n_1^2 + n_2^3 + n_3^2)}}\right|$

This formula is given in the Edexcel formulae booklet. If you use a formula from the booklet, it is sensible to quote it in your solution. The distance of a point from a plane is defined to be the shortest distance from the point to the plane; that is the perpendicular distance from the point to the plane.

Hence the distance from (0, 0, 0) to $2x - 3y - 2z = -7$

is $\left|\dfrac{7}{\sqrt{(2^2 + (-3)^2 + (-2)^2)}}\right| = \dfrac{7}{\sqrt{17}}$

**f**




Let the angle between $AD$ and $\Pi$ be $\theta$

$$AD^2 = (3-0)^2 + (4-1)^2 + (1-2)^2 = 9 + 9 + 1 = 19$$
$$AD = \sqrt{19}$$
$$\sin\theta = \frac{\left(\frac{1}{\sqrt{17}}\right)}{\sqrt{19}} = 0.055641\ldots$$
$$\theta = 3.2^\circ \ (1 \text{ d.p.})$$

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 18**

**Question:**

Relative to a fixed origin $O$ the lines $l_1$ and $l_2$ have equations

$l_1 : \mathbf{r} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + s(-2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$, $l_2 : \mathbf{r} = -\mathbf{j} + 7\mathbf{k} + t(-\mathbf{i} + \mathbf{j} - \mathbf{k})$,

where $s$ and $t$ are variable parameters.

**a** Show that the lines intersect and are perpendicular to each other.

**b** Find a vector equation of the straight line $l_3$ which passes through the point of intersection of $l_1$ and $l_2$ and the point with position vector $4\mathbf{i} + \lambda\mathbf{j} - 3\mathbf{k}$, where $\lambda$ is a real number.

The line $l_3$ makes an angle $\theta$ with the plane containing $l_1$ and $l_2$.

**c** Find $\sin\theta$ in terms of $\lambda$.

Given that $l_1, l_2$ and $l_3$ are coplanar,

**d** find the value of $\lambda$. **[E]**

**Solution:**

**a** Equating the $x$ components

$-1-2s=-t$ ①

Equating the $y$ components

$2+s=-1+t$ ②

① + ② $1-s=-1 \Rightarrow s=2$

Substitute $s=2$ into ② $4=-1+t \Rightarrow t=5$

Checking the $z$ components

For $l_1$: $-4+3s=-4+6=2$

For $l_2$: $7-t=7-5=2$

These are the same, so $l_1$ and $l_2$ intersect.

> To show that two lines intersect, you find the values of the two parameters, here $s$ and $t$, which make two of the components equal and then you show that these values make the third components equal.

The lines $l_1$ and $l_2$ are parallel to $-2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $-\mathbf{i}+\mathbf{j}-\mathbf{k}$ respectively.

$(-2\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}-\mathbf{k})=2+1-3=0$

Hence $l_1$ is perpendicular to $l_2$.

> As the scalar product $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$, where $\theta$ is the angle between the vectors, if, for non-zero vectors, the scalar produce is zero then $\cos\theta=0$ and $\theta=90°$

**b** Substituting $s=2$ into the equation for $l_1$, the common point has position vector

$-\mathbf{i}+2\mathbf{j}-4\mathbf{k}+2(-2\mathbf{i}+\mathbf{j}+3\mathbf{k})=-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}$

Using $\mathbf{r}=\mathbf{a}+u(\mathbf{b}-\mathbf{a})$, an equation of $l_3$ is

$$\mathbf{r}=-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}+u(4\mathbf{i}+\lambda\mathbf{j}-3\mathbf{k}-(-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}))$$
$$=-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}+u(9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k})$$

> $\mathbf{r}=\mathbf{a}+u(\mathbf{b}-\mathbf{a})$ is the appropriate form of the equation of a straight line going through two points with position vectors $\mathbf{a}$ and $\mathbf{b}$.
> Here
> $\mathbf{a}=-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}$
> $\mathbf{b}=4\mathbf{i}+\lambda\mathbf{j}-3\mathbf{k}$.

**c** A vector $\mathbf{n}$ perpenticular to the plane, $\Pi$ say, containing $l_1$ and $l_2$ is

$$\mathbf{n}=(-\mathbf{i}+\mathbf{j}-\mathbf{k})\times(-2\mathbf{i}+\mathbf{j}+3\mathbf{k})$$
$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 1 & -1\\ -2 & 1 & 3\end{vmatrix}=4\mathbf{i}+5\mathbf{j}+\mathbf{k}$$

Let the angle between $l_3$ and $\Pi$ be $\theta$

$|\mathbf{n}|^2 = 4^2 + 5^2 + 1^2 = 42$

$|9\mathbf{i} + (\lambda - 4)\mathbf{j} - 5\mathbf{k}| = 9^2 + (\lambda - 4)^2 + (-5)^2$

$= 81 + \lambda^2 - 8\lambda + 16 + 25 = \lambda^2 - 8\lambda + 122$

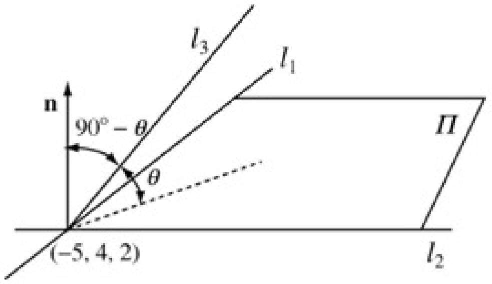
$\mathbf{n} \cdot (9\mathbf{i} + (\lambda - 4)\mathbf{j} - 5\mathbf{k}) = |\mathbf{n}||(9\mathbf{i} + (\lambda - 4)\mathbf{j} - 5\mathbf{k})|$
$\cos(90° - \theta)$

$(4\mathbf{i} + 5\mathbf{j} + \mathbf{k}) \cdot (9\mathbf{i} + (\lambda - 4)\mathbf{j} - 5\mathbf{k})$

$= \sqrt{42} \times \sqrt{(\lambda^2 - 8\lambda + 122)} \sin\theta$

$$\sin\theta = \frac{4 \times 9 + 5(\lambda - 4) + 1 \times (-5)}{\sqrt{42}\sqrt{(\lambda^2 - 8\lambda + 122)}} =$$

$$= \frac{5\lambda + 11}{\sqrt{42}\sqrt{(\lambda^2 - 8\lambda + 122)}}$$




The cosine of the angle between $\mathbf{n}$ and $l_3$ can be found using the scalar product of $\mathbf{n}$ and a vector parallel to $l_3$. This cosine is $\sin\theta$.

**d** If $l_1, l_2$ and $l_3$ are coplanar then $\theta = 0$ and $\sin\theta = 0$

Hence $5\lambda + 11 = 0 \Rightarrow \lambda = \frac{-11}{5}$

Looking at the diagram in part **b** above, if $l_3$ lies in the plane $\Pi$, then $\theta = 0$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 19**

**Question:**

Referred to a fixed origin $O$, the planes $\Pi_1$ and $\Pi_2$ have equations $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+2\mathbf{k})=9$ and $\mathbf{r}\cdot(4\mathbf{i}+3\mathbf{j}-\mathbf{k})=8$ respectively.

**a** Determine the shortest distance from $O$ to the line of intersection of $\Pi_1$ and $\Pi_2$.

**b** Find, in vector form, an equation of the plane $\Pi_3$ which is perpendicular to $\Pi_1$ and $\Pi_2$ and passes through the point with position vector $2\mathbf{j}+\mathbf{k}$.

**c** Find the position vector of the point that lies in $\Pi_1$, $\Pi_2$ and $\Pi_3$. **[E]**

**Solution:**

**a** The Cartesian equations of the planes are

$\Pi_1 : 2x - y + 2z = 9$ ①

$\Pi_2 : 4x + 3y - z = 8$ ②

① + 2 × ②

$10x + 5y = 25$

$2x + y = 5$

Let $x = t$, then $y = 5 - 2x = 5 - 2t$

From ②

Points on the line of intersection of the two planes can be found by solving simultaneously the Cartesian equations of the two planes. As there are 2 equations in 3 unknowns, there are infinitely many solutions. A free choice can be made for one variable, here $x$ is given the parameter $t$, and the other variables can then be found in terms of $t$.

$z = 4x + 3y - 8$

$= 4t + 3(5 - 2t) - 8 = 7 - 2t$

The general point on the line of intersection of the planes has coordinates $(t, 5 - 2t, 7 - 2t)$

This is the equivalent of the parametric equations of the common line $x = t, y = 5 - 2t, z = 7 - 2t$.
The equivalent vector equation of this line is $\mathbf{r} = 5\mathbf{j} + 7\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$.

The distance, $y$ say, from $O$ to this general point is given by

$y^2 = t^2 + (5 - 2t)^2 + (7 - 2t)^2$

$= t^2 + 25 - 20t + 4t^2 + 49 - 28t + 4t^2$

$= 9t^2 - 48t + 74$ ③

Differentiating both sides of ③ with respect to $t$

$$2y\frac{dy}{dt} = 18t - 48$$

A calculus method of finding the minimum distance is shown here. You could instead use the property that, at the shortest distance, the position vector of the point is perpendicular to the common line. This method is illustrated in Question **13**.

At a minimum distance $\frac{dy}{dt} = 0$

$18t - 48 = 0 \Rightarrow t = \frac{48}{18} = \frac{8}{3}$

Substituting into ③

$$y^2 = 9 \times \left(\frac{8}{3}\right)^2 - 48 \times \frac{8}{3} + 74$$

$= 64 - 128 + 74 = 10$

The shortest distance from $O$ to the line of intersection of the planes is $\sqrt{10}$.

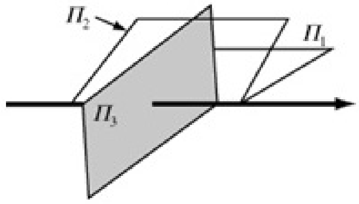
**b** The line of intersection of $\Pi_1$ and $\Pi_2$ has vector equation $\mathbf{r} = 5\mathbf{j} + 7\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$

Hence the vector $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ is perpendicular to $\Pi_3$.

An equation of $\Pi_3$ is

$$\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = (2\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$$
$$= -4 - 2 = -6$$

The common line of $\Pi_1$ and $\Pi_2$ is a normal to the plane $\Pi_3$ which is perpendicular to $\Pi_1$ and $\Pi_2$.




**c** Substituting $(t, 5 - 2t, 7 - 2t)$ into $x - 2y - 2z = -6$

$t - 2(5 - 2t) - 2(7 - 2t) = -6$

$t - 10 + 4t - 14 + 4t = -6 \Rightarrow 9t = 18 \Rightarrow t = 2$

The position vector of the common point is

$t\mathbf{i} + (5 - 2t)\mathbf{j} + (7 - 2t)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

The point that lies on the three planes is given by substituting the general point on the line of intersection of $\Pi_1$ and $\Pi_2$ into the Cartesian equation of $\Pi_3$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 20**

**Question:**

Vector equations of the two straight lines $l$ and $m$ are respectively

$\mathbf{r} = \mathbf{j} + 3\mathbf{k} + t(2\mathbf{i} + \mathbf{j} - \mathbf{k})$,

$\mathbf{r} = \mathbf{i} + \mathbf{j} - \mathbf{k} + u(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$.

**a** Show that these lines do not intersect.

The point $A$ with parameter $t_1$ lies on $l$ and the point $B$ with parameter $u_1$ lies on $m$.

**b** Write down the vector $\overrightarrow{AB}$ in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}, t_1$ and $u_1$.

Given that the line $AB$ is perpendicular to both $l$ and $m$,

**c** find the values of $t_1$ and $u_1$ and show that, in this case, the length of $AB$ is $\frac{7}{\sqrt{5}}$. **[E]**

**Solution:**

**a** Equating the $x$ components

$2t = 1 - 2u$ ①

Equating the $y$ components

$1 + t = 1 + u \Rightarrow t = u$ ②

Substituting ② into ①

$2u = 1 - 2u \Rightarrow u = \frac{1}{4}$

As $t = u, t = \frac{1}{4}$

Checking the $z$ components

For $l$: $3 - t = 3 - \frac{1}{4} = \frac{11}{4}$

For $m$: $-1 + u = -1 + \frac{1}{4} = -\frac{3}{4}$

$\frac{11}{4} \neq -\frac{3}{4}$, so the lines do not intersect.

To show that two lines do not intersect, you find the values of the two parameters, here $t$ and $u$, which make two of the components equal and then you show that, with these values, the third components are not equal.

**b** $\overrightarrow{OA} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + t_1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\overrightarrow{OB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + u_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + u_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - t_1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 1 - 2t_1 - 2u_1 \\ -t_1 + u_1 \\ -4 + t_1 + u_1 \end{pmatrix}$

$= (1 - 2t_1 - 2u_1)\mathbf{i} + (-t_1 + u_1)\mathbf{j} + (-4 + t_1 + u_1)\mathbf{k}$

c If $\overrightarrow{AB}$ is perpendicular to $l$

$$\begin{pmatrix}1-2t_1-2u_1\\-t_1+u_1\\-4+t_1+u_1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}=0$$

$$2-4t_1-4u_1-t_1+u_1+4-t_1-u_1=0$$

$$6t_1+4u_1=6 \quad ③$$

If $\overrightarrow{AB}$ is perpendicular to $m$

$$\begin{pmatrix}1-2t_1-2u_1\\-t_1+u_1\\-4+t_1+u_1\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=0$$

$$-2-4t_1+4u_1-t_1+u_1-4+t_1+u_1=0$$

$$4t_1+6u_1=6 \quad ④$$

$3\times ④ - 2\times ③$

$$10u_1=6\Rightarrow u_1=\frac{3}{5}$$

Substituting $u_1=\frac{3}{5}$ into ④

$$4t_1+\frac{18}{5}=6\Rightarrow t_1=\frac{6-\frac{18}{5}}{4}=\frac{3}{5}$$

$$\overrightarrow{AB}=\begin{pmatrix}1-2t_1-2u_1\\-t_1+u_1\\-4+t_1+u_1\end{pmatrix}=\begin{pmatrix}1-\frac{6}{5}-\frac{6}{5}\\-\frac{3}{5}+\frac{3}{5}\\-4+\frac{3}{5}+\frac{3}{5}\end{pmatrix}=\begin{pmatrix}-\frac{7}{5}\\0\\-\frac{14}{5}\end{pmatrix}$$

$$\left|\overrightarrow{AB}\right|^2=\left(-\frac{7}{5}\right)^2+\left(-\frac{14}{5}\right)^2=\frac{245}{25}=\frac{49}{5}$$

The length of $AB$ is given by

$$\left|\overrightarrow{AB}\right|=\sqrt{\left(\frac{49}{5}\right)}=\frac{7}{\sqrt{5}}\text{, as required.}$$

As $\overrightarrow{AB}$ is perpendicular to $l$, the scalar product of $\overrightarrow{AB}$ with the direction of $l$, which is $\begin{pmatrix}2\\1\\-1\end{pmatrix}$, is zero. This gives one equation in $t_1$ and $u_1$. Carrying out the same process with the direction of $m$, gives you a second equation in $t_1$ and $u_1$. You solve these simultaneous equations for $t_1$ and $u_1$ and use these values to find $\overrightarrow{AB}$. The magnitude of this vector is the length you have been asked to find.

This length is the shortest distance between the two skew lines. This question illustrates one of the methods by which this shortest distance can be found.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 21**

**Question:**

$$A=\begin{pmatrix}1 & 1 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$$

Prove by induction, that for all positive integers $n$, $A^n=\begin{pmatrix}1 & n & \frac{1}{2}(n^2+3n)\\ 0 & 1 & n\\ 1 & 0 & 1\end{pmatrix}$. **[E]**

**Solution:**

$$\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2+3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

Let $n = 1$

> You start all inductions by showing that the formula you are asked to prove is true for a small number, usually 1.

$$\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1^2+3\times 1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{A}$$

The formula is true for $n = 1$.
Assume the formula is true for $n = k$.
That is

$$\mathbf{A}^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

> This is the **induction hypothesis**. You assume that the formula is true for $n = k$ and show that this implies that the formula is true for $n = k+1$.

$$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}$$

$$= \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1+k & 2+k+\frac{1}{2}(k^2+3k) \\ 0 & 1 & 1+k \\ 0 & 0 & 1 \end{pmatrix}$$

> This term is obtained by multiplying the first row of the first matrix by the third row of the second matrix, that is
>
> $$\begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$
>
> $$= 1\times 2 + k\times 1 + \frac{1}{2}(k^2+3k)\times 1$$

$$2+k+\frac{1}{2}(k^2+3k) = \frac{1}{2}k^2 + \frac{3k}{2} + k + 2$$

$$= \frac{1}{2}(k^2+5k+4)$$

$$= \frac{1}{2}(k^2+2k+1+3k+3)$$

$$= \frac{1}{2}\left((k+1)^2 + 3(k+1)\right)$$

> Keep in mind what you are aiming for as you work out the algebra. You are looking to prove that the formula is true for $n = k+1$, so you are trying to reach $\frac{1}{2}(n^2+3n)$ with the $n$ replaced by $k+1$.

$$\mathbf{A}^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}\left((k+1)^2 + 3(k+1)\right) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$

This is the formula with $k+1$ substituted for $n$.

Hence, the formula is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the formula is true for all positive integers $n$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 22**

**Question:**

$$A = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix}$$, where $k$ is a real constant.

**a** Find the values of $k$ for which A is singular.
Given that A is non-singular,
**b** find, in terms of $k$, $A^{-1}$. **[E]**

**Solution:**

**a** $\det \mathbf{A} = k\begin{vmatrix}-1 & k\\ 1 & 0\end{vmatrix} - 1\begin{vmatrix}0 & k\\ 9 & 0\end{vmatrix} + (-2)\begin{vmatrix}0 & -1\\ 9 & 1\end{vmatrix}$

$= k\times(-k) - 1\times(-9k) + (-2)\times 9$

$= -k^2 + 9k - 18 = 0$

$k^2 - 9k + 18 = (k-3)(k-6) = 0$

$k = 3, 6$

The $2\times 2$ determinants are worked out using the formula $\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc$, which you learnt in book FP1.

A singular matrix is a matrix without an inverse. The determinant of a singular matrix is 0.

**b** The matrix of the minors, $\mathbf{M}$ say, is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}-1 & k\\ 1 & 0\end{vmatrix} & \begin{vmatrix}0 & k\\ 9 & 0\end{vmatrix} & \begin{vmatrix}0 & -1\\ 9 & 1\end{vmatrix} \\ \begin{vmatrix}1 & -2\\ 1 & 0\end{vmatrix} & \begin{vmatrix}k & -2\\ 9 & 0\end{vmatrix} & \begin{vmatrix}k & 1\\ 9 & 1\end{vmatrix} \\ \begin{vmatrix}1 & -2\\ -1 & k\end{vmatrix} & \begin{vmatrix}k & -2\\ 0 & k\end{vmatrix} & \begin{vmatrix}k & 1\\ 0 & -1\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -k & -9k & 9 \\ 2 & 18 & k-9 \\ k-2 & k^2 & -k \end{pmatrix}$$

The matrix of the cofactors, $\mathbf{C}$ say, is given by

$$\mathbf{C} = \begin{pmatrix} -k & 9k & 9 \\ -2 & 18 & -k+9 \\ k-2 & -k^2 & -k \end{pmatrix}$$

As you have worked out the determinant of $\mathbf{A}$ in part **a**, the remaining steps for working out an inverse of a $3\times 3$ matrix are:
**1** Work out the matrix of the minors.
**2** Obtain the matrix of cofactors by adjusting the signs of the minors using the pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.
**3** Transpose the matrix of the cofactors.
**4** Divide the transpose of the matrix of cofactors by the determinant of the matrix.

The transpose of the matrix of the cofactors is given by

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & -k+9 & -k \end{pmatrix}$$

The inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{-k^2+9k-18}\begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & -k+9 & -k \end{pmatrix}$$

You have worked out the determinant of $\mathbf{A}$ in part **a**. It is perfectly acceptable to leave your answer in this form. You do not have to divide every individual term in the matrix by $-k^2+9k-18$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 23**

**Question:**

The matrix $\mathbf{M}$ is given by

$\mathbf{M} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix}$, where $p$, $a$, $b$ and $c$ are constants and $a > 0$. Given that $\mathbf{MM}^{\mathrm{T}} = k\mathbf{I}$

for some constant $k$, find

**a** the value of $p$,
**b** the value of $k$,
**c** the values of $a$, $b$ and $c$,
**d** det $\mathbf{M}$. **[E]**

**Solution:**

**a** $\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} 1 & 3 & a \\ 4 & 0 & b \\ -1 & p & c \end{pmatrix}$

$$\mathbf{M}\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix}\begin{pmatrix} 1 & 3 & a \\ 4 & 0 & b \\ -1 & p & c \end{pmatrix}$$

$$= \begin{pmatrix} 1+16+1 & 3-p & a+4b-c \\ 3-p & 9+p^2 & 3a+pc \\ a+4b-c & 3a+pc & a^2+b^2+c^2 \end{pmatrix}$$

$$= k\mathbf{I} = k\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$\begin{pmatrix} 18 & 3-p & a+4b-c \\ 3-p & 9+p^2 & 3a+pc \\ a+4b-c & 3a+pc & a^2+b^2+c^2 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} *$$

If two matrices are equal, then all of the corresponding elements in the matrices must be equal. Potentially, there are 9 equations here. This equation has 5 unknowns and you pick out 5 equations which you can solve to find the unknowns.

Equating the second elements in the top row of *

$3-p=0 \Rightarrow p=3$

**b** Equating the first elements in the top row of *

$k=18$

**c** Equating each of the terms in the third row of * and using $p=3$ and $k=18$

$a+4b-c=0$ ①

$3a+3c=0$ ②

$a^2+b^2+c^2=18$ ③

You solve these 3 simultaneous equations by finding $b$ and $c$ in terms of $a$ and, then, eliminating $b$ and $c$. It is sensible to find $a$ first as that is the unknown for which you are given the additional information that $a>0$.

From ② $c=-a$

Substituting $c=-a$ into ①

$a+4b+a=0 \Rightarrow 4b=-2a \Rightarrow b=-\frac{1}{2}a$

Substituting $c=-a$ and $b=-\frac{1}{2}a$ into ③

$a^2+\frac{1}{4}a^2+a^2=18$

$\frac{9a^2}{4}=18 \Rightarrow a^2=\frac{18\times 4}{9}=8$

As $a>0$

$a=\sqrt{8}=2\sqrt{2}$

$b=-\frac{1}{2}a=-\sqrt{2}$

$c=-a=-2\sqrt{2}$

**d** $\quad \mathbf{M} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & 3 \\ 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \end{pmatrix}$

$$\det \mathbf{M} = 1\begin{vmatrix} 0 & 3 \\ -\sqrt{2} & -2\sqrt{2} \end{vmatrix} - 4\begin{vmatrix} 3 & 3 \\ 2\sqrt{2} & -2\sqrt{2} \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 2\sqrt{2} & -\sqrt{2} \end{vmatrix}$$

$$= 1 \times 3\sqrt{2} - 4 \times \left(-6\sqrt{2} - 6\sqrt{2}\right) + (-1) \times \left(-3\sqrt{2}\right)$$

$$= 3\sqrt{2} + 48\sqrt{2} + 3\sqrt{2} = 54\sqrt{2}$$

The $2 \times 2$ determinants are worked out using the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, which you learnt in book FP1.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 24**

**Question:**

**a** Given that $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$, find $A^2$.

**b** Using $A^3 = \begin{pmatrix} 10 & 9 & 23 \\ 5 & 9 & 14 \\ 9 & 5 & 19 \end{pmatrix}$, show that $A^3 - 5A^2 + 6A - \mathbf{I} = 0$.

**c** Deduce that $A(A - 2I)(A - 3I) = \mathbf{I}$.

**d** Hence find $A^{-1}$. **[E]**

**Solution:**

**a** $A^2 = A \cdot A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1+0+2 & 1+2+0 & 2+1+4 \\ 0+0+1 & 0+4+0 & 0+2+2 \\ 1+0+2 & 1+0+0 & 2+0+4 \end{pmatrix}$

$= \begin{pmatrix} 3 & 3 & 7 \\ 1 & 4 & 4 \\ 3 & 1 & 6 \end{pmatrix}$

As an example, the third element in the third row is found by multiplying the third row of the first matrix by the third column of the second matrix. That is

$(1\ 0\ 2)\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 1\times 2 + 0\times 1 + 2\times 2$

$= 2+0+4 = 6$

**b** $A^3 - 5A^2 + 6A - I$

$= \begin{pmatrix} 10 & 9 & 23 \\ 5 & 9 & 14 \\ 9 & 5 & 19 \end{pmatrix} - 5\begin{pmatrix} 3 & 3 & 7 \\ 1 & 4 & 4 \\ 3 & 1 & 6 \end{pmatrix} + 6\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 10-15+6-1 & 9-15+6-0 & 23-35+12-0 \\ 5-5+0-0 & 9-20+12-1 & 14-20+6-0 \\ 9-15+6-0 & 5-5+0-0 & 19-30+12-1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{O}$, as required.

When a matrix is multiplied by a scalar, each element in the matrix is multiplied by the scalar so

$6\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 12 \\ 0 & 12 & 6 \\ 6 & 0 & 12 \end{pmatrix}$

**c** $A^3 - 5A^2 + 6A - I = 0$

$A^3 - 5A^2 + 6A = I$

$A(A^2 - 5A + 6I) = I$

$A(A - 2I)(A - 3I) = I$, as required.

The rules for factorising expressions with matrices are essentially the same as those for factorising ordinary polynomials, so if $x^2 - 5x + 6 = (x-2)(x-3)$, then $A^2 - 5A + 6I = (A - 2I)(A - 3I)$. The **I**s are needed to preserve the dimensions of the matrices.

**d** Comparing $\mathbf{A}(\mathbf{A}-2\mathbf{I})(\mathbf{A}-3\mathbf{I})=\mathbf{I}$ with the definition of the inverse matrix $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$

$$\mathbf{A}^{-1}=(\mathbf{A}-2\mathbf{I})(\mathbf{A}-3\mathbf{I})$$
$$=\mathbf{A}^2-5\mathbf{A}+6\mathbf{I}$$

An alternative method is to work out the matrices $(\mathbf{A}-2\mathbf{I})$ and $(\mathbf{A}-3\mathbf{I})$ and multiply them together. The method shown here is a little quicker unless you have a calculator which multiplies matrices.

Hence

$$\mathbf{A}^{-1}=\begin{pmatrix}3&3&7\\1&4&4\\3&1&6\end{pmatrix}-5\begin{pmatrix}1&1&2\\0&2&1\\1&0&2\end{pmatrix}+6\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}3-5+6&3-5+0&7-10+0\\1-0+0&4-10+6&4-5+0\\3-5+0&1-0+0&6-10+6\end{pmatrix}$$

$$=\begin{pmatrix}4&-2&-3\\1&0&-1\\-2&1&2\end{pmatrix}$$

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 25**

**Question:**

Given that $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, use matrix multiplication to find

**a** $A^2$,
**b** $A^3$.
**c** Prove by induction that $A^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 2^n - 1 \\ 0 & 0 & 1 \end{pmatrix}, n \geq 1$.
**d** Find the inverse of $A^n$. **[E]**

**Solution:**

**a** $\mathbf{A}^2 = \mathbf{A}\cdot\mathbf{A} = \begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&4&2+1\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&4&3\\0&0&1\end{pmatrix}$

**b** $\mathbf{A}^3 = \mathbf{A}^2\cdot\mathbf{A} = \begin{pmatrix}1&0&0\\0&4&3\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&8&4+3\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&8&7\\0&0&1\end{pmatrix}$

Using $\mathbf{A}^3 = \mathbf{A}\cdot\mathbf{A}^2$ will give you the same result.

**c** $\mathbf{A}^n = \begin{pmatrix}1&0&0\\0&2^n&2^n-1\\0&0&1\end{pmatrix}$

Let $n = 1$

You start all inductions by showing that the formula you are asked to prove is true for a small number, usually 1.

$\mathbf{A}^1 = \begin{pmatrix}1&0&0\\0&2^1&2^1-1\\0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix} = \mathbf{A}$

The formula is true for $n = 1$.
Assume the formula is true for $n = k$.
That is

$\mathbf{A}^k = \begin{pmatrix}1&0&0\\0&2^k&2^k-1\\0&0&1\end{pmatrix}$

$\mathbf{A}^{k+1} = \mathbf{A}^k\cdot\mathbf{A}$

$= \begin{pmatrix}1&0&0\\0&2^k&2^k-1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&1\\0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&2^k\times 2&2^k+2^k-1\\0&0&1\end{pmatrix}$

$2^k\times 2 = 2^k\times 2^1 = 2^{k+1}$

$2^k + 2^k - 1 = 2\times 2^k - 1 = 2^{k+1} - 1$

The second element in the second row is found by multiplying the second row of the first matrix by the second column of the second matrix. That is

$\begin{pmatrix}0&2^k&2^k-1\end{pmatrix}\begin{pmatrix}0\\2\\0\end{pmatrix} = 0\times 0 + 2^k\times 2 + (2^k-1)\times 0$

$= 2^k\times 2 = 2^{k+1}.$

The third element in the second row is found by multiplying the second row of the first matrix by the third column of the second matrix. That is

$\begin{pmatrix}0&2^k&2^k-1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix} = 0\times 0 + 2^k\times 1 + (2^k-1)\times 1$

$= 2^k + 2^k - 1 = 2^{k+1} - 1.$

Hence $\mathbf{A}^{k+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{k+1} & 2^{k+1}-1 \\ 0 & 0 & 1 \end{pmatrix}$

This is the formula with $k+1$ substituted for $n$.
Hence, the formula is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.
By mathematical induction the formula is true for all positive integers $n$.

**d** $\det(\mathbf{A}^n) = 2^n$

The matrix of the minors of $\mathbf{A}^n$, $\mathbf{M}$ say, is given by

$$\mathbf{M} = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2^n-1 & 2^n \end{pmatrix}$$

The matrix of the cofactors, $\mathbf{C}$ say, is given by

$$\mathbf{C} = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1-2^n & 2^n \end{pmatrix}$$

The transpose of the matrix of the cofactors is given by

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 1-2^n \\ 0 & 0 & 2^n \end{pmatrix}$$

The inverse of $\mathbf{A}^n$ is given by

$$(\mathbf{A}^n)^{-1} = \frac{1}{\det(\mathbf{A}^n)}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{2^n}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 1-2^n \\ 0 & 0 & 2^n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^n} & \frac{1-2^n}{2^n} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-n} & 2^{-n}-1 \\ 0 & 0 & 1 \end{pmatrix}$$

A possible alternative approach is to note that the form of $\mathbf{A}^n$,

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 2^n-1 \\ 0 & 0 & 1 \end{pmatrix}$,

suggests that the inverse of $\mathbf{A}^n$, which is $(\mathbf{A}^n)^{-1} = \mathbf{A}^{-n}$, using the laws of indices, might be found by changing the $n$ to $-n$, giving $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-n} & 2^{-n}-1 \\ 0 & 0 & 1 \end{pmatrix}$, which is the correct answer.
Relations of this kind are commonly true for the powers of matrices but, in itself, this is not a sufficient argument. However, if you now verified that this was the inverse by multiplying $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 2^n-1 \\ 0 & 0 & 1 \end{pmatrix}$ by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-n} & 2^{-n}-1 \\ 0 & 0 & 1 \end{pmatrix}$ and obtaining the identity matrix, this would be acceptable.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 26**

**Question:**

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & u \end{pmatrix}, u \neq 1$$

**a** Show that $\det A = 2(u-1)$.

**b** Find the inverse of A.

The image of the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when transformed by the matrix $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & 6 \end{pmatrix}$ is $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$.

**c** Find the values of $a$, $b$ and $c$. **[E]**

**Solution:**

**a**

$$\det \mathbf{A} = 3\begin{vmatrix}1 & 1\\3 & u\end{vmatrix} - 1\begin{vmatrix}1 & 1\\5 & u\end{vmatrix} + (-1)\begin{vmatrix}1 & 1\\5 & 3\end{vmatrix}$$

$$= 3(u-3) - 1(u-5) - 1\times(-2)$$

$$= 3u - 9 - u + 5 + 2 = 2u - 2$$

$$= 2(u-1), \text{ as required}$$

Each $2\times 2$ determinant is evaluated using the formula $\begin{vmatrix}a & b\\c & d\end{vmatrix} = ad - bc$.

**b** The matrix of the minors, $\mathbf{M}$ say, is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix}1 & 1\\3 & u\end{vmatrix} & \begin{vmatrix}1 & 1\\5 & u\end{vmatrix} & \begin{vmatrix}1 & 1\\5 & 3\end{vmatrix} \\ \begin{vmatrix}1 & -1\\3 & u\end{vmatrix} & \begin{vmatrix}3 & -1\\5 & u\end{vmatrix} & \begin{vmatrix}3 & 1\\5 & 3\end{vmatrix} \\ \begin{vmatrix}1 & -1\\1 & 1\end{vmatrix} & \begin{vmatrix}3 & -1\\1 & 1\end{vmatrix} & \begin{vmatrix}3 & 1\\1 & 1\end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} u-3 & u-5 & -2 \\ u+3 & 3u+5 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

The minor of an element is found by deleting the row and the column in which the element lies.
For example, to find the minor of $b$ in $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, delete the row and column through $b$ $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.
The minor is the determinant of the elements left, $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$.

The matrix of the cofactors, $\mathbf{C}$ say, is given by

$$\mathbf{C} = \begin{pmatrix} u-3 & -u+5 & -2 \\ -u-3 & 3u+5 & -4 \\ 2 & -4 & 2 \end{pmatrix}$$

The transpose of the matrix of the cofactors is given by

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} u-3 & -u-3 & 2 \\ -u+5 & 3u+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

The inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{2(u-1)}\begin{pmatrix} u-3 & -u-3 & 2 \\ -u+5 & 3u+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

c With $u = 6$

$$\mathbf{A}^{-1} = \frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

The matrix in part **c** $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & 6 \end{pmatrix}$ is the matrix **A** of parts **a** and **b** with $u = 6$. To find the object vector when you are given the image vector, you will need the inverse matrix with $u = 6$.

$$\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

This equation expresses the information that the image of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, under the transformation represented by the matrix **A**, is $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$.

Multiplying both sides on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

Hence, as $\mathbf{AA}^{-1} = \mathbf{I}$,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 9-9+12 \\ -3+23-24 \\ -6-4+12 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 12 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1.2 \\ -0.4 \\ 0.2 \end{pmatrix}$$

$a = 1.2, b = -0.4, c = 0.2$

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 27**

**Question:**

The transformation $R$ is represented by the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix}$, and

where $a$, $b$ and $c$ are constants. Given that $\mathbf{M} = \mathbf{M}^{-1}$,

**a** find the values of $a$, $b$ and $c$,

**b** evaluate the determinant of $\mathbf{M}$,

**c** find an equation satisfied by all the points which remain invariant under $R$. **[E]**

**Solution:**

**a** $\mathbf{MM} = \mathbf{I}$

$$\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By definition, $\mathbf{MM}^{-1} = \mathbf{I}$. As you have been given that $\mathbf{M} = \mathbf{M}^{-1}$, it follows that $\mathbf{MM} = \mathbf{I}$. This matrix is self-inverse.

$$\begin{pmatrix} 9+2a & 3a+ab & 0 \\ 6+2b & 2a+b^2 & 0 \\ 4c & ac & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If two matrices are equal, then all of the corresponding elements in the matrices must be equal. Potentially, there are 9 equations here. This question has 3 unknowns and you pick out 3 equations which you can solve to find the unknowns.

Equating the first elements in the first row
$9 + 2a = 1 \Rightarrow a = -4$
Equating the first elements in the second row
$6 + 2b = 0 \Rightarrow b = -3$
Equating the first elements in the third row
$4c = 0 \Rightarrow c = 0$

**b** Using the values of $a$, $b$ and $c$ found in part **a**

$$\mathbf{M} = \begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det \mathbf{M} = 3\begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix}$$
$$= 3\times(-3) + 4\times 2 = -1$$

**c** Let the point which is invariant under $R$ have position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The vector of an invariant point is unchanged when multiplied by the matrix representing the transformation.

$$\begin{pmatrix} 3x-4y \\ 2x-3y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The top and middle elements give the same equation and this is the equation satisfied by the invariant points. Equating the lowest elements gives $z = z$. This is an identity, always satisfied, and gives you no further information.

Equating the top elements
$3x - 4y = x \Rightarrow 2x - 4y = 0 \Rightarrow x = 2y$

Equating the middle elements
$2x - 3y = y \Rightarrow 2x - 4y = 0 \Rightarrow x = 2y$

An equation satisfied by all the invariant points is $x = 2y$.

The transformation is 3-dimensional and $x = 2y$ represents a plane of points.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 28**

**Question:**

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{M}$.

The vector $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is transformed by $T$ to $\begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}$, the vector $\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ is transformed to $\begin{pmatrix} -1 \\ 9 \\ 0 \end{pmatrix}$

and the vector $\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$ is transformed to $\begin{pmatrix} -\alpha+1 \\ 5 \\ 2\alpha+2 \end{pmatrix}$, where $\alpha(\alpha \neq -1)$ is a constant.

**a** Find $\mathbf{M}$.

The plane $\Pi_1$ has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$, where $\lambda$ and $\mu$ are parameters,

and $T$ transforms $\Pi_1$ to the plane $\Pi_2$.

**b** Find a Cartesian equation of $\Pi_2$. **[E]**

**Solution:**

**a** Let $\mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a-b \\ 2d-e \\ 2g-h \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}$$

This equation expresses the information that $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is transformed by $T$, the transformation represented by $\mathbf{M}$, to $\begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}$. Equating the 3 elements gives 3 equations. The other two vector transformations, similarly, give 6 more equations and all 9 equations are needed to find the 9 elements of $\mathbf{M}$.

Equating the top elements

$2a - b = -5$ ①

Equating the middle elements

$2d - e = -1$ ②

Equating the lowest elements

$2g - h = 0$ ③

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -b+2c \\ -e+2f \\ -h+2i \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 0 \end{pmatrix}$$

Equating the top elements

$-b + 2c = -1$ ④

Equating the middle elements

$-e + 2f = 9$ ⑤

Equating the lowest elements

$-h + 2i = 0$ ⑥

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} \alpha \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a\alpha+c \\ d\alpha+f \\ g\alpha+i \end{pmatrix} = \begin{pmatrix} -\alpha+1 \\ 5 \\ 2\alpha+2 \end{pmatrix}$$

Equating the top elements

$a\alpha + c = -\alpha + 1$ ⑦

Equating the middle elements

$d\alpha + f = 5$ ⑧

Equating the lowest elements

$g\alpha + i = 2\alpha + 2$ ⑨

Taking equations ①, ④ and ⑦

$2a - b = -5$ ①

$-b + 2c = -1$ ④

$a\alpha + c = -\alpha + 1$ ⑦

These 3 equations are 3 simultaneous equations in $a$, $b$ and $c$. You solve them by eliminating $a$ and $b$ from the equations and finding $c$.

① − ④

$2a - 2c = -4 \Rightarrow a = c - 2$

Substitute $a = c - 2$ into ⑦

$$(c-2)\alpha + c = -\alpha + 1$$
$$c\alpha + c - \alpha - 1 = 0$$
$$c(\alpha+1) - 1(\alpha+1) = 0$$
$$(c-1)(\alpha+1) = 0$$

The condition $\alpha \neq -1$ is important in this question. If $\alpha = -1$, the equations could not be solved. You will notice the importance of this condition again later in the question. As frequently happens in mathematics, this special case is of considerable interest and is worth further investigation but this goes beyond the specification for this module.

As $\alpha \neq -1$

$$c = 1$$
$$a = c - 2 = 1 - 2 = -1$$

From ①

$b = 2a + 5 = -2 + 5 = 3$

Taking equations ②, ⑤ and ⑧

$2d - e = -1$ ②

$-e + 2f = 9$ ⑤

$d\alpha + f = 5$ ⑧

② − ⑤

$2d - 2f = -10 \Rightarrow f = d + 5$

Substitute

$$f = d + 5$$
$$d\alpha + d + 5 = 5$$
$$d(\alpha + 1) = 0$$

As $\alpha \neq -1$

$d = 0$

$f = d + 5 = 0 + 5 = 5$

From ②

$e = 2d + 1 = 0 + 1 = 1$

Taking equations ③, ⑥ and ⑨

$2g - h = 0$ ③

$-h + 2i = 0$ ⑥

$g\alpha + i = 2\alpha + 2$ ⑨

③ − ⑥

$2g - 2i = 0 \Rightarrow g = i$

Substituting $g = i$ into ⑨

$$i\alpha + i = 2\alpha + 2$$
$$i(\alpha+1) - 2(\alpha+1) = 0$$
$$(i-2)(\alpha+1) = 0$$

As $\alpha \neq -1$

$i = 2$

$g = i = 2$

From ③

$h = 2g = 4$

Hence

$$\mathbf{M} = \begin{pmatrix} -1 & 3 & 1 \\ 0 & 1 & 5 \\ 2 & 4 & 2 \end{pmatrix}$$

**b** Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+2\lambda \\ -\lambda-\mu \\ 1+2\mu \end{pmatrix}$$

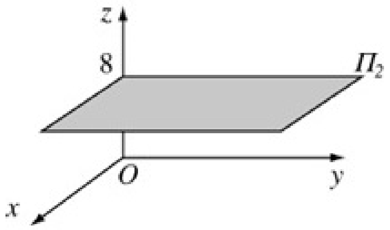
This general point is transformed by $T$ to

$$\begin{pmatrix} -1 & 3 & 1 \\ 0 & 1 & 5 \\ 2 & 4 & 2 \end{pmatrix}\begin{pmatrix} 3+2\lambda \\ -\lambda-\mu \\ 1+2\mu \end{pmatrix} = \begin{pmatrix} -1(3+2\lambda)+3(-\lambda-\mu)+1(1+2\mu) \\ 1(-\lambda-\mu)+5(1+2\mu) \\ 2(3+2\lambda)+4(-\lambda-\mu)+2(1+2\mu) \end{pmatrix}$$

$$= \begin{pmatrix} -2-5\lambda-\mu \\ 5-\lambda+9\mu \\ 8 \end{pmatrix}$$

An equation of $\Pi_2$ is $z = 8$.

As $\lambda$ and $\mu$ are parameters, the $x$- and $y$- coordinates of $\Pi_2$ can take all real values; there are no restrictions on these coordinates. However, the $z$-coordinate is 8, so the equation of $\Pi_2$ is $z = 8$. This is a plane parallel to the plan $Oxy$.



# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 29**

**Question:**

The transformation $S:\mathbb{R}^3 \to \mathbb{R}^3$ maps the point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ onto the point $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ where

$a = x + y - z$
$b = \quad y + z$
$c = \quad\quad z,$

The matrix of this transformation is $\mathbf{A}$.

**a** By solving the given equations for $x$, $y$ and $z$ in terms of $a$, $b$ and $c$, or otherwise, write down the matrix $\mathbf{A}^{-1}$.

The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ has matrix $\mathbf{B} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$

**b** Given that $\mathbf{BB}^{\mathrm{T}} = k\mathbf{I}$, find the value of $k$.

$U$ is the composite transformation consisting of $T$ followed by $S$.

**c** Find the point whose image under $U$ is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. **[E]**

**Solution:**

**a** $a = x + y - z$ ①

$b = y + z$ ②

$c = z$ ③

③ can be written as $z = c$

Substituting $z = c$ into ②

$b = y + c \Rightarrow y = b - c$

If $\mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then this set of 3 equations can be written as $\mathbf{a} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Substituting $z = c$ and $y = b - c$ into ①

$a = x + b - c - c \Rightarrow x = a - b + 2c$

Hence the three equations can be written as

$x = a - b + 2c$

$y = b - c$

$z = c$

or in vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

As $\mathbf{a} = \mathbf{A}\mathbf{x} \Leftrightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{a}$, then if you can find a matrix, $\mathbf{C}$ say, such that $\mathbf{x} = \mathbf{C}\mathbf{a}$, then $\mathbf{C} = \mathbf{A}^{-1}$.

Hence

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

**b** $$\mathbf{B}\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4+4 & 2+2-4 & 2-4+2 \\ 2+2-4 & 4+1+4 & 4-2-2 \\ 2-4+2 & 4-2-2 & 4+4+1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = 9 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 9\mathbf{I}$$

$\mathbf{B}\mathbf{B}^{\mathrm{T}} = 9\mathbf{I}$ can be rewritten as $\mathbf{B}\left(\frac{1}{9}\mathbf{B}^{\mathrm{T}}\right) = \mathbf{I}$. As, by definition, $\mathbf{B}\mathbf{B}^{-1} = \mathbf{I}$, in this case $\mathbf{B}^{-1} = \frac{1}{9}\mathbf{B}^{\mathrm{T}}$.

Hence

$k = 9$

c From part b $\mathbf{B}^{-1} = \frac{1}{9}\mathbf{B}^{\mathrm{T}} = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$

The matrix representing $U$ is $\mathbf{AB}$

The order of multiplying matrices is important. The matrix applied first, $\mathbf{B}$ representing the transformation $T$, is on the right. The matrix applied second, $\mathbf{A}$ representing the transformation $S$ is on the left. You learnt a similar rule, when applying functions, in module C3: fg means 'do g first, then f'.

Let the point whose image under $U$ is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ have vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then

$$\mathbf{AB}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

This equation expresses the information that the combined transformation $U = ST$ transforms $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. You use inverse matrices to solve this equation for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Multiplying both sides on the left by $(\mathbf{AB})^{-1}$

$$(\mathbf{AB})^{-1}.\mathbf{AB}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\mathbf{AB})^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Using $(\mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I}$ and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{B}^{-1}\mathbf{A}^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 3-2+2 \\ -6+1+2 \\ 6+2+1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 3 \\ -3 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

The point whose image under $U$ is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ has vector $\begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$.

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 30**

**Question:**

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$$

**a** Find the eigenvalues of $\mathbf{M}$.

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{M}$. There is a line through the origin for which every point on the line is mapped onto itself under $T$.

**b** Find the Cartesian equation of this line. **[E]**

**Solution:**

**a**

$$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4-\lambda & -5 \\ 6 & -9-\lambda \end{pmatrix}$$

$$\det(\mathbf{M} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & -5 \\ 6 & -9-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-9-\lambda) - (-5)\times 6$$

$$= -36 - 4\lambda + 9\lambda + \lambda^2 + 30$$

$$= \lambda^2 + 5\lambda - 6 = 0$$

$$(\lambda + 6)(\lambda - 1) = 0$$

$$\lambda = -6, 1$$

The eigenvalues of $\mathbf{M}$ are $-6$ and $1$.

The eigenvalues of a square matrix are found by solving the polynomial $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$. You find the determinant of a $2\times 2$ matrix using the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

**b** Let the line through the origin have equation $y = mx$. If $t$ is a real parameter, the point $(t, mt)$ lies on the line. Under $T$, the point $(t, mt)$ is mapped onto itself.

$$\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} t \\ mt \end{pmatrix}$$

$$\begin{pmatrix} 4t - 5mt \\ 6t - 9mt \end{pmatrix} = \begin{pmatrix} t \\ mt \end{pmatrix}$$

An alternative method is to use the fact that a line of invariant points is determined by the eigenvector corresponding to $\lambda = 1$. The general eigenvector is $t\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $(5t, 3t)$ always lies on $y = \frac{3}{5}x$.

Equating the upper elements

$$4t - 5mt = t$$

$$5mt = 3t \Rightarrow m = \frac{3}{5}$$

$t = 0$ is an answer but that gives you no additional information as the point $(0, 0)$ lies on $y = \frac{3}{5}x$.

An equation of the line of invariant points is $y = \frac{3}{5}x$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 31**

**Question:**

A transformation $T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented by the matrix $\mathbf{A}=\begin{pmatrix} k & 2 \\ 2 & -1 \end{pmatrix}$, where $k$ is a constant.

For the case $k=-4$,

**a** find the image under $T$ of the line with equation $y=2x+1$.

For the case $k=2$, find

**b** the two eigenvalues of $\mathbf{A}$,

**c** a Cartesian equation of the two lines passing through the origin which are invariant under $T$. **[E]**

**Solution:**

**a** If $t$ is a real parameter, the point $(t, 2t+1)$ lies on the line with equation $y = 2x+1$, for all $t$.
With $k = -4$,
$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} t \\ 2t+1 \end{pmatrix} = \begin{pmatrix} -4t+4t+2 \\ 2t-2t-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
The image under $T$ of the line with equation $y = 2x+1$ is the point with coordinates $(2, -1)$.

The whole of the line is mapped onto a single point. Usually a line is mapped onto a line but it is not always the case. Here the determinant of the matrix is 0 and the matrix is singular. With a singular matrix, a line may map onto a single point.

**b** With $k = 2$
$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}$$
$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$
$$(2-\lambda)(-1-\lambda) - 4 = -2 - 2\lambda + \lambda + \lambda^2 - 4 = 0$$
$$\lambda^2 - \lambda - 6 = 0$$
$$(\lambda+2)(\lambda-3) = 0$$
$$\lambda = -2, 3$$
The eigenvalues of $\mathbf{A}$ are $-2$ and 3.

$\lambda^2 - \lambda - 6$ is the **characteristic polynomial** of the matrix $\mathbf{A}$.

**c** Using the eigenvalues from part **b**
With $\lambda = -2$,
$$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$
Equating the upper elements
$2x + 2y = -2x \Rightarrow y = -2x$
With $\lambda = 3$,
$$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$
Equating the upper elements
$$2x + 2y = 3x \Rightarrow y = \frac{1}{2}x$$
The Cartesian equations of the lines are
$y = \frac{1}{2}x$ and $y = -2x$.

Vectors directed along the invariant lines are eigenvectors.
A Cartesian equation of the invariant line corresponding to an eigenvalue $\lambda$ can be found by writing the equation for an eigenvector,
$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix},$$
and equating either the upper or the lower elements. Either of the elements will give you the same equation.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 32**

**Question:**

The eigenvalues of the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$, are $\lambda_1$ and $\lambda_2$, where $\lambda_1 < \lambda_2$.

**a** Find the value of $\lambda_1$ and the value of $\lambda_2$.

**b** Find $\mathbf{M}^{-1}$.

**c** Verify that the eigenvalues of $\mathbf{M}^{-1}$ are $\lambda_1^{-1}$ and $\lambda_2^{-1}$.

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{M}$. There are two lines, passing through the origin, each of which is mapped onto itself under the transformation $T$.

**d** Find Cartesian equations for each of these lines. **[E]**

**Solution:**

**a** $\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}4-\lambda & -2\\ 1 & 1-\lambda\end{pmatrix}$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}4-\lambda & -2\\ 1 & 1-\lambda\end{vmatrix}=0$$

$$(4-\lambda)(1-\lambda)+2=4-4\lambda-\lambda+\lambda^2+2=0$$

$$\lambda^2-5\lambda+6=0$$

$$(\lambda-2)(\lambda-3)=0$$

$$\lambda=2,3$$

As $\lambda_1<\lambda_2$

$\lambda_1=2, \lambda_2=3$

**b** $\mathbf{M}^{-1}=\frac{1}{4+2}\begin{pmatrix}1 & 2\\ -1 & 4\end{pmatrix}=\frac{1}{6}\begin{pmatrix}1 & 2\\ -1 & 4\end{pmatrix}$

> You use the formula for the inverse of a $2\times2$ matrix, $\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$. The formulae for the determinant and the inverse of a $2\times2$ matrix are parts of the FP1 specification and these formulae may be tested on an FP3 paper.

**c** $\mathbf{M}^{-1}-\lambda\mathbf{I}=\begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{6}-\lambda & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}-\lambda\end{pmatrix}$

$$\begin{vmatrix}\frac{1}{6}-\lambda & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}-\lambda\end{vmatrix}=\left(\frac{1}{6}-\lambda\right)\left(\frac{2}{3}-\lambda\right)+\frac{1}{18}=0$$

$$\frac{1}{9}-\frac{1}{6}\lambda-\frac{2}{3}\lambda+\lambda^2+\frac{1}{18}=0$$

$$2-3\lambda-12\lambda+18\lambda^2+1=0$$

$$18\lambda^2-15\lambda+3=0$$

$$6\lambda^2-5\lambda+1=(2\lambda-1)(3\lambda-1)=0$$

$\lambda=\frac{1}{2},\frac{1}{3}=\lambda_1^{-1},\lambda_2^{-1}$, as required

> It would also be acceptable to substitute $\lambda=\frac{1}{2}$ and $\lambda=\frac{1}{3}$ into this determinant, evaluate the determinant and show that both substitutions gave 0. For example
>
> $$\begin{vmatrix}\frac{1}{6}-\frac{1}{2} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}-\frac{1}{2}\end{vmatrix}=\begin{vmatrix}-\frac{1}{3} & \frac{1}{3}\\ -\frac{1}{6} & \frac{1}{6}\end{vmatrix}=-\frac{1}{18}+\frac{1}{18}=0.$$

**d** Using the eigenvalues from part **a**

With $\lambda = 2$, $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$

Equating the upper elements

$4x - 2y = 2x \Rightarrow y = x$

With $\lambda = 3$, $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$

Equating the upper elements

$4x - 2y = 3x \Rightarrow y = \frac{1}{2}x$

The Cartesian equations of the lines are

$y = \frac{1}{2}x$ and $y = -2x$.

> If you equated the lower elements, you would obtain $x + y = 2y \Rightarrow y = x$. Equating the upper and the lower elements gives the same answer.

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 33**

**Question:**

Find the eigenvalues and corresponding eigenvectors for the matrix $\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$

**[E]**

**Solution:**

Let $\mathbf{A}=\begin{pmatrix}2 & -3 & 1\\ 3 & 1 & 3\\ -5 & 2 & -4\end{pmatrix}$, then

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2 & -3 & 1\\ 3 & 1 & 3\\ -5 & 2 & -4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2 & -3 & 1\\ 3 & 1 & 3\\ -5 & 2 & -4\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}2-\lambda & -3 & 1\\ 3 & 1-\lambda & 3\\ -5 & 2 & -4-\lambda\end{pmatrix}$$

As the eigenvalues are the solutions of $\det(\mathbf{A}-\lambda\mathbf{I})=0$, the first step is to find $(\mathbf{A}-\lambda\mathbf{I})$. With practice, this step can just be written down and this working will not be shown in the working to later questions.

$$\det(\mathbf{A}-\lambda\mathbf{I})=(2-\lambda)\begin{vmatrix}1-\lambda & 3\\ 2 & -4-\lambda\end{vmatrix}-(-3)\begin{vmatrix}3 & 3\\ -5 & -4-\lambda\end{vmatrix}+1\begin{vmatrix}3 & 1-\lambda\\ -5 & 2\end{vmatrix}$$

$$=(2-\lambda)\left[(1-\lambda)(-4-\lambda)-6\right]+3(-12-3\lambda+15)+(6+5-5\lambda)$$

$$=(2-\lambda)(\lambda^2+3\lambda-10)-9\lambda+9+11-5\lambda$$

$$=2\lambda^2+6\lambda-20-\lambda^3-3\lambda^2+10\lambda+20-14\lambda$$

$$=-\lambda^3-\lambda^2+2\lambda$$

This cubic is the **characteristic polynomial** of the matrix $\mathbf{A}$.

Using $\det(\mathbf{A}-\lambda\mathbf{I})=0$

$$-\lambda^3-\lambda^2+2\lambda=0$$

$$-\lambda(\lambda^2+\lambda-2)=-\lambda(\lambda-1)(\lambda+2)=0$$

$$\lambda=0,1,-2$$

The eigenvalues of the matrix are $-2, 0$ and 1.

**To find an eigenvector corresponding to** $-2$.

$$\begin{pmatrix}2 & -3 & 1\\ 3 & 1 & 3\\ -5 & 2 & -4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=-2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

An eigenvector corresponding to an eigenvalue $\lambda$ is a solution of $\mathbf{A}\mathbf{x}=\lambda\mathbf{x}$. There are always an infinite number of such solutions but they are all parallel.

$$\begin{pmatrix}2x-3y+z\\ 3x+y+3z\\ -5x+2y-4z\end{pmatrix}=\begin{pmatrix}-2x\\ -2y\\ -2z\end{pmatrix}$$

Equating the top elements

$2x-3y+z=-2x \Rightarrow 4x-3y+z=0$ ①

Equating the middle elements

$3x+y+3z=-2y \Rightarrow 3x+3y+3z=0$ ②

① + ②

Equating the three elements would give three equations. However two of the equations will usually be sufficient to find an eigenvector.

$7x+4z=0$

Let $x=4$, then $z=-7$

From ②

$y=-x-z=-4+7=3$

At this stage there is a free choice of one variable and the other variables can then be evaluated. Here $x$ has been chosen as 4 as this avoids fractions but any value, other than 0, could be chosen.

An eigenvector corresponding to the eigenvalue $-2$ is $\begin{pmatrix} 4 \\ 3 \\ -7 \end{pmatrix}$.

Any non-zero multiple of this vector is also a correct answer.

**To find an eigenvector corresponding to 0**

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

You now repeat the procedure you used to find an eigenvector corresponding to $-2$ to find eigenvectors corresponding to 0 and 1.

$$\begin{pmatrix} 2x-3y+z \\ 3x+y+3z \\ -5x+2y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the top elements

$2x-3y+z=0$ ①

Equating the middle elements

$3x+y+3z=0$ ②

① $+3\times$ ②

$11x+10z=0$

Let $x=10$, then $z=-11$

From ②

$y=-3x-3z=-30+33=3$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 10 \\ 3 \\ -11 \end{pmatrix}$.

Again any non-zero multiple of the vector is correct. Zero is impossible as, by definition, eigenvectors are non-zero but note that, as here, an eigenvalue may be zero.

**To find an eigenvector corresponding to 1**

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x-3y+z \\ 3x+y+3z \\ -5x+2y-4z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the top elements

$2x-3y+z=x \Rightarrow x-3y+z=0$ ①

Equating the middle elements

$3x+y+3z=y \Rightarrow 3x+3z=0 \Rightarrow x+z=0$ ②

In this case there is no $y$ in equation 2 so it is not necessary to eliminate a variable between the equations and you can make a choice of $x$ (or $z$) immediately.

Let $x=1$, then $z=-1$

From ①

$1-3y-1=0 \Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 34**

**Question:**

Given that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{A}$ where $\mathbf{A} = \begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix}$,

**a** find the eigenvalue of $\mathbf{A}$ corresponding to $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$,

**b** find the value of $p$ and the value of $q$.

The image of the vector $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ when transformed by $\mathbf{A}$ is $\begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$.

**c** Using the values of $p$ and $q$ from part **b**, find the values of the constants $l$, $m$ and $n$.

**[E]**

**Solution:**

**a** $\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \lambda\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

If the column vector **x** is an eigenvector of the matrix **A** then for some number $\lambda$, $\mathbf{Ax} = \lambda\mathbf{x}$.

$\begin{pmatrix} 4-p \\ q+4 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ -\lambda \end{pmatrix}$

Equating the three elements of these column vectors gives you three equations from which you can find the values of $\lambda$, $p$ and $q$.

Equating the lowest elements

$-2 = -\lambda \Rightarrow \lambda = 2$

**b** Equating the top elements

$4 - p = 0 \Rightarrow p = 4$

Equating the middle elements

$q + 4 = \lambda = 2 \Rightarrow q = -2$

**c** Using the results of part **b**

$\begin{pmatrix} 3 & 4 & 4 \\ -1 & -2 & -4 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$

As $\mathbf{A}\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$, then an alternative method is to find $\mathbf{A}^{-1}$ and use $\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$. However an inverse matrix is quite complicated to work out and, in this question, you have not been asked to find it. If the question does not require you to find the inverse, the method illustrated here, using simultaneous equations and the ordinary processes of algebra, is often carried out more quickly. If you use the inverse matrix then

$\mathbf{A}^{-1} = -\frac{1}{6}\begin{pmatrix} -2 & -8 & -8 \\ -1 & 5 & 8 \\ 1 & 1 & -2 \end{pmatrix}$.

$\begin{pmatrix} 3l+4m+4n \\ -l-2m-4n \\ l+m+3n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$

Equating the elements of the vectors

$3l + 4m + 4n = 10$ ①

$-l - 2m - 4n = -4$ ②

$l + m + 3n = 3$ ③

① $+3\times$ ②

$-2m - 8n = -2$ ④

② + ③

$-m - n = -1$ ⑤

$2\times$ ⑤ − ④

$6n = 0 \Rightarrow n = 0$

Substitute $n = 0$ into ⑤

$-m = -1 \Rightarrow m = 1$

Substitute $n = 0$ and $m = 1$ into ③

$l + 1 + 0 = 3 \Rightarrow l = 2$

$l = 2, m = 1, n = 0$

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 35**

**Question:**

$$A = \begin{pmatrix} 5 & 1 & -2 \\ -1 & 6 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

**a** Show that 3 is an eigenvalue of $\mathbf{A}$.

**b** Find the other two eigenvalues of $\mathbf{A}$.

**c** Find also a normalised eigenvector corresponding to the eigenvalue 3. **[E]**

**Solution:**

**a** Substitute $\lambda = 3$ into $\begin{vmatrix} 5-\lambda & 1 & -2 \\ -1 & 6-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix}$

If 3 is an eigenvalue of $\mathbf{A}$, then $\lambda = 3$ must satisfy the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. So to solve part **a**, it is sufficient to substitute $\lambda = 3$ into this determinant and show that its value is 0.

$$\begin{vmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= 2\times(-1) - 1\times 0 + (-2)\times(-1) = -2 + 2 = 0$$

Hence 3 is an eigenvalue of $\mathbf{A}$.

**b** $\begin{vmatrix} 5-\lambda & 1 & -2 \\ -1 & 6-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix}$

$$= (5-\lambda)\begin{vmatrix} 6-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 0 & 3-\lambda \end{vmatrix} + (-2)\begin{vmatrix} -1 & 6-\lambda \\ 0 & 1 \end{vmatrix}$$

$$= (5-\lambda)\left[(6-\lambda)(3-\lambda) - 1\right] + (3-\lambda) + 2$$

$$= (5-\lambda)\left[18 - 9\lambda + \lambda^2 - 1\right] + 5 - \lambda$$

$$= (5-\lambda)\left[\lambda^2 - 9\lambda + 17\right] + 5 - \lambda$$

$$= 5\lambda^2 - 45\lambda + 85 - \lambda^3 + 9\lambda^2 - 17\lambda + 5 - \lambda$$

$$= 90 - 63\lambda + 14\lambda^2 - \lambda^3$$

The eigenvalues of $\mathbf{A}$ are the solutions of

$\lambda^3 - 14\lambda^2 + 63\lambda - 90 = 0$

Let

$\lambda^3 - 14\lambda^2 + 63\lambda - 90 = (\lambda - 3)(\lambda^2 + a\lambda + 30)$

As you know $\lambda = 3$ is a solution of this equation, you can factorise this cubic either by long division or, as is shown here, by equating coefficients.

Equating the coefficients of $\lambda^2$

$-14 = -3 + a \Rightarrow a = -11$

Hence

$$(\lambda - 3)(\lambda^2 - 11\lambda + 30) = (\lambda - 3)(\lambda - 5)(\lambda - 6) = 0$$

$$\lambda = 3, 5, 6$$

The other two eigenvalues of $\mathbf{A}$ are 5 and 6.

c To find an eigenvector corresponding to 3.

$$\begin{pmatrix} 5 & 1 & -2 \\ -1 & 6 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 5x + y - 2z \\ -x + 6y + z \\ y + 3z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the lowest elements

$y + 3z = 3z \Rightarrow y = 0$ ①

Equating any 2 of the 3 elements will give you sufficient information to solve the question. Here the lowest elements give a particularly simple equation, so these have been used first.

Equating the top elements and substituting $y = 0$

$5x - 2z = 3x \Rightarrow 2x = 2z \Rightarrow x = z$ ②

Let $z = 1$, then $x = 1$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

The length of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is $\sqrt{(1^2 + 0^2 + 1^2)} = \sqrt{2}$

A normalised eigenvector is an eigenvector of length 1. To normalise an eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, you divide each of the components of the vector by the length (or magnitude) of the vector, $\sqrt{(x^2 + y^2 + z^2)}$.

A normalised eigenvector corresponding to the eigenvalue 3 is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Either of these forms is acceptable as an answer.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 36**

**Question:**

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & k \end{pmatrix}$$

**a** Show that $\det A = 20 - 4k$.

**b** Find $A^{-1}$.

Given that $k = 3$ and that $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of A,

**c** find the corresponding eigenvalue.
Given that the only other distinct eigenvalue of A is 8,

**d** find a corresponding eigenvector. **[E]**

**Solution:**

**a** $\det \mathbf{A} = 3\begin{vmatrix} 0 & 2 \\ 2 & k \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & k \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix}$

$= 3(0-4) - 2(2k-8) + 4(4-0)$

$= -12 - 4k + 16 + 16 = 20 - 4k$, as required.

**b** The matrix of the minors, $\mathbf{M}$ say, is given by

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & 2 \\ 2 & k \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 4 & k \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix} \\ \cdots & \begin{vmatrix} 3 & 4 \\ 4 & k \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} \\ \cdots & \cdots & \begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} \end{pmatrix}$$

As the matrix $\mathbf{A}$ is symmetric, the matrix of the minors is symmetric and you do not have to work out separately the three minors marked … in this matrix.

$$= \begin{pmatrix} -4 & 2k-8 & 4 \\ 2k-8 & 3k-16 & -2 \\ 4 & -2 & -4 \end{pmatrix}$$

The matrix of the cofactors, $\mathbf{C}$ say, is given by

$$\mathbf{C} = \begin{pmatrix} -4 & -2k+8 & 4 \\ -2k+8 & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

The matrix of the cofactors is obtained from the matrix of the minors by changing the signs of the elements marked with a negative sign in this pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

As $\mathbf{C}$ is symmetric $\mathbf{C}^{\mathrm{T}} = \mathbf{C}$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}} = \frac{1}{\det \mathbf{A}}\mathbf{C}$$

$$= \frac{1}{20-4k}\begin{pmatrix} -4 & -2k+8 & 4 \\ -2k+8 & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

**c** If $k = 3, \mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

The eigenvalue corresponding to $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ is given by

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \lambda\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\lambda \\ -\lambda \end{pmatrix}$$

If the column vector $\mathbf{x}$ is an eigenvector of the matrix $\mathbf{A}$ then the corresponding eigenvalue $\lambda$ is given by $\mathbf{Ax} = \lambda\mathbf{x}$.

Equating the middle elements

$-2 = 2\lambda \Rightarrow \lambda = -1$

The corresponding eigenvalue is $-1$.

**d** To find an eigenvector corresponding to 8

$$\begin{pmatrix} 3x+2y+4z \\ 2x+2z \\ 4x+2y+3z \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix}$$

Equating any two of the three elements will enable you to find an eigenvector.

Equating the top elements

$3x+2y+4z = 8x \Rightarrow -5x+2y+4z = 0$ ①

Equating the lowest elements

$4x+2y+3z = 8z \Rightarrow 4x+2y-5z = 0$ ②

② − ①

$9x - 9z = 0 \Rightarrow x = z$

Let $z = 2$, then $x = 2$

Here you have a free choice of either $x$ or $z$. This choice has been made to avoid fractions but any value of $z$ could be chosen.

Substitute $x = 2$ and $z = 2$ into ①

$-10 + 2y + 8 = 0 \Rightarrow 2y = 2 \Rightarrow y = 1$

An eigenvector corresponding to the eigenvalue 8 is $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

Any non-zero multiple of this vector is also a correct answer.

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 37**

**Question:**

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}$$

**a** Verify that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and find the corresponding eigenvalue.

**b** Show that 9 is another eigenvalue of $\mathbf{A}$ and find the corresponding eigenvector.

**c** Given that the third eigenvector of $\mathbf{A}$ is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$. **[E]**

**Solution:**

**a** $\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}2+4\\-10+4\\8-8+3\end{pmatrix}=\begin{pmatrix}6\\-6\\3\end{pmatrix}$

$=3\begin{pmatrix}2\\-2\\1\end{pmatrix}$

Hence $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and the corresponding eigenvalue is 3.

An eigenvector is a vector whose direction is not changed by the transformation. So to verify that a column vector $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, you have to show that for some constant $\lambda, \mathbf{Ax}=\lambda\mathbf{x}$. $\lambda$, the magnification factor of the vector under the transformation, is the eigenvalue corresponding to $\mathbf{x}$.

**b** Substitute $\lambda=9$ into

$\begin{vmatrix}1-\lambda&0&4\\0&5-\lambda&4\\4&4&3-\lambda\end{vmatrix}$

An eigenvalue is a solution of $\det(\mathbf{A}-\lambda\mathbf{I})=0$. To verify that 9 is an eigenvalue it is sufficient to substitute $\lambda=9$ into this determinant and show that its value is 0.

$\begin{vmatrix}1-9&0&4\\0&5-9&4\\4&4&3-9\end{vmatrix}=\begin{vmatrix}-8&0&4\\0&-4&4\\4&4&-6\end{vmatrix}$

$=(-8)\begin{vmatrix}-4&4\\4&-6\end{vmatrix}-0\begin{vmatrix}0&4\\4&-6\end{vmatrix}+4\begin{vmatrix}0&-4\\4&4\end{vmatrix}$

$=(-8)(24-16)-0+4(0+16)$

$=-8\times8+4\times16=-64+64=0$

Hence 9 is an eigenvalue of $\mathbf{A}$.

To find an eigenvector corresponding to 9.

$\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=9\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\begin{pmatrix}x+4z\\5y+4z\\4x+4y+3z\end{pmatrix}=\begin{pmatrix}9x\\9y\\9z\end{pmatrix}$

Equating the top elements
$x+4z=9x\Rightarrow -8x+4z=0\Rightarrow z=2x$
Let $x=1$, then $z=2$
Equating the middle elements
$5y+4z=9y\Rightarrow 4z=4y\Rightarrow y=z$
As $z=2, y=2$

At this stage there is a free choice of one variable and the other variables can then be evaluated. Here $x$ has been chosen as 1, as this avoids fractions, but any value, other than 0, could have been chosen,

An eigenvector corresponding to the eigenvalue 9 is $\begin{pmatrix}1\\2\\2\end{pmatrix}$.

Any non-zero multiple of this vector is also a correct answer.

**c** $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2-8 \\ 5-8 \\ 8+4-6 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 6 \end{pmatrix}$

$= -3\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

The diagonal elements of the matrix **D** are the eigenvalues and you will need the eigenvalue corresponding to $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ to write down **D**.

The eigenvalue corresponding to $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ is $-3$.

The lengths of the vector $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is

$\sqrt{\left(2^2 + (-2)^2 + 1^2\right)} = \sqrt{9} = 3$

A normalised eigenvector is an eigenvector of length 1.
To normalise an eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, you divide each of the components of the vector by the length (or magnitude) of the vector, $\sqrt{\left(x^2 + y^2 + z^2\right)}$. In this case the length of all three vectors is 3.

Similarly the lengths of $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ are 3.

Normalised eigenvectors are $\begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$ and $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$.

$\mathbf{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

If normalised eigenvectors are used to form **P** then the diagonal elements of **D** are the corresponding eigenvalues and this is the safest way to complete the question. However, any three distinct eigenvectors of the same magnitude can be used and the diagonal elements will be multiples of the eigenvalues. There are infinitely many possible answers. One other possible answer is

$\mathbf{P} = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 27 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & -27 \end{pmatrix}$.

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 38**

**Question:**

$$\mathbf{A} = \begin{pmatrix} 6 & 2 & -3 \\ 2 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix}$$

Given that $\lambda = -1$ and $\lambda = 8$ are two eigenvalues of **A**,

**a** find the third eigenvalue of **A**.

**b** Find the normalised eigenvector corresponding to the eigenvalue $\lambda = 8$.

Given that $\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$ are normalised eigenvectors corresponding to the other two eigenvalues,

**c** find a matrix **P** such that $\mathbf{P}^T\mathbf{AP}$ is a diagonal matrix.

**d** Find $\mathbf{P}^T\mathbf{AP}$. **[E]**

**Solution:**

**a** $\det(\mathbf{A}-\lambda\mathbf{I})=0$

$$\begin{vmatrix} 6-\lambda & 2 & -3 \\ 2 & -\lambda & 0 \\ -3 & 0 & 2-\lambda \end{vmatrix}$$

$$=(6-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ -3 & 2-\lambda \end{vmatrix} + (-3)\begin{vmatrix} 2 & -\lambda \\ -3 & 0 \end{vmatrix}$$

$$=(6-\lambda)(-2\lambda+\lambda^2)-2(4-2\lambda)+(-3)(-3\lambda)$$

$$=-12\lambda+6\lambda^2+2\lambda^2-\lambda^3-8+4\lambda+9\lambda$$

$$=-\lambda^3+8\lambda^2+\lambda-8=0$$

$$\lambda^3-8\lambda^2-\lambda+8=0$$

$$\lambda^2(\lambda-8)-1(\lambda-8)=0$$

$$(\lambda^2-1)(\lambda-8)=0$$

$$(\lambda-1)(\lambda+1)(\lambda-8)=0$$

$$\lambda=1,-1,8$$

The third eigenvalue is 1.

> As you know that $-1$ and 8 are roots of this equation you could just write down that the factors of the cubic are $(\lambda+1)(\lambda-8)(\lambda-1)$. However it is a good idea to factorise fully the expression, as shown here, to check that you have not made an error.

**b**

$$\begin{pmatrix} 6 & 2 & -3 \\ 2 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 6x+2y-3z \\ 2x \\ -3x+2z \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix}$$

Equating the middle elements

$2x=8y \Rightarrow x=4y$

Let $y=1$, then $x=4$

Equating the lowest elements

$-3x+2z=8z \Rightarrow 3x=-6z \Rightarrow x=-2z$

As $x=4$

$4=-2z \Rightarrow z=-2$

> In general, the simplest equations are those with the fewest variables in them, so it is sensible to equate the middle and lowest terms.

An eigenvector corresponding to $\lambda = 8$ is $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

The length of $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(4^2 + 1^2 + (-2)^2\right)} = \sqrt{21}$.

A normalised vector corresponding to $\lambda = 8$ is

$$\begin{pmatrix} \frac{4}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \\ \frac{-2}{\sqrt{21}} \end{pmatrix}$$

A normalised eigenvector is an eigenvector of length 1.

To normalise an eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, you divide each of the components of the vector by the length (or magnitude) of the vector, $\sqrt{\left(x^2 + y^2 + z^2\right)}$. The vector $\begin{pmatrix} \frac{-4}{\sqrt{21}} \\ \frac{-1}{\sqrt{21}} \\ \frac{2}{\sqrt{21}} \end{pmatrix}$ is also correct.

**c** $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{21}} \end{pmatrix}$

**d** To find the eigenvalue corresponding to $\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$

$\mathbf{P}^\mathrm{T}\mathbf{AP}$ is a diagonal matrix with the eigenvalues on the diagonals in the order corresponding to the order of the normalised eigenvectors used to form $\mathbf{P}$. At this stage, you do not know if $\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$ corresponds to 1 or $-1$, and you must establish which before proceeding.

$$\begin{pmatrix} 6 & 2 & -3 \\ 2 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} \frac{6+4-9}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{-3+6}{\sqrt{14}} \end{pmatrix} = 1 \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$$

1 is the eigenvalue corresponding to $\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$

Hence $\mathbf{P}^\mathrm{T}\mathbf{AP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 39**

**Question:**

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix}$$

**a** Find the eigenvalues and corresponding eigenvectors of $\mathbf{M}$.

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{M}$.

**b** Find Cartesian equations of the image of the line $\frac{x}{2} = y = \frac{z}{-1}$ under this transformation. **[E]**

**Solution:**

**a** $\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}1-\lambda & 0 & 1\\ 0 & 2-\lambda & 0\\ 4 & 3 & 1-\lambda\end{pmatrix}$

The eigenvalues are the solutions of the equation $\det(\mathbf{M}-\lambda\mathbf{I})=0$.

$$\det(\mathbf{M}-\lambda\mathbf{I})=(1-\lambda)\begin{vmatrix}2-\lambda & 0\\ 3 & 1-\lambda\end{vmatrix}-0\begin{vmatrix}0 & 0\\ 4 & 1-\lambda\end{vmatrix}+1\begin{vmatrix}0 & 2-\lambda\\ 4 & 3\end{vmatrix}$$
$$=(1-\lambda)(2-\lambda)(1-\lambda)-0-4(2-\lambda)$$
$$=(2-\lambda)(1-\lambda)^2-4(2-\lambda)$$
$$=(2-\lambda)\left((1-\lambda)^2-4\right)=(2-\lambda)(\lambda^2-2\lambda-3)$$
$$=(2-\lambda)(\lambda-3)(\lambda+1)=0$$
$$\lambda=-1,2,3$$

$(2-\lambda)$ is a common factor of this expression. Taking this factor outside the expression at this stage avoids having to factorise a cubic later.

The eigenvalues of $\mathbf{M}$ are $-1, 2$ and 3.

**To find an eigenvector corresponding to $-1$**

$$\begin{pmatrix}1&0&1\\0&2&0\\4&3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-1\begin{pmatrix}x\\y\\z\end{pmatrix}$$
$$\begin{pmatrix}x+z\\2y\\4x+3y+z\end{pmatrix}=\begin{pmatrix}-x\\-y\\-z\end{pmatrix}$$

Equating the middle elements
$2y=-y\Rightarrow 3y=0\Rightarrow y=0$
Equating the top elements
$x+z=-x\Rightarrow z=-2x$
Let $x=1$, then $z=-2$

In general, start by equating the elements with the fewest variables. Here the middle elements contain only $y$.

An eigenvector corresponding to the eigenvalue $-1$ is $\begin{pmatrix}1\\0\\-2\end{pmatrix}$.

**To find an eigenvector corresponding to 2**

$$\begin{pmatrix}1&0&1\\0&2&0\\4&3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=2\begin{pmatrix}x\\y\\z\end{pmatrix}$$
$$\begin{pmatrix}x+z\\2y\\4x+3y+z\end{pmatrix}=\begin{pmatrix}2x\\2y\\2z\end{pmatrix}$$

Equating the middle elements here gives $2y=2y$. This is a simple identity and gives you no information so you must use the other elements.

Equating the top elements
$x+z=2x\Rightarrow x=z$
Let $z=1$, then $x=1$
Equating the lowest elements
$4x+3y+z=2z\Rightarrow 3y=z-4x$
As $x=1$ and $z=1$
$3y=1-4\Rightarrow y=-1$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

**To find an eigenvector corresponding to 3**

$$\begin{pmatrix}1&0&1\\0&2&0\\4&3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x+z\\2y\\4x+3y+z\end{pmatrix}=\begin{pmatrix}3x\\3y\\3z\end{pmatrix}$$

Equating the middle elements

$2y=3y\Rightarrow y=0$

Equating the top elements

$x+z=3x\Rightarrow z=2x$

Let $x=1$, then $z=2$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix}1\\0\\2\end{pmatrix}$.

**b** Let $\frac{x}{2}=y=\frac{z}{-1}=t$

The $x=2t, y=t, z=-t$

$(2t,t,-t)$ is the parametric form of the general point on the line

Although there are other methods, when finding the images of lines under three-dimensional linear transformations, it is usually sensible to find the parametric form of a general point on the line and to obtain the image of the general point by matrix multiplication.

$$\begin{pmatrix}1&0&1\\0&2&0\\4&3&1\end{pmatrix}\begin{pmatrix}2t\\t\\-t\end{pmatrix}=\begin{pmatrix}2t-t\\2t\\8t+3t-t\end{pmatrix}=\begin{pmatrix}t\\2t\\10t\end{pmatrix}$$

The image of the line under this transformation is $x=t, y=2t, z=10t$

Hence

$$x=\frac{y}{2}=\frac{z}{10}=t$$

Eliminating $t$ gives Cartesian equations of the line.

Cartesian equations of the image of the line are

$$x=\frac{y}{2}=\frac{z}{10}.$$

# Solutionbank FP3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 40**

**Question:**

**a** Show that 9 is an eigenvalue of the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}$.

**b** Find the other two eigenvalues of the matrix.

**c** Find also normalised eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ corresponding to each of these eigenvalues.

**d** Verify that the matrix **P** with columns $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ is an orthogonal matrix. **[E]**

**Solution:**

**a** $\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 5-\lambda & 0 \\ 2 & 0 & 7-\lambda \end{vmatrix}$

$= (6-\lambda)\begin{vmatrix} 5-\lambda & 0 \\ 0 & 7-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -2 & 0 \\ 2 & 7-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 5-\lambda \\ 2 & 0 \end{vmatrix}$

$= (6-\lambda)(5-\lambda)(7-\lambda) + 2(-14+2\lambda) + 2(-10+2\lambda)$

$= (6-\lambda)(5-\lambda)(7-\lambda) - 28 + 4\lambda - 20 + 4\lambda$

$= (6-\lambda)(5-\lambda)(7-\lambda) + 8\lambda - 48$

$= (6-\lambda)(5-\lambda)(7-\lambda) + 8(\lambda - 6)$

$= (6-\lambda)(35 - 12\lambda + \lambda^2 - 8)$

$= (6-\lambda)(27 - 12\lambda + \lambda^2)$

$= (6-\lambda)(3-\lambda)(9-\lambda)$

$(6-\lambda)(3-\lambda)(9-\lambda) = 0$

$\lambda = 3, 6, 9$

9 is an eigenvalue of the matrix.

**b** The other eigenvalues are 3 and 6

**c** **To find an eigenvector corresponding to 3**

$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} 6x - 2y + 2z \\ -2x + 5y \\ 2x + 7z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$

Equating the middle terms

$-2x + 5y = 3y \Rightarrow 2y = 2x \Rightarrow y = x$

Let $x = 2$, then $y = 2$

Equating the lowest terms

$2x + 7z = 3z \Rightarrow 2x = -4z \Rightarrow x = -2z$

As $x = 2$, $z = -1$

You can find an eigenvector by equating any two of the elements. You can then choose a non-zero value for any one of the variables and use it to calculate the values of the other variables. Here $x$ has been chosen to be 2 in order to avoid fractions.

An eigenvector corresponding to 3 is $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$.

The length of $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ is $\sqrt{(2^2 + 2^2 + (-1)^2)} = \sqrt{9} = 3$

A normalised eigenvector is an eigenvector of length 1. To normalise an eigenvector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, you divide each of the components of the vector by the length (or magnitude) of the vector, $\sqrt{(x^2 + y^2 + z^2)}$.

A normalised eigenvector, $\mathbf{x}_1$ say, corresponding to 3 is $\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$.

$\begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ is also correct.

**To find an eigenvector corresponding to 6**

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 6x-2y+2z \\ -2x+5y \\ 2x+7z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the middle terms

$-2x+5y=6y \Rightarrow y=-2x$

Let $x=-1$, then $y=2$

Equating the lowest terms

$2x+7z=6z \Rightarrow z=-2x$

As $x=-1$, $z=2$

An eigenvector corresponding to 6 is $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

The length of $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ is $\sqrt{\left((-1)^2+2^2+2^2\right)}=\sqrt{9}=3$

A normalised eigenvector, $\mathbf{x}_2$ say, corresponding to 6 is

$\begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$.

Again the negative of this vector is correct. There are always two normalised vectors corresponding to an eigenvalue. These two are essentially the same vector in opposite directions.

**To find an eigenvector corresponding to 9.**

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 6x-2y+2z \\ -2x+5y \\ 2x+7z \end{pmatrix} = \begin{pmatrix} 9x \\ 9y \\ 9z \end{pmatrix}$$

Equating the middle terms

$-2x+5y=9y \Rightarrow -2x=4y \Rightarrow x=-2y$

Let $y=-1$, then $x=2$

Equating the lowest terms

$2x+7z=9z \Rightarrow x=z$

As $x=2$, $z=2$

An eigenvector corresponding to 9 is $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

The length of $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is $\sqrt{\left(2^2+(-1)^2+2^2\right)}=\sqrt{9}=3$

A normalised eigenvector, $\mathbf{x}_3$ say, corresponding to 9 is $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

**d** $\quad \mathbf{P} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

There are two ways of testing that a matrix is orthogonal. One is to show that $\mathbf{PP}^{\mathrm{T}}=\mathbf{I}$ and the other is to show that the 3 normalised column vectors are orthogonal to each other. The second method has been used here. The scalar product of each of the three pairs of vectors is shown to be zero and, as all of the vectors are non-zero, this shows the vectors are mutually orthogonal.

$$\mathbf{x}_1.\mathbf{x}_2 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{2}{3}\times\left(-\frac{1}{3}\right)+\frac{2}{3}\times\frac{2}{3}+\left(-\frac{1}{3}\right)\times\frac{2}{3}$$

$$= -\frac{2}{9}+\frac{4}{9}-\frac{2}{9}=0$$

Hence $\mathbf{x}_1$ is orthogonal (perpendicular) to $\mathbf{x}_2$.

$$\mathbf{x}_1 \cdot \mathbf{x}_3 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{2}{3}\times\frac{2}{3}+\frac{2}{3}\times\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)\times\frac{2}{3}$$

$$= \frac{4}{9}-\frac{2}{9}-\frac{2}{9}=0$$

Hence $\mathbf{x}_1$ is orthogonal (perpendicular) to $\mathbf{x}_3$.

Orthogonal and perpendicular have the same meaning.

$$\mathbf{x}_2 \cdot \mathbf{x}_3 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \left(-\frac{1}{3}\right)\times\frac{2}{3}+\frac{2}{3}\times\left(-\frac{1}{3}\right)+\frac{2}{3}\times\frac{2}{3}$$

$$= -\frac{2}{9}-\frac{2}{9}+\frac{4}{9}=0$$

Hence $\mathbf{x}_2$ is orthogonal (perpendicular) to $\mathbf{x}_3$.

The matrix $\mathbf{P}$ is an orthogonal matrix.